Small-angle impurity scattering and the spin Hall conductivity in two-dimensional semiconductor systems

A. V. Shytov,[1] E. G. Mishchenko,[2] H.-A. Engel,[3] and B. I. Halperin[3]
[1]*Physics Department, Brookhaven National Laboratory, Upton, New York 11973-5000, USA*
[2]*Department of Physics, University of Utah, Salt Lake City, Utah 84112, USA*
[3]*Lyman Laboratory, Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA*
(Received 27 September 2005; revised manuscript received 28 December 2005; published 10 February 2006)

An arbitrarily small concentration of impurities can affect the spin Hall conductivity in a two-dimensional semiconductor system. We develop a Boltzmann-like equation that can be used for impurity scattering with an arbitrary angular dependence, and for an arbitrary angular dependence of the spin-orbit field $\mathbf{b}(\mathbf{k})$ around the Fermi surface. For a model applicable to a two-dimensional hole system in GaAs, if the impurity scattering is not isotropic, we find that the spin Hall conductivity depends on the derivative of b with respect to the energy, on deviations from a parabolic band structure, and on the angular dependence of the scattering. In principle, the resulting spin Hall conductivity can be larger or smaller than the "intrinsic value," and can have an opposite sign. In the limit of small-angle scattering, in a model appropriate for small hole concentrations, where the band is parabolic and $b \propto k^3$, the spin Hall conductivity has an opposite sign from the intrinsic value, and has a larger magnitude. Our analysis assumes that the spin-orbit splitting b and the transport scattering rate τ^{-1} are both small compared to the Fermi energy, but the method is valid for an arbitrary value of $b\tau$.

DOI: 10.1103/PhysRevB.73.075316 PACS number(s): 73.23.−b, 72.25.−b, 73.50.Bk

I. INTRODUCTION

An electric current passing through a semiconductor can induce spin polarization near lateral edges of the sample, with an opposite sign at opposite edges. Such an effect, commonly referred to as the *spin Hall effect*, had been predicted more than 30 years ago,[1] to our knowledge, and was observed recently.[2–4] This effect results from the coupling between spin and momentum of an itinerant electron. Two specific mechanisms have been considered for the spin Hall effect. The "extrinsic" mechanism couples the spin to the momentum during events of impurity scattering in the Mott skew scattering channel.[1,5,6] As a result, scattered electrons of different spin polarizations propagate towards opposite boundaries of the system. As opposed to the extrinsic, the "intrinsic" effect originates from a spin-split band structure[7,8] rather than from impurity scattering.

Experiments have been performed on both electron-doped[2,4] and hole-doped[3] GaAs-based semiconductor structures of different dimensionalities. A spin Hall effect in three-dimensional n-doped GaAs films[2] has been explained in terms of extrinsic skew and sidejump impurity scattering,[9,10] while the intrinsic mechanism was estimated[11] to yield the spin accumulation smaller by an order of magnitude than actually observed in the experiment. The two-dