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Renormalization Group Equations and the Lifshitz Point In Noncommutative Landau-Ginsburg Theory

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Abstract

A one-loop renormalization group (RG) analysis is performed for noncommutative Landau-Ginsburg theory in an arbitrary dimension. We adopt a modern version of the Wilsonian RG approach, in which a shell integration in momentum space bypasses the potential IR singularities due to UV-IR mixing. The momentum-dependent trigonometric factors in interaction vertices, characteristic of noncommutative geometry, are marginal under RG transformations, and their marginality is preserved at one loop. A negative Θ-dependent anomalous dimension is discovered as a novel effect of the UV-IR mixing. We also found a noncommutative Wilson-Fisher (NCWF) fixed point in less than four dimensions. At large noncommutativity, a momentum space instability is induced by quantum fluctuations, and a consequential first-order phase transition is identified together with a Lifshitz point in the phase diagram. In the vicinity of the Lifshitz point, we introduce two critical exponents ν_m and β_k, whose values are determined to be 1/4 and 1/2, respectively, at mean-field level.

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I. INTRODUCTION

Field theory on a noncommutative (NC) space, or simply noncommutative field theory (NCFT), has attracted much interest recently[1]. The fact that NCFT arises in string/M(atrix) theory [14,15] suggests that space or spacetime noncommutativity should be a general feature of quantum gravity for generic points inside the moduli space of M-theory. As non-gravitational theory that has stringy features, studying NCFT is expected to shed new light on string/M theory. Moreover, as a natural deformation of usual quantum field theory, NCFT is of interest in its own right. In particular, NCFT is expected to be relevant to planar quantum Hall systems, since a charge in the lowest Landau level (LLL) in a strong perpendicular magnetic field can be viewed as living in a noncommutative space: The guiding-center coordinates for the cyclotron motion of the charge in the LLL are known not to commute [16,17].

Of great importance both for quantum gravity and for the applications in condensed matter physics, one of the central issues about quantum dynamics of NCFT is to understand its low energy effective theory. At first glance, the answer to this question seems to be trivial according to the following reasoning: Low energies correspond to large distances. When the distance scale under consideration is much larger than the length scale given by the coordinate noncommutativity, the effects of the latter should be negligible. In other words, the effects of coordinate noncommutativity should be restricted to distances comparable to the noncommutative length scale. Therefore, it seems that noncommutative effects should disappear at sufficiently low energies. If this were true, one would expect that a strong magnetic field, which gives rise to a small noncommutative magnetic length, should have no effects in the large-distance physics of the charged particles in the LLL. From our experience with the quantum Hall systems we know that this is certainly incorrect. On one hand, a feature of the Laughlin wave function [18], which is crucial for its success, is that it is a

[1] For an incomplete list, see e.g. [1–12]. For a more complete list, please see [13].

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wave function in the LLL. On the other hand, many people suspect that the problems with the Chern-Simons theory for half-filled LLL are related to the fact that it fails to properly incorporate the LLL condition [17]. Thus, our experience with quantum Hall physics seems to indicate that coordinate noncommutativity should have characteristic observable effects in large distance phenomena.

To see noncommutativity effects at large distances or at low energies requires a careful analysis of the effects of quantum fluctuations, involving renormalization and renormalization group (RG) flow. In studying one-loop renormalization of noncommutative scalar theory, Minwalla, Raamsdonk and Seiberg (MRS) have discovered a novel effect [5], called UV-IR mixing, which is characteristic of coordinate noncommutativity. The physical essence of UV-IR mixing can be seen as follows: Suppose we consider a noncommutative two dimensional Euclidean space, with

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu} \qquad \mu, \nu = 1, 2, \tag{1}$$

where the antisymmetric $\Theta^{\mu\nu} = \Theta\varepsilon^{\mu\nu}$ $(\mu, \nu = 1, 2)$ contains a real parameter Θ of dimension length squared. Then, we have the uncertainty relation like, say, $\Delta x^1 \Delta x^2 \geq \Theta$. For a wave packet on this noncommutative plane, if we make its size in x^1-direction small, then its size in x^2-direction will become big. So the UV effects in x^1-direction are entangled with the IR effects in x^2-direction. Conceptually, in our opinion, it must be the general notion of this UV-IR mixing that makes noncommutativity effects possible to manifest themselves at large distances.

Let us examine the concrete effects of UV-IR mixing found in Ref. [5], to see whether it could help us understand the RG flow of NCFT to low energies. There they found IR singularities emerging as a trade-off of the UV divergences in loop momentum integration in non-planar diagrams. After removing the cut-off the corrected propagator will have an IR-singular piece even in the massive case. More precisely, the non-planar part of the one-loop corrections to the ϕ-propagator in ϕ^4 theory in D dimensions, is given by

$$\Sigma_{np}(p) = g \int^\Lambda \frac{d^D k}{(2\pi)^D} \frac{e^{ip\wedge k}}{k^2 + m^2}$$

$$= \frac{g}{(4\pi)^{D/2}} \int_0^\infty \frac{d\alpha}{\alpha^{D/2}} \exp\{-\alpha m^2 - \frac{p \circ p}{4\alpha} - \frac{1}{\Lambda^2 \alpha}\}, \tag{2}$$

where Λ is a UV regulator, g^2 the coupling constant and $p \circ q = -p_\mu q_\nu \Theta_{\mu\lambda} \Theta^\nu_\lambda$. This results in a corrected inverse propagator of the form [20]

$$\Gamma^{(2)}(p) = p^2 + m^2 + 2gI_2(0) + gI_2(p), \tag{3}$$

with

$$I_2(p) = (2\pi)^{-D/2} m^{\frac{D-2}{2}} \left(p \circ p + \frac{4}{\Lambda^2}\right)^{\frac{2-D}{4}} K_{\frac{D-2}{4}} \left(m\sqrt{p \circ p + \frac{4}{\Lambda^2}}\right), \tag{4}$$

$K_\nu(x)$ being a modified Bessel function. It is easy to see that the limit $\Lambda \to \infty$ does not commute with the low momentum limit $p \to 0$. If one insists to take the limit $\Lambda \to \infty$ first, then $\Gamma^{(2)}(p)$ acquires an IR singularity that grows to infinity as $p \to 0$. This seems to make a renormalization group analysis difficult to carry out.

However, an RG analysis is essential for a good understanding of the phase structure and the low energy effective theory of NCFT, especially for potential applications in the quantum Hall systems. In this paper we attempt to present a Wilsonian RG analysis for NCFT, in the form recently proposed by Shankar [21] in the context of condensed matter physics and by Polchinski [22] in the context of high energy physics. We observe that in this modern RG approach to low energy effective theory, the momentum cut-off, Λ, that defines the theory is thought of as the momentum above which the dynamical degrees of freedom have been integrated out. Therefore one should *not* take the cut-off to infinity. Rather, it should be kept finite while we do RG transformations to eliminate degrees of freedom between Λ and Λ/s with $s > 1$. Therefore, it is the integration in momentum space over the shell between Λ/s and Λ (with $s > 1$) that is relevant to the RG transformation. This is a shell in momentum space with finite radius; obviously this shell integration bypasses the potential IR singularity found in Ref. [5]. It remains to see how noncommutativity effects manifest themselves at low energies (with increasing s). This is the subject of this paper.

For simplicity, in this paper, we treat a Euclidean version of NC ϕ^4 theory as a noncommutative Landau-Ginsburg (NCLG) theory. Namely, we explicitly introduce temperature

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into the theory through the "mass" term: $m^2 = a_2(T - T_c)$, where $a_2 > 0$ is a constant and T_c is the critical temperature. Other coefficients in front of ϕ^n do not explicitly depend on temperature.

In performing RG analysis, a crucial issue is how to deal with the RG transformation of the momentum dependent Moyal phase or trigonometric factors in the interaction vertices (see eq. (18) below), which arise from coordinate noncommutativity (1), or equivalently from the Moyal's star product (6). One thought might be that when we scale momentum $k \to k' = sk$ under RG transformations, the noncommutativity parameters $\Theta^{\mu\nu}$ appearing in these factors, since they are of dimension of length squared, should transform like $\Theta^{\mu\nu} \to \Theta'^{\mu\nu} = s^{-2}\Theta^{\mu\nu}$, and therefore should eventually flow to zero at low energy (with $s \to \infty$). This is equivalent to renormalizing the NC-parameters in defining the renormalized interaction vertices. However, this is in conflict with the notion of noncommutative geometry, i.e. with the idea that $\Theta_{\mu\nu}$ are among the parameters that describe geometry of the space (or spacetime), which are unaffected by any non-gravitational dynamics. Therefore, after RG transformation the interaction vertices should still be written in the form of a star product of fields with the *same* $\Theta_{\mu\nu}$, rather than the re-scaled ones. As we will see below, this leads to a classification of relevance of operators according to the original star product rather than one that eventually flows to the usual product. This will have an important implication to the RG flow of interactions at low energy: Namely, interactions in the effective low-energy action are always those with an invariable star product; they can never flow to the interactions with the usual product. In this sense we say that the Moyal trigonometric factors in interaction vertices in NCFT is marginal at tree level, and remains marginal at one loop level with the same $\Theta^{\mu\nu}$.

The main results of the present paper reveal a new manifestation of the general notion of UV-IR mixing: Namely, in or near four Euclidean dimensions, non-planar contributions from the noncommutative interaction vertices at one loop induce a renormalization of the quadratic kinetic term in the low energy effective action. This effect is distinct for NCFT, because it is well-known that in ordinary scalar theory in or near four dimensions there is

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no wave function renormalization at all at one loop. On one hand, the new effect indicates a momentum space instability for sufficiently large values of noncommutativity parameter Θ. On the other hand, it gives rise to a *negative, and* Θ*-dependent* critical exponent (or anomalous dimension) η at the noncommutative Wilson-Fisher (NCWF) fixed point in less than four dimensions. This effect certainly makes noncommutativity visible at distances that are much larger than the noncommutative scale, an astonishing entanglement between the UV (at noncommutative scale) and the IR (at the scale of the correlation length). When the noncommutativity exceeds a critical value θ_c, the large distance asymptotic behavior of the correlation function would change its form from a usual negative power law to a positive power law, this signals a phase transition. In momentum space, this phenomenon corresponds to a susceptibility $G(k)$ that changes from the $1/k^2$ to $1/k^4$ behavior, implying an instability in momentum space: The coefficient of the k^2 term for kinetic energy in the effective action becomes zero or negative, so that the dynamical behavior is dominated by the k^4 term, provided it has a positive coefficient to maintain the system stable. This instability leads to the appearance of a Lifshitz (multi-critical) point in the phase diagram of the system, as is known in the theory of phase transitions and critical phenomena [30,31]. In this way, we have shown a quantum fluctuation induced modulated phase in the NCLG theory. The scaling behavior and corresponding critical exponents in the vicinity of the Lifshitz point are also studied.

The present paper is the long version of a previous short letter [23], presenting more calculations and adding more detailed discussion on the Lifshitz point. Before our letter, Ref. [20] had considered the possibility of a transition to a non-uniform phase in the NCLG theory, and conjectured the appearance of a Lifshitz point in the phase diagram. Our reasoning for the transition to a modulated phase is different from theirs, and our demonstration of the existence of the Lifhsitz point is based on a solid RG analysis. After our letter, Ref. [37] confirmed our way of doing renormalization in NCFT, and also addressed the issue of the Wilsonian RG equation. We will discuss the differences between these papers and ours in the final section.

The paper is organized as follows: In Sec. II, we present an introduction to RG analysis, *à la* Shankar and Polchinski, for the quadratic terms in the NCLG theory, determining the RG transformations for the field and the mass parameter at tree level. This section is elementary and experts in condensed matter RG can just skip it. Then in Sec. III we study the peculiarities due to noncommutativity in classifying the relevance of quartic perturbations at tree level. One loop RG equations for quadratic and quartic vertices are discussed in Sec. IV; in particular, the anomalous dimension of the field and rescaling of the momentum dependence of the Moyal phase factor from the star product are further discussed at one-loop level. The subsequent section, Sec. V, is devoted to exploring the consequences of this novel one-loop anomalous dimension (wave function renormalization) and to discussing the NCWF fixed point. In Sec. VI, we present an analysis of the fluctuation induced first-order phase transition and the Lifshitz point. The final section (Sec. VII) is devoted to conclusions and discussions.

II. FIXED POINT AND QUADRATIC TERMS

We start with the NCLG theory in D dimensions:

$$S = -\int d^D x [\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} \phi * \phi * \phi * \phi], \tag{5}$$

where the star product of two functions is defined as

$$(f * g)(x) = e^{\frac{i}{2}\Theta^{\mu\nu}\partial^y_\mu \partial^z_\nu} f(y) g(z)|_{y=z=x}, \tag{6}$$

with $\Theta^{\mu\nu}$ real and antisymmetric. For convenience, we choose the noncommutativity matrix to be

$$\Theta^{12} = \Theta^{34} = \Theta \tag{7}$$

and other independent components being zero. The generalization to anisotropic cases is straightforward.

In the following, we present a Wilsonian RG analysis in the modern formulation recently proposed by Shankar [21] and Polchinski [22]. The basic procedure is as follows: One starts with a free system with momentum cut-off Λ, and identifies the RG scaling laws so as to make the free action invariant (a fixed point of RG transformations in the low energy regime). After doing so, we turn on the interactions, consider them as perturbations to the fixed-point action, and examine the interaction vertices both at the tree and one-loop levels. According to their behavior under RG scaling, the interactions can be classified as relevant, irrelevant, or marginal ones. Only the relevant and marginally relevant interactions can possibly change the fixed point, i.e. change the low energy physics. So *in the low energy effective field theory, we keep only the relevant and marginally relevant interactions.* Now let us apply this general procedure to the D dimensional action (5).

We rewrite the free field action S_0 in momentum space as

$$S_0 = -\frac{1}{2}\int_{|k|<\Lambda} \frac{d^D k}{(2\pi)^D} k^2 \phi(-k)\phi(k). \tag{8}$$

First separate the field variable $\phi(k)$ into slow modes $\phi_s(k)$ and fast modes $\phi_f(k)$:

$$\phi_s(k) = \phi(k), \qquad |k| < \frac{\Lambda}{s}, \tag{9}$$

$$\phi_f(k) = \phi(k), \qquad \frac{\Lambda}{s} \le |k| \le \Lambda, \tag{10}$$

where s is a number greater than unity. Then let us integrate out the fast modes in the partition function $Z = \int \mathcal{D}\phi e^S$; since they are quadratic in the action, the result is a constant and does not contribute to the critical properties. We are interested in the resulting action for the slow modes; after rescaling $k' = sk$ to make the cut-off back to Λ, it becomes

$$\begin{aligned} S_0'(\phi_s) &= -\frac{1}{2}\int_{|k|<\frac{\Lambda}{s}} \frac{d^D k}{(2\pi)^D} k^2 \phi_s(-k)\phi_s(k), \\ &= -\frac{1}{2} s^{-D-2} \int_{|k'|<\Lambda} \frac{d^D k'}{(2\pi)^D} k'^2 \phi_s(-\frac{k'}{s})\phi_s(\frac{k'}{s}). \end{aligned} \tag{11}$$

To make this action invariant, we redefine the field variable as

$$\phi'(k') = s^{-\frac{D}{2}-1}\phi_s(\frac{k'}{s}) = s^{-\frac{D}{2}-1}\phi_s(k). \tag{12}$$

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With the definition (12), eq. (11) reads

$$S_0'(\phi_s) = S_0(\phi') = S^*, \tag{13}$$

which tells us that the free action S_0 is a fixed point under the following RG transformations

$$k' = sk, \qquad \phi'(k') = s^{-\frac{D}{2}-1}\phi_s(k). \tag{14}$$

Having the fixed point in hand, we can immediately proceed to investigate the relevance of the mass term in the action (5). After integrating out the fast modes we have

$$\begin{aligned} S_2'(\phi_s) &= -\frac{1}{2}m^2 \int_{|k|<\frac{\Lambda}{s}} \frac{d^D k}{(2\pi)^D}\phi_s(-k)\phi_s(k), \\ &= -\frac{1}{2}m^2 s^{-D} \int_{|k'|<\Lambda} \frac{d^D k'}{(2\pi)^D}\phi_s(-\frac{k'}{s})\phi_s(\frac{k'}{s}), \\ &= -\frac{1}{2}m^2 s^2 \int_{|k'|<\Lambda} \frac{d^D k'}{(2\pi)^D}\phi'(-k')\phi'(k'). \end{aligned} \tag{15}$$

From the last equality sign, we derive the following scaling relation for the mass parameter

$$r' \equiv m'^2 = s^2 r \equiv s^2 m^2. \tag{16}$$

Recalling that $s > 1$, the mass term is enhanced under RG transformations (14). Namely it flows away from zero – its value at the RG fixed point. We conclude that the mass term is relevant under the RG flow to the low energy regime. This finishes the RG analysis for the quadratic actions at tree level. Note that noncommutativity does not show up in the quadratic action, consequently above analysis is identical to those in ordinary space.

III. CLASSIFICATION OF RELEVANCE FOR QUARTIC PERTURBATIONS: TREE LEVEL

In this section, we study the quartic perturbations in the action (5):

$$S_4 = -\frac{1}{4!}\int_\Lambda \phi(4)\phi(3)\phi(2)\phi(1)\, u(4321), \tag{17}$$

where the interaction function $u(4321)$ is given by

$$u(4321) = gI(4321) \equiv \frac{g}{3}\Big[\cos(\frac{k_1 \wedge k_2}{2})\cos(\frac{k_3 \wedge k_4}{2}) + \cos(\frac{k_1 \wedge k_3}{2})\cos(\frac{k_2 \wedge k_4}{2}) + \cos(\frac{k_1 \wedge k_4}{2})\cos(\frac{k_2 \wedge k_3}{2})\Big], \tag{18}$$

with $k_1 \wedge k_2 = \Theta^{\mu\nu} k_{1\mu} k_{2\nu}$. We write the interaction function in this way so as to make the vertex totally symmetric; we will see that this representation has some advantage in counting the symmetry factors. The short-hand notation of the integral in eq. (17) is understood as

$$\int_\Lambda = \int_{|k_i|<\Lambda} \prod_{i=1}^{4} \frac{d^D k_i}{(2\pi)^D} \delta^{(D)}(k_1 + k_2 + k_3 + k_4). \tag{19}$$

Unlike the quadratic part discussed in the preceding section, the slow modes and fast modes get mixed up in the quartic case. To perform the mode elimination, in general we have to go back to the perturbative expansion. Here we use the cumulant expansion widely used in the study of critical phenomena [19].

To see how the cumulant expansion works, we cast $S_4(\phi)$ into $S_4(\phi_s, \phi_f)$ and perform path integral over fast modes, to obtain the partition function in the form

$$\begin{aligned} Z &= \int \mathcal{D}\phi_s \mathcal{D}\phi_f e^{S_0(\phi_s)} e^{S_0(\phi_f)} e^{S_4(\phi_s,\phi_f)}, \\ &= \int \mathcal{D}\phi_s e^{S^r(\phi_s)}, \end{aligned} \tag{20}$$

where the reduced action $S^r(\phi_s)$ is defined as

$$\begin{aligned} e^{S^r(\phi_s)} &\equiv e^{S_0(\phi_s)} \frac{\int \mathcal{D}\phi_f e^{S_0(\phi_f)} e^{S_4(\phi_s,\phi_f)}}{\int \mathcal{D}\phi_f e^{S_0(\phi_f)}}, \\ &= e^{S_0(\phi_s)} \left\langle e^{S_4(\phi_s,\phi_f)} \right\rangle_{0f}, \\ &= e^{S_0(\phi_s) + S_4'(\phi_s)}. \end{aligned} \tag{21}$$

Here we have used the fact that the functional integral in the denominator is only a constant, so we can freely drop it; and $< ... >_{0f}$ denotes the average over the fast modes weighted by the free action of fast modes.

The basic idea of the cumulant expansion is to relate the mean of exponential to the exponential of means, namely

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$$\Big\langle e^{S_4(\phi_s,\phi_f)}\Big\rangle_{0f} = \exp\Big[\Big\langle S_4(\phi_s,\phi_f)\Big\rangle_{0f} + \frac{1}{2}\Big(\Big\langle S_4^2(\phi_s,\phi_f)\Big\rangle_{0f} - \Big\langle S_4(\phi_s,\phi_f)\Big\rangle_{0f}^2\Big) + \cdots\Big]. \quad (22)$$

In this way, we get a perturbative series for $S_4'(\phi_s)$:

$$S_4'(\phi_s) = \Big\langle S_4(\phi_s,\phi_f)\Big\rangle_{0f} + \frac{1}{2}\Big(\Big\langle S_4^2(\phi_s,\phi_f)\Big\rangle_{0f} - \Big\langle S_4(\phi_s,\phi_f)\Big\rangle_{0f}^2\Big) + \cdots. \quad (23)$$

Now let us apply this perturbative expansion to The quartic interactions in NC ϕ^4 theory. Consider first the leading term in $S_4'(\phi_s)$:

$$\Big\langle S_4(\phi_s,\phi_f)\Big\rangle_{0f} = -\frac{1}{4!}\Big\langle\int(\phi_s+\phi_f)_4(\phi_s+\phi_f)_3(\phi_s+\phi_f)_2(\phi_s+\phi_f)_1 u(4321)\Big\rangle_{0f}. \quad (24)$$

The right side contains 16 terms in total, which include 8 terms each containing an odd number of fast modes, one term purely fast modes, one term purely slow modes, and the remaining six terms two fast and two slow modes. It is easy to see that the terms involving an odd number of fast modes vanish due to symmetry of the integral. The term involving only fast modes gives us a constant anyway, so we are not interested in it. In the following, we only need to concentrate on the last two classes, i.e. the term only involving slow modes and the six terms involving two fast and two slow modes.

The term with only slow modes gives us the tree level contribution to S_4', since the average over fast modes gives us unity:

$$\begin{aligned} S'_{4,tree} &= -\frac{1}{4!}\int_{|k|<\frac{\Lambda}{s}} \phi_{4s}\phi_{3s}\phi_{2s}\phi_{1s}u(4321), \quad (25)\\ &= -\frac{1}{4!}s^{-3D}\int_{|k'|<\Lambda} \phi_{4s}(\frac{k_4'}{s})\phi_{3s}(\frac{k_3'}{s})\phi_{2s}(\frac{k_2'}{s})\phi_{1s}(\frac{k_1'}{s})u(\frac{4'}{s}\frac{3'}{s}\frac{2'}{s}\frac{1'}{s}),\\ &= -\frac{1}{4!}s^{4-D}\int_{|k'|<\Lambda} \phi_{4s}'(k')\phi_{3s}'(k')\phi_{2s}'(k')\phi_{1s}'(k')u(\frac{4'}{s}\frac{3'}{s}\frac{2'}{s}\frac{1'}{s}),\\ &= -\frac{1}{4!}\int_{|k'|<\Lambda} \phi_{4s}'(k')\phi_{3s}'(k')\phi_{2s}'(k')\phi_{1s}'(k')u'(4'3'2'1'). \end{aligned}$$

In the second and third line, the RG transformation (14) have been used. Therefore, at tree level, we derive the scaling for the interaction function $u(4321)$ as follows

$$u'(4'3'2'1') = s^{4-D}u(\frac{4'}{s}\frac{3'}{s}\frac{2'}{s}\frac{1'}{s}) = s^{4-D}u(4321). \quad (26)$$

If one follows the usual procedure in the RG analysis on ordinary space, he/she would expand the cosine factors in the interaction function (18) in powers of momenta, and apply the above RG transformation to each term. At tree level this would lead to

$$u'(4'3'2'1') = s^{4-D}g - \frac{gs^{2-D}}{12}[k'_1 \wedge k'_2 + k'_3 \wedge k'_4] + \cdots . \tag{27}$$

Here the ellipsis represents terms with s^{-D} or higher negative powers of s, which have no chance to be relevant in any dimension D. Thus, except for the Θ-independent constant term, all the higher-order Θ-dependent terms would be *irrelevant* under the RG transformation, so that noncommutativity at tree level would be irrelevant to the low energy physics. Certainly this is in conflict with the general notion of UV-IR mixing.

What was wrong in the above procedure is apparently that one should *not* have expanded the Moyal factor $I(1'2'3'4')$ into power series of momenta. We should bear in mind that this momentum dependent factor originates from the star product, i.e. from coordinate noncommutativity, an intrinsic geometric feature of the NC space that ought to be respected by any non-gravitational dynamics. In other words, *dynamics of non-gravitational systems ought to respect NC geometry.* In our opinion, it is this constraint from NC geometry on dynamics that constitutes the main new feature for the RG analysis on an NC space.

So the problem with the expansion (27) is just that it did not respect the star product in the NC space, that coherently organizes infinitely many higher-order derivative terms. Though each of them may behave like an irrelevant operator in ordinary sense, however their coherent sum may give rise to non-trivial effects. That a coherent structure in the interactions may change the relevance of an interaction operator has been known to happen in ordinary field theory. One famous example is the 1 + 1 dimension sine-Gordon model $S = \int d^2x[\frac{1}{2}\partial_\mu\Phi\partial_\mu\Phi + g\cos(\beta\Phi)]$. If we expand the interaction term as an infinite Taylor series, then a simple scaling analysis tells us that every term is a relevant operator. However, it is well-known that this is not quite right. Actually the coherent structure $\cos(\beta\Phi)$ has scaling dimension $\Delta = \frac{\beta^2}{4\pi}$, so whether it is relevant depends on whether $\Delta > 2$ (irrelevant) or $\Delta < 2$ (relevant), and $\Delta = 2$ (marginal). Similarly in NCFT, it is the structure of the

Moyal's star product, dictated by the intrinsic NC geometry, that neatly and coherently organizes infinitely many higher derivative terms, modifying the classification of relevance of operators in NCFT.

With these points in mind, one should define the renormalized interactions in terms of star products and classify their relevance, not that of ordinary interactions. Indeed, at least at one loop, it has been shown [5] that counterterms have the same star product structure. Thus, the operators allowed to appear in the Wilsonian effective action must be always of the the form of a star product with the same Θ parameter. The most general quartic terms are always of the form of Eq. (18), with the coefficient g becoming a function of momenta: $u(4321) = g(4321)\, I(4321)$. We should classify the relevance of the interaction operator by expanding only the coefficient $g(k_1, k_2, k_3, k_4)$ into powers of momenta, while keeping the Moyal factor $I(4321)$ intact as if it is *marginal*.

To summarize, the difference of our treatments from the usual RG analysis in the commutative case is that we keep the star product structure of the quartic interaction intact, and apply RG transformations only to the coefficient function $g(k_1, k_2, k_3, k_4)$. Then define their relevance, irrelevance etc as usual. Thus at tree level $g' = s^{4-D} g$, namely, it is a relevant operator in less than four dimension, it is irrelevant in higher than four dimension and it is marginal in the four dimension. In next section, we will study one-loop effects, and see that the "marginality" of the Moyal factor will be preserved at one loop level; i.e. it is indeed protected from quantum fluctuations intrinsically by NC geometry.

IV. ONE LOOP RG ANALYSIS

A. One loop corrections to the quadratic action

Now let us proceed to examine the terms with two fast and two slow modes in Eq. (24). Of these six terms, one typical term is of the form

$$\int_{|k|<\frac{\Lambda}{s}} \frac{d^D k_1}{(2\pi)^D} \int_{|k|<\frac{\Lambda}{s}} \frac{d^D k_2}{(2\pi)^D} \int_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{d^D k_3}{(2\pi)^D} \int_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{d^D k_4}{(2\pi)^D} \phi_{4f}(k_4)\phi_{3f}(k_3)\phi_{2s}(k_2)\phi_{1s}(k_1). \quad (28)$$

Since the free action of fast modes is of the form $\int_{\frac{\Lambda}{s}<|k|<\Lambda} k^2\phi_f(-k)\phi_f(k)$ and the interaction function $u(4321)$ is symmetrized, so all the six terms give the same contribution. Again, using the symmetry in the integral over the fast modes, we should take

$$k_3 = -k_4 = p, \qquad k_1 = -k_2 = k \tag{29}$$

to yield nonzero result. So we are led to the following simple result

$$\begin{aligned} &-6\times\frac{g}{4!3}\int_{|k|<\frac{\Lambda}{s}}\frac{d^Dk}{(2\pi)^D}\phi_s(-k)\phi_s(k)\int_{\frac{\Lambda}{s}<|k|<\Lambda}\frac{d^Dp}{(2\pi)^D}\Big\langle\phi_f(-p)\phi_f(p)\Big\rangle_{0f}[2+\cos(k\wedge p)], \\ &= -\frac{1}{2}\int_{|k|<\frac{\Lambda}{s}}\frac{d^Dk}{(2\pi)^D}\phi_s(-k)\phi_s(k)\Gamma_2(k), \end{aligned} \tag{30}$$

where the self-energy $\Gamma_2(k)$ is defined by

$$\Gamma_2(k) = \frac{g}{6}\int_{\frac{\Lambda}{s}<|p|<\Lambda}\frac{d^Dp}{(2\pi)^D}\Big[\frac{2}{p^2}+\frac{\cos(k\wedge p)}{p^2}\Big]. \tag{31}$$

In passing, we would like to remark that the above result can be summarized in a Feynman diagram, the tadpole diagram in Fig.1, the only difference from the familiar rule in quantum field theory being that the loop momentum only takes its value in a thin momentum shell $[\frac{\Lambda}{s},\Lambda]$. In addition, since the mass term is a relevant operator, we can also start from the full quadratic action to do the perturbation theory, the only change is the propagator, namely, $1/p^2$ is changed to $1/(p^2+m^2)$. This change will be used later to get the RG equation for the mass term.

The planar contribution to the quadratic action from the first term in eq. (31) is

$$\begin{aligned} &-\frac{g}{12}\int_{|k|<\frac{\Lambda}{s}}\frac{d^Dk}{(2\pi)^D}\phi_s(-k)\phi_s(k)K_D\Lambda^{D-2}(1-\frac{1}{s}), \\ &= -\frac{g}{12}\int_{|k'|<\Lambda}\frac{d^Dk'}{(2\pi)^D}\phi'_s(-k')\phi'_s(k')K_D\Lambda^{D-2}s^2(1-\frac{1}{s}), \end{aligned} \tag{32}$$

where the integration variable p in the integrand has been replaced by its upper limit Λ, because the thickness, $\Lambda(1-\frac{1}{s})$, of the integration shell in momentum space is very small. Moreover, $K_D = S_D/(2\pi)^D$ and S_D is the surface area of a unit sphere in D dimensions. Clearly this gives a correction to the mass term (15).

Now let us calculate non-planar contribution to the quadratic action from the second term in eq. (31):

$$-\frac{g}{12}\int_{|k|<\frac{\Lambda}{s}}\frac{d^Dk}{(2\pi)^D}\phi_s(-k)\phi_s(k)I(k). \tag{33}$$

where $I(k)$ has been defined as

$$I(k)=\int_{\frac{\Lambda}{s}<|p|<\Lambda}\frac{d^Dp}{(2\pi)^D}\frac{e^{ik\wedge p}}{p^2}. \tag{34}$$

To calculate this integral, we take $\Theta^{\mu\nu}$ to be of the form of eq. (7), then $k\wedge p$ is calculated as

$$k\wedge p=\Theta\sum_{i=0}^{\frac{D}{2}-1}(k_{2i+1}p_{2i+2}-k_{2i+2}p_{2i+1}). \tag{35}$$

To be more specific, we take $n=2$ as a example and then generalize it to the more general case. We write 4-dimensional vectors k and p as

$$\begin{aligned} P&=\vec{p}_{\|}+\vec{p}_{\perp}, & \vec{p}_{\|}&=p_1\vec{e}_1+p_2\vec{e}_2, & \vec{p}_{\perp}&=p_3\vec{e}_3+p_4\vec{e}_4, \\ K&=\vec{k}_{\|}+\vec{k}_{\perp}, & \vec{k}_{\|}&=k_1\vec{e}_1+k_2\vec{e}_2, & \vec{k}_{\perp}&=k_3\vec{e}_3+k_4\vec{e}_4. \end{aligned} \tag{36}$$

Then we choose polar coordinates in 12-plane and 34-plane with the azimuthal angles φ_1 and φ_2, respectively. The four dimensional integral (34) can be cast into

$$I(k)=\frac{1}{(2\pi)^4}\int dp_{\|}p_{\|}d\varphi_1\int dp_{\perp}p_{\perp}d\varphi_2\frac{e^{i\Theta p_{\|}k_{\|}\sin\varphi_1}e^{i\Theta p_{\perp}k_{\perp}\sin\varphi_2}}{p_{\|}^2+p_{\perp}^2}. \tag{37}$$

After performing the integral over azimuthal angles, we get

$$I(k)=\frac{(2\pi)^2}{(2\pi)^4}\int dp_{\|}p_{\|}\int dp_{\perp}p_{\perp}\frac{J_0(\Theta p_{\|}k_{\|})J_0(\Theta p_{\perp}k_{\perp})}{p_{\|}^2+p_{\perp}^2}. \tag{38}$$

Noting that the integral over p has the shell constraint, namely, we have

$$\left(\frac{\Lambda}{s}\right)^2\le p_{\|}^2+p_{\perp}^2\le\Lambda^2. \tag{39}$$

Introducing polar coordinate on the $(p_{\|},p_{\perp})-$ plane with the azimuthal angle φ, we get

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$$
\begin{aligned}
I(k) &= \tfrac{1}{4\pi^2}\int_{\Lambda/s}^{\Lambda} dpp \int_0^{\pi/2} d\varphi \sin\varphi\cos\varphi J_0(\Theta k_{\|} p\cos\varphi) J_0(\Theta k_{\perp} p \sin\varphi), \\
&= \frac{\Lambda^2}{8\pi^2}(1-\frac{1}{s})\int_0^{\pi/2} d\varphi \sin 2\varphi J_0(\Theta k_{\|}\Lambda\cos\varphi) J_0(\Theta k_{\perp}\Lambda\sin\varphi).
\end{aligned} \quad (40)
$$

By expanding the Bessel function up to second order, i.e.

$$J_0(x) = 1 - \frac{1}{4}x^2 + O(x^4), \quad (41)$$

and calculate the integral over the angular variable ,we get

$$I(k) = K_4\Lambda^2(1-\frac{1}{s})(1-\frac{1}{8}\Theta^2\Lambda^2k^2). \quad (42)$$

The above procedure can be easily generalized to the case of $D = 2n$, the result turns out to be

$$I(k) = K_D\Lambda^{D-2}(1-\frac{1}{s})(1-\frac{1}{8}\Theta^2\Lambda^2k^2). \quad (43)$$

The constant term on the right side contributes to mass renormalization. Combining it with the contribution from eq. (32), we get the one-loop correction to the scaling relation (16):

$$r' = s^2 r + \frac{gK_D\Lambda^D}{2(\Lambda^2+m^2)}(s^2 - s). \quad (44)$$

Setting $s = 1 + t$, where t is infinitesimal, and taking derivative with respect to t, then we get the one-loop β-function for the mass term:

$$\frac{dr}{dt} = 2r + \frac{u}{2}K_D(1-r), \quad (45)$$

where $u = g\Lambda^{4-D}$ has been used.

Moreover, the appearance of a momentum dependent term in eq. (43) is a distinct feature due to coordinate noncommutativity, since in contrast, in the usual commutative case (with $\Theta = 0$) $I(k)$ does not depend on momentum at all. However, only a quadratic momentum dependence is marginally relevant, so we need Taylor expand eq. (41) only up to second order at this moment. By substituting the second term of eq. (43) into eq. (33), we see that the noncommutative ϕ^4 interactions induce a renormalization of the kinetic term at one

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loop. Using field theory terminology, it gives rise to *wave function renormalization* for the ϕ-field, that explicitly *depends on noncommutativity.* This is one of our key observations in this paper. With this term, the Gaussian fixed point action S_0 is modified to

$$S_0' = -\frac{1}{2}\Big[1 - \frac{u}{48}K_D(\Theta\Lambda^2)^2 t\Big]\int_0^{\frac{\Lambda}{s}} k^2\phi_s(-k)\phi_s(k) \tag{46}$$

Using $1+\gamma t \approx s^\gamma$, and introducing a dimensionless $\theta = \Theta\Lambda^2$ for noncommutativity, we see that the ϕ-field acquires an anomalous dimension, modifying eq. (14) to

$$\phi'(k') = s^{-\frac{D+2-\gamma(u,\theta)}{2}}\phi(k) \tag{47}$$

with the one-loop γ depending on noncommutativity:

$$\gamma(u,\theta) = -\frac{1}{48}uK_D\theta^2. \tag{48}$$

This is a novel result, because in ordinary Landau-Ginsburg theory the anomalous dimension vanishes at one loop. Our result is a consequence of the UV/IR mixing in NCLGM. Also unusual is that the this anomalous dimension is *negative.* It may significantly affect the stability of the symmetric phase, as will become clear in the next section.

B. One loop corrections to the quartic interaction

We have seen that the leading term in the cumulant expansion produces the one-loop effects for the quadratic terms, but provides only tree level information on interaction vertices. To deal with the one-loop effects for vertices, we need to invoke the sub-leading terms in eq. (23):

$$\frac{1}{2}\Big[\Big\langle S_4^2(\phi_s,\phi_f)\Big\rangle_{0f} - \Big\langle S_4(\phi_s,\phi_f)\Big\rangle_{0f}^2\Big]. \tag{49}$$

By repeating the arguments used before for the renormalization of the quadratic terms, we see that the contributions to the vertices come from the terms of the following form:

$$\int_{|k|<\frac{\Lambda}{s}}\phi_s(4)\phi_s(3)\phi_s(2)\phi_s(1)\Big\langle\int_{\frac{\Lambda}{s}<|k|<\Lambda}\phi_f(8)\phi_f(7)\phi_f(6)\phi_f(5)\mathcal{P}(k_8,k_7,\cdots,k_1)\Big\rangle_{0f}, \tag{50}$$

where $\mathcal{P}(k_8, k_7, \cdots, k_1)$ is composed of trigonometric functions, whose specific form will be given later. Repeating the same reasoning in the preceding section, we know that the non-vanishing contributions can only come from the Feynman diagrams shown in Fig. 2.

Since the interaction function has been symmetrized, we can easily count the symmetry factor for each diagram to be 1/2. If we assume the momentum flow into the diagram is Q, the loop momentum is q, then the function $\mathcal{P}$ for the first diagram is given by [24]

$$\begin{aligned}
\mathcal{P}\left(p_1, p_2, p_3, p_4; q\right) &= \frac{g^2}{9}\cos(\frac{p_1\wedge p_2}{2})\cos(\frac{p_3\wedge p_4}{2})\Big[1+\cos q\wedge(p_1+p_2)\Big] \qquad (51)\\
&+\frac{g^2}{18}\cos(\frac{p_3\wedge p_4}{2})\Big[\cos(\frac{p_1\wedge p_2}{2}+q\wedge p_1)+\cos(\frac{p_1\wedge p_2}{2}-q\wedge p_2)\Big]\\
&+\frac{g^2}{18}\cos(\frac{p_1\wedge p_2}{2})\Big[\cos(\frac{p_3\wedge p_4}{2}-q\wedge p_3)+\cos(\frac{p_3\wedge p_4}{2}+q\wedge p_4)\Big]\\
&+\frac{g^2}{36}\Big[\cos[\frac{p_1\wedge p_2-p_3\wedge p_4}{2}+q\wedge(p_1+p_3)]+\cos[\frac{p_1\wedge p_2+p_3\wedge p_4}{2}+q\wedge(p_1+p_4)]\Big].
\end{aligned}$$

By proper permutation of the external momenta, we can get the $\mathcal{P}$ function for the other two diagrams. Noting that p_i $(i = 1, 2, 3, 4)$ are slow mode momenta and the loop integral is only within the momentum shell $[\frac{\Lambda}{s}, \Lambda]$, we can put the external momenta to zero whenever they wedge the loop momentum [19,25]. This approximation greatly simplifies the above $\mathcal{P}$ function, and leads to

$$\mathcal{P}(p_1, p_2, p_3, p_4) = \frac{g^2}{2}\cos(\frac{p_1\wedge p_2}{2})\cos(\frac{p_3\wedge p_4}{2}). \qquad (52)$$

After permutation on the external momenta, we can get a similar structure for the other two diagrams in Fig. 2. Thus the one loop corrections to the quartic interactions are given by

$$\begin{aligned}
&\frac{g^2}{2}\int_{|k|<\frac{\Lambda}{s}} I(4321)\phi_s(4)\phi_s(3)\phi_s(2)\phi_s(1)\int_{\frac{\Lambda}{s}}^{\Lambda}\frac{d^Dp}{(2\pi)^D}\frac{1}{(p^2+r)^2}, \qquad (53)\\
&=\frac{g^2}{2}\frac{K_D\Lambda^D}{(\Lambda^2+r)^2}(1-\frac{1}{s})\int_{|k|<\frac{\Lambda}{s}} I(4321)\phi_s(4)\phi_s(3)\phi_s(2)\phi_s(1),\\
&=\frac{g^2}{2}\frac{K_D\Lambda^D}{(\Lambda^2+r)^2}s^{4-D}(1-\frac{1}{s})\int_{|k|<\Lambda} I(\frac{4'}{s}\frac{3'}{s}\frac{2'}{s}\frac{1'}{s})\phi'(4')\phi'(3')\phi'(2')\phi'(1').
\end{aligned}$$

where $I(4321) = 3u(4321)/g$ is the Moyal factor defined by the square bracket in eq. (18). Now we come to the question of how to scale the $u(4321)$ again. To get the one loop

RG for the quartic vertex, we have to correctly handle the scaling for the trigonometric factor $I(4321)$. At tree level, we have maintained that this factor is untouched when we do rescaling. We need to check the consistency of this treatment when the loop effects are included. To get the one loop RG, we set $s = 1 + t$ with t is small and positive. We consider one term in $I(4321)$, say

$$\cos(\frac{k_1' \wedge k_2'}{2s^2})\cos(\frac{k_3' \wedge k_4'}{2s^2}) = \cos(\frac{k_1' \wedge k_2'}{2} - tk_1' \wedge k_2')\cos(\frac{k_3' \wedge k_4'}{2} - tk_3' \wedge k_4') \tag{54}$$

The small t expansion gives

$$\cos(\frac{k_1' \wedge k_2'}{2})\cos(\frac{k_3' \wedge k_4'}{2}) + tk_3' \wedge k_4' \cos(\frac{k_1' \wedge k_2'}{2})\sin(\frac{k_3' \wedge k_4'}{2}) + tk_1' \wedge k_2' \cos(\frac{k_3' \wedge k_4'}{2})\sin(\frac{k_1' \wedge k_2'}{2}). \tag{55}$$

Therefore, after rescaling, some extra terms are generated. However, all extra terms are irrelevant in the sense of RG, since they all contain higher order derivatives (e.g. $k_1 \wedge k_2$ etc.). We can discard them in the low energy effective action. Thus the consistency check within one loop is passed; namely, we do not need to scale the momentum dependence in the star product. In passing we would like to emphasize that the irrelevant terms in eq. (55) is due to a small t expansion, not a small external momentum expansion as in eq. (27). After this discussion on the star product, the RG equation of the quartic vertex is readily derived; the result is actually the same as its commutative counterpart:

$$\frac{du}{dt} = (4 - D)u - \frac{3}{2}K_D u^2. \tag{56}$$

However, we should notice that the NC effective theory is different from its commutative counterpart, because we need one more parameter, the noncommutative parameter Θ, to specify the star product in the interactions, unless the low energy effective theory is a free theory, as in the cases with $D \geq 4$ as discussed in the next section.

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V. STABILITY ANALYSIS OF THE GAUSSIAN FIXED POINT

In this section, we will concentrate on the physical implications of our RG analysis presented in the preceding sections. Since the physics critically depends on the dimensionality

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of space, we will discuss the $D \geq 4$ and $D < 4$ cases separately.

A. Above the critical dimension: $D \geq 4$

According to eq. (56) the upper critical dimension in NCLG remains to be $D = 4$. When $D > 4$, the quartic coupling u flows to zero, so it is irrelevant. Thus, it is the unique, trivial Gaussian fixed point, $u^* = 0 = r^*$, that controls the IR asymptotically free low-energy behavior, with the same sets of critical exponents as usual in mean field theory: $\nu = 1/2$ and $\eta = \gamma(u^*, \theta) = 0$.

However, this is not the whole story, when we consider approaching to the critical point $r = 0$. In fact, the modified scaling law (47) gives a two-point correlation function behaving like

$$\langle \phi(x)\phi(0) \rangle \sim \frac{1}{x^{D-2+\gamma}}. \tag{57}$$

Due to the minus sign in (48), for very large noncommutative parameter θ, the above correlation function at large distance does not become zero, which signals an instability in the system. The critical value, θ_c, is given by the condition

$$u\theta_c^2 = \frac{48(D-2)}{K_D}. \tag{58}$$

More precisely, the parameter space for the NCLG is now three-dimensional, described by (u, r, θ). The condition (58) gives us a surface in parameter space. To access the Gaussian fixed point, we have to fine-tune the parameter θ to make $\theta < \theta_c(u)$. Of course, the closer it gets to the fixed point, the less important is the condition (58), since θ_c is pushed to infinity when the Gaussian fixed point is reached.

In the critical dimension $D = 4$, the one-loop non-zero anomalous dimension (48) is expected to modify the logarithmic corrections to the scaling laws at criticality.

B. Below the critical dimension: physics in $4-\varepsilon$ dimension

According to eq. (56), if the dimension is slightly lower than four, the Gaussian fixed point becomes unstable, and we have a new IR stable fixed point, the noncommutative counterpart of the Wilson-Fisher (NCWF) fixed point. Besides the noncommutativity parameter θ, its position in (r,u)-space for small $\varepsilon \equiv 4-D$ is the same as in ordinary Landau-Ginsburg model (OLGM):

$$u^* = \frac{16\pi^2}{3}\varepsilon, \qquad r^* = -\frac{1}{6}\varepsilon. \tag{59}$$

At this fixed point, the critical exponent ν is unchanged: $\nu = \frac{1}{2} + \frac{\varepsilon}{12}$, but the one-loop critical exponent η becomes non-vanishing:

$$\eta = \gamma(u^*,\theta) = -\frac{\varepsilon\theta^2}{72}. \tag{60}$$

This result is characteristic of the NCLG. We would like to stress three important aspects of the critical exponent (60): (1) It starts at order of ε, while in OLGM it starts at order of ε^2. (2) It is *negative*, while positive in OLGM. (3) It looks like non-universal because of its dependence on the dimensionless noncommutativity parameter θ. However, in the present case, the NCWF fixed point had better be viewed as a line of fixed points labeled by θ. Since θ originates in the microscopic sector of the system, its appearance in the macroscopic critical exponent is a genuine manifestation of *UV/IR mixing*, namely, the fingerprint of a "high-energy" parameter in low-energy phenomena.

To see how the anomalous dimension (60) affects the stability of the NCWF fixed point, let us examine the two-point correlation function

$$\langle\phi(x)\phi(0)\rangle \sim \frac{1}{x^{2-\varepsilon(1+\theta^2/72)}}. \tag{61}$$

The criterion to maintain the stability of the NCWF fixed point is that the above correlation function should tend to zero at large distances. Now we have an additional parameter θ as a new knob to tune the system. If it is too large, the correlation function may become finite or even divergent. The critical value is given by

$$\theta_c = 12/\sqrt{\varepsilon}, \tag{62}$$

which depends only on ε. Therefore, if $\theta < \theta_c$, the NCWF fixed point is stable. On the contrary, if $\theta > \theta_c$ the NCWF fixed point will no longer be stable. This is reflected in the phase diagram, Fig. 3.

This situation is similar to previous RG analyses in the literature for one dimensional and three dimensional interacting fermion systems. There the RG analysis was used to show a similar instability for the Fermi liquid fixed point. To get the picture of the Luttinger liquid (in $1d$) and BCS superconductivity (in $3d$), one had to determine nonperturbatively the underlying physics for the new phase. Here to gain knowledge of the new phase for $\theta > \theta_c$, we also need extra efforts. This will be presented in the next section.

VI. FIRST ORDER PHASE TRANSITION AT LARGE θ

In the previous section, based on a one loop RG analysis we found that the critical exponent η takes a negative value $-\varepsilon\theta^2/72$, which depends on space dimensionality and noncommutativity parameters. In this section, we are going to demonstrate that system experiences a first order phase transition to a modulated phase at the large $\theta_c = 12/\sqrt{\varepsilon}$ and critical temperature.

At first sight, one may wonder whether the appearance of a negative η is a bad news for NCFT. Indeed, in the pure commutative ϕ^4 theory, one can prove [26–28] that the positive measure of the Källen-Lehmann spectral representation has already implied a non-negative anomalous dimension, namely $\eta \geq 0$. However, this is not the case when the scalar field couples to a gauge field, say the case a superconductor is placed in a magnetic field. In this case, a negative anomalous dimension for the scalar field is unanimously established both from perturbative calculations and Monte Carlo simulations performed on a lattice (see Ref. [28] and references therein). More recent works showed that a negative anomalous dimension actually signals that the system experiences a first order phase transition through the Lifshitz scenario [30]. On the other hand, the proof [26–28] of the non-negative anomalous dimension

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is not applicable to systems that are either non-local or have massless gauge field degrees of freedom [30]. Thus the negative η appeared in our NCLG theory does not mean that the theory is ill-defined; it is due to the non-locality of the NCLG. On the contrary, as we will show in this section, such a negative η implied that NCLG theory will also experience a first order phase transition to a modulated phase through the Lifshitz scenario. Such a phase transition is a manifestation of the UV/IR mixing in the NCLG theory.

To see what is going on, we Fourier transform the correlation function (61), resulting in the asymptotic behavior of Green's function (susceptibility)

$$\begin{aligned} G(k) &\sim k^{2-D-\varepsilon(1+\theta^2/72)}, \\ &= k^{-2-\varepsilon\theta^2/72}. \end{aligned} \tag{63}$$

With the critical value $\theta_c = 12/\sqrt{\varepsilon}$, the asymptotic behavior of $G(k)$ becomes

$$G(k) \sim \frac{1}{k^4}. \tag{64}$$

This result is remarkable. While we are originally dealing with a system with kinetic energy $\sim k^2$, the result (64) tells us that, due to the quantum fluctuations, the dispersion law is actually modified to be dominated by k^4 in the long wavelength limit. Normally as long as the k^2 term has a positive coefficient, it maintains the system stable, and dominates the low energy dynamical behavior. Therefore, the scenario consistent with (64) at θ_c should be that the k^2-part of the kinetic energy vanishes exactly at the critical θ_c, so that the long wavelength behavior of the system at θ_c is dominated by the k^4-term. Thus we are led to the following effective action in the low energy limit

$$S = -\frac{1}{2}\int^{\Lambda} \frac{d^D k}{(2\pi)^D}[(1-au\theta^2)k^2 + bk^4 + m^2]\phi(-k)\phi(k) - \frac{1}{4!}\int^{\Lambda} u(4321)\phi(4)\phi(3)\phi(2)\phi(1), \tag{65}$$

where coefficients a, b are positive numbers to maintain the stability of the system. In the four and above dimensions, the RG analysis in previous sections tells us the fixed point action is Gaussian, namely $u^* = 0$; in this case, we don't need to worry about the higher derivative terms at all, since the k^2-term is positive. However, in $4-\varepsilon$ dimension, we have a nontrivial NCWF fixed point, which makes $u^* = 16\pi^2\varepsilon/3$ nonzero. In this case, the value of a is determined to be $3/2304\pi^2$ within the accuracy of the ε-expansion. It is necessary to keep track of the k^4 term, because the sign of the k^2 term may possibly change at large θ. Actually, if we go back to eq. (40) , expand Besell function $J_0(x)$ to higher order, and take the integral over angular variable, one can immediately see that we indeed have positive coefficient for k^4 term. (In contrast, in the ordinary φ^4 theory, the sign of the k^4 term is negative even up to two loops [34].) The value of the coefficients are not really important: As we will show below, for the physical picture we are going to extract to work, the crucial thing is just the sign of a and b in eq. (65).

The commutative counterpart of action (65) has been studied in the context of phase transition [29]. The most interesting thing is that it allows the existence of a multi-critical point, more specifically, the so-called Lifshitz point in the phase diagram of the system. A Lifshitz point is defined as *a special tri-critical point in the phase diagram, at which the coefficient of* k^2 *vanishes and that of* k^4 *is positive to maintain the stability of systems* [31]. Therefore, the value $\theta_c = 12/\sqrt{\varepsilon}$ in $4-\varepsilon$ dimensions discussed in the previous section is nothing but a Lifshitz point. At this point the correlation function (61) qualitatively changes its asymptotic behavior, a signal of a phase transition. This phase transition is first-order, because it is not related to a change of symmetry, which is calibrated by parameter m^2 in Landau-Ginsburg theory.

It is easy to see that before the parameter θ reaches its critical value θ_c from below, the k^2 term has a positive coefficient, thus the system does not suffer any instability, and the only possible phase transition is the second order one that occurs at $m^2 = 0$. The ordered phase is uniform. When the parameter θ exceeds θ_c, the k^2 term has a negative coefficient and the system suffers an instability near $k = 0$ in momentum space. Fortunately, we have the k^4 term with a positive coefficient, which can help stabilize the system. As a consequence, the system develops a soft mode which is associated with the minimum of kinetic energy. This minimum of kinetic energy occurs at

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$$k_c = \sqrt{\frac{au\theta^2 - 1}{2b}}, \tag{66}$$

which corresponds to a minimal kinetic energy

$$E_k^{min} = m^2 - \frac{(1 - au\theta^2)^2}{4b}. \tag{67}$$

The appearance of this soft mode also induces a shift in the critical temperature for the second order phase transition from $m^2 = 0$ to

$$m^2 = \frac{(1 - au\theta^2)^2}{4b}. \tag{68}$$

As usual in the Landau-Ginsburg theory, we take $m^2 = a_2(T - T_c^{(0)})$, then the shifted critical temperature is given by

$$T_c = T_c^{(0)} + \frac{(1 - au\theta^2)^2}{4a_2 b}. \tag{69}$$

Because the energy is minimized at the nonzero momentum k_0, the ordering below $\tilde{T}_c$ for $\theta > \theta_c$ is characterized by the following modulating order parameter

$$\phi(r) = \phi_0 \cos(\mathbf{p} \cdot \mathbf{r} + \varphi_0), \tag{70}$$

where ϕ_0 is the amplitude and $|\mathbf{p}| = k_c$ the wavevector of this modulating phase, and φ_0 an initial phase. Therefore, there are two ordering phases (uniform phase at $\theta < \theta_c$ and a modulated phase at $\theta > \theta_c$) in NCLG. The existence of a modulated phase (70) is due to the strong quantum fluctuations at large noncommutativity, i.e., strong quantum fluctuations violates the uniform ordering phase and eventually the system enters a finite wavelength modulated phase to get stabilized.

Before we discuss the property around the Lifshitz point, we would like to point out one basic difference between our noncommutative modulated phase (NCMP) and the ordinary modulated phase (OMP). As pointed out in Ref. [32], the parity of a field theory lives on NC space is automatically violated. Thus NCMP is also a parity violated phase, so it is different from ordinary modulated phase.

To describe the critical behavior in the vicinity of the Lifshitz point, we need to introduce additional critical exponents. In the NCMP, the dominant momentum dependence is k^4, implying that we need a new exponent ν_m to characterize the phase transition between NCMP and the disordered phase ($T > T_c$). A simple dimensional analysis gives the asymptotic of the correlation length ξ as

$$\xi \sim m^{-\frac{1}{2}} \sim (T - T_c)^{-\frac{1}{4}}. \tag{71}$$

Therefore, the mean field critical component ν_m is

$$\nu_m = \frac{1}{4}. \tag{72}$$

Here we only extract the mean field exponent; to go beyond this result, it is a folklore in statistical physics that we have to invoke RG again. We expect that the result would be similar to the case from the uniform ordered phase to the disordered phase; namely, after running the RG, ν_m also picks up an ε-dependent correction. What we would like to stress is that ν_m is different from $\nu = 1/2$ for the transition from the uniform ordering phase to the disordered phase even at the mean field level. In general, when Lifshitz point is approached, there will be a crossover from $\nu_m = 1/4$ to $\nu = 1/2$. This point can also be seen from the RG analysis in Sec. V that the anomalous dimension η increases with noncommutativity and makes the susceptibility $1/k^2$ cross over to $1/k^4$ at the critical point.

To further characterize how the phase boundary between the modulated and the uniformly ordered phases is approached, we introduce a new critical exponent β_k. Approaching the Lifshitz point on the NCMP segment of the second order phase transition line, we expect that the wave vector $\mathbf{p}$ of NCMP will be related to the dimensionless variable θ by

$$p \sim |\theta - \theta_c|^{\beta_k} \qquad (\theta > \theta_c). \tag{73}$$

By noting that the coefficient b is not singular at θ_c, and expanding (66) around θ_c, we get

$$p \sim |\theta - \theta_c|^{\beta_k} \sim |\theta - \theta_c|^{1/2}. \tag{74}$$

Therefore, at mean field level, we have

$$\beta_k = \frac{1}{2}. \tag{75}$$

Of course, higher order fluctuations may well alter this value and give rise to singularities in the shape of the phase boundaries at the Lifshitz point. Similar behavior was discussed for bi-critical points in Ref. [33].

In passing, several remarks are in order. (1) In ordinary Landau-Ginsburg theory, we do not have such a first order phase transition in the low temperature phase even if we include quantum fluctuations up to two loops, because there quantum fluctuations can not induce a positive coefficient for k^4, thus the system can not stabilize at finite momentum [34]; (2) the standard way to study the Lifshitz point is to consider the commutative version of eq. (65) as the starting point with the desired coefficient. Namely, the higher order derivative terms in quadratic action have already been included at the tree level. Meanwhile, in the NCLG theory, we do not have such a tree level higher order derivative terms in the quadratic action. Rather we have infinitely many higher order derivative terms in the interaction part which are hidden in the star product. Upon expanding the star product, each order of the expansion involves four field variables and their higher order derivatives. Therefore, intuitively it is the contraction of two fields with higher order derivatives and the distinct UV/IR mixing of NCFT that make the desired higher order derivatives terms for the quadratic action possible. After recognizing this feature of the star product, it seems that the appearance of the Lifshitz point and the first order phase transition in the NCLG is very natural.

VII. CONCLUSIONS AND DISCUSSIONS

In this paper, we have presented a Wilsonian RG analysis for noncommutative landau-Ginsburg (NCLG) theory in arbitrary dimensions. The key point of this paper is that to classify the relevance or irrelevance of interaction operators in a way consistent with the intrinsic constraints from noncommutative (NC) geometry. More precisely, we propose *not to apply RG transformations to the Moyal factor* in the interaction functions, though they are momentum dependent. The reasoning behind this proposal is that NC parameters, though

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dimensionful, are not renormalized, so that they do not transform under RG transformations. Therefore the Moyal factor coming from the star product structure that coherently organizes infinitely many higher order derivative terms, is not renormalized. The consistency of such a classification has been checked at one loop. With the star products preserved at one loop, RG analysis can be done readily.

For the NCLG model, the upper critical dimension remains to be four, the same as its commutative cousin. In a dimension higher than four, there is no nontrivial infrared fixed point, and the low energy behavior of the theory is mainly controlled by the Gaussian fixed point. Thus the noncommutativity effects do not show up in $D > 4$ dimensions. Exactly in $D = 4$, although we still do not have a nontrivial fixed point, we argue that noncommutativity effects should appear in logarithmic corrections to the scaling behaviors. In less than four dimension, by invoking a dimensional reduction scheme, we developed an ε-expansion in NCLG, a nontrivial stable fixed point appears besides the unstable Gaussian fixed point, since the star product will appear in the fixed point action, we call it as a noncommutative Wilson-Fisher (NCWF) fixed point in contrast to WF fixed point in the ordinary space. We emphasize that this is a new fixed point in the sense that we need at least one additional NC parameter $\theta = \Theta\Lambda^2$, to characterize it, besides the usual mass parameter $r = m^2/\Lambda^2$, and interaction strength $u = g\Lambda^{D-4}$.

A new manifestation of the UV/IR mixing in the NC field theories is uncovered in $D < 4$ dimension: The one loop self-energy is no longer merely a constant that renormalizes the mass term. It also contains a momentum dependent part, which gives rise to an anomalous dimension to the field. This results in a negative critical exponent $\eta = -\varepsilon\theta^2/72$ near the NCWF fixed point. The fact that the "macroscopic" critical exponent η depends on θ, the "microscopic" defining parameter for NC geometry, is a new manifestation of the UV-IR mixing. Naively this might seem to indicate a non-universal behavior of the exponent. In our opinion, we would rather view the NC parameter θ as one of the parameters that characterize the RG fixed point or the universality class, in addition to the usual u^* and r^*. Therefore, more precisely, the NCWF fixed point is not really a point; we have the NCWF

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fixed points, forming a fixed-point line labeled by the NC parameter θ. In this sense, θ represents a marginal parameter.

Here we would like to mention that the authors of Ref. [37] have presented a proof of renormalizability of the NCLG model based on the Wilsonian RG equations. The renormalization conditions proposed in this paper maintain the star product with unrenormalized NC parameters, in line with the spirit of our RG analysis reported in our short paper [23]. However, the authors argued later in the same paper that at sufficiently low momenta the ordinary Wilsonian RG equations with no θ-dependence should be recovered, so there should be an intermediate momentum region in which somehow the Wilsonian RG equations with and without θ-dependence should overlap. We do not agree with this opinion. Indeed, for the nonlocal theories in which the nonlocality is always of a finite range, so that there is no UV-IR mixing, we do agree that at distances much larger than the nonlocality scale, the ordinary Wilsonian RG equations should be valid without any trace of the nonlocality scale. However, in our opinion, this argument does not apply to an NC space with constant coordinate commutators: Due to coordinate-coordinate uncertainty relations, the nonlocal effects are *not* of a finite range in space and we do have UV-IR mixing. So it is natural to expect that in such NC space, nonlocal effects show up at long distances due to UV-IR mixing, and the Wilsonian RG equations always exhibit θ-dependence even at very low momenta.

Moreover, we have shown that for large $\theta > \theta_c = 12/\sqrt{\varepsilon}$, the qualitative behavior of the correlation function is changed. In momentum space, this change reflects in the momentum dependence of the susceptibility $G(k)$ that changes from $1/k^2$ to $1/k^4$. This signals a momentum space instability, that yields a soft mode in the kinetic energy, in which the kinetic energy is minimized at non-zero momentum k_0. Therefore, the NCLG model may exhibit a new modulated order, which is characterized by the wavelength $\lambda_0 = 2\pi/k_0$. A first order phase transition from the uniform order to a modulated order at θ_c and $T = T_c$, which we identify as a Lifshitz point, is developed for large θ in less than four dimension.

In contrast to the self-consistent Hartree-Fock calculation developed in Ref. [20], our RG analysis does not support the stripe phase scenario in the critical dimension $D = 4$.

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We note that the authors of Ref. [20] conjectured the possibility of a Lifshitz point in the NCLG model. Using our RG analysis, we have proved that indeed it does appear in $4-\varepsilon$ dimension, and that such a Lifshitz point appears exactly when the k^2 coefficient vanishes while the k^4 term in kinetic energy is positive to maintain stability. Due to the existence of such a modulated phase, we need a new critical exponent ν_m to describe the Lifshitz point, in the mean field level, $\nu_m = 1/4$. Concomitantly, the mean field phase transition temperature from ordered phase to disordered phase is shifted. To further describe the first order phase transition between modulated phase and uniform ordered phase, we introduced a new critical component β_k to describe the behavior of k_c near the Lifshitz point. Within mean field approximation, β_k takes value 1/2.

Finally, several remarks on possible applications and implications of our work are in order. First, the successful generalization of the Wilsonian RG to NC field theory seems to imply that it should be possible to generalize field theory RG to NC field theory as well. Since the star product structure should be intact during renormalization, the definition of the beta functions and anomalous dimensions should remain the same as their commutative counterparts. However, the central issue is how to incorporate the nonlocality in NCFT into the generalized Callan-Symanzik equations. The simplest and natural guess for the latter would be

$$[\Lambda\frac{\partial}{\partial\Lambda} + \beta_u\frac{\partial}{\partial u} - \frac{N}{2}\gamma_\phi]\Gamma^{(N)}(k_i, u, \Lambda) = 0, \tag{76}$$

with the Moyal phase factor

$$\exp(-\frac{i}{2}\sum_{i<j} k_i \wedge k_j) \tag{77}$$

included in the $\Gamma^{(N)}(k_i, u, \Lambda)$, where u's are coupling constants. However, we should admit that such a proposal is still very speculative. To our current knowledge, how to make sense of field theory renormalizability in NCFT (especially in NC Yang-Mills) to arbitrary orders is still a big question, since the limit of pushing cut-off to infinity generates new IR singularities [5] due to UV/IR mixing. Though, as we have seen, the Wilsonian RG scheme can bypass

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this obstacle by introducing both UV cut-off and IR cutoff, the Callan-Symanzik equations contain more information than the leading asymptotic behavior near the stable fixed point and, therefore, need more efforts to establish.

Secondly, it seems to impossible to introduce operator product expansion (OPE) in a NCFT due to its non-locality. However, perhaps we should not be too disappointed if we recall that in an ordinary field theory there is a beautiful relation between one loop Wilsonian RG equations and OPE coefficients, namely, OPE coefficients uniquely determine the one loop Wilsonian RG equations [38]. If we formally reverse the above logic, it seems that there should exist an appropriate modification of the OPE, since we already know how to make sense of the RG equations for the NCLG model. Of course, by no means it would be an easy job. Curiosity is the drive for advances in science, NCFT provides us new challenges and also new opportunities to understand nonlocal field theories which have great potetential applications both in condensed matter physics and string theory.

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FIG. 1. Tadpole contributions to quadratic action

FIG. 2. One loop corrections to the quartic vertex

FIG. 3. Phase diagram along parameter θ line

FIG. 4. Phase diagram for the Lifshitz scenario. Up: Lifshitz point in the phase diagram; down: scaling of critical wave vector K_c

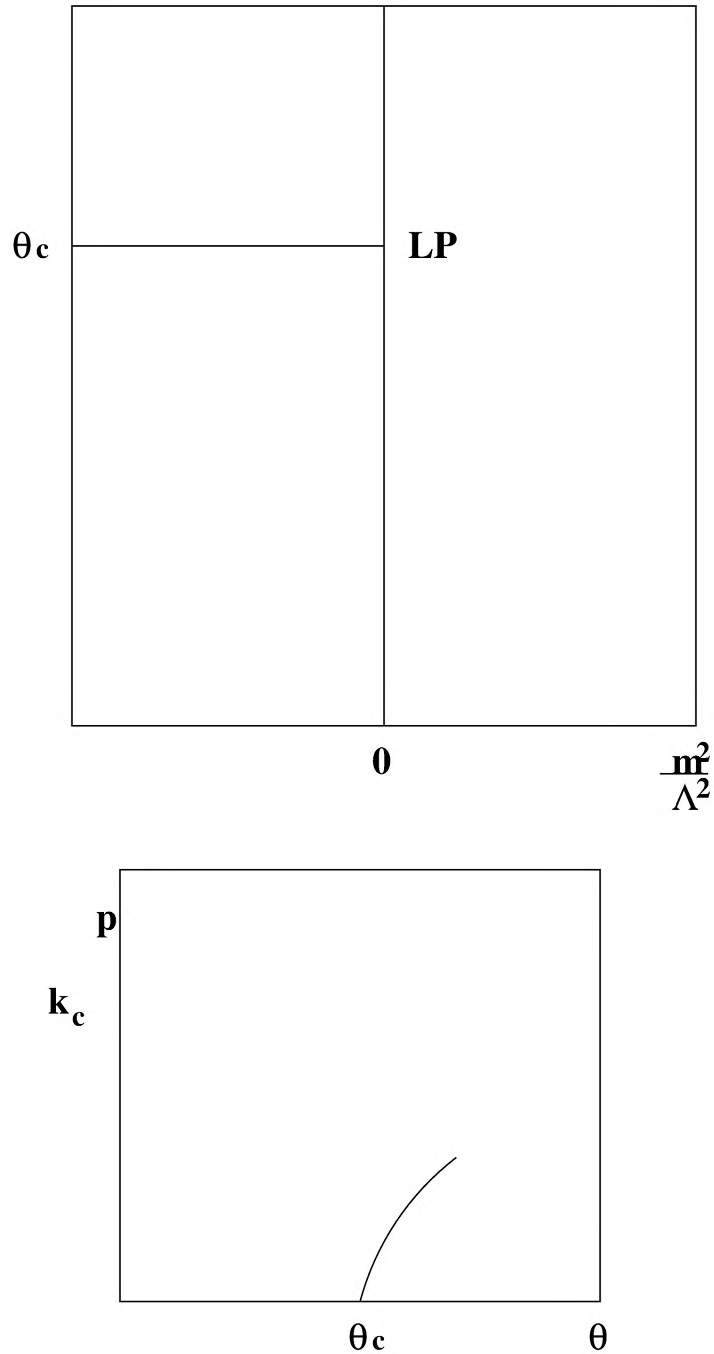

Fig. 4

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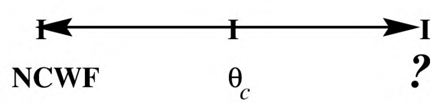
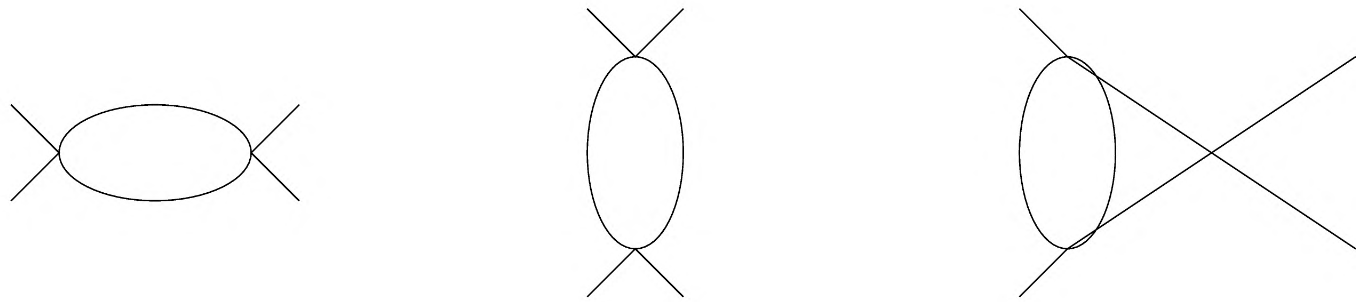


Fig. 3



(a) (b) (c)

Fig. 2

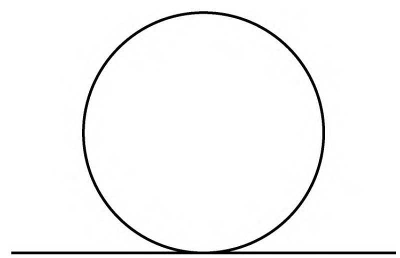


Fig. 1

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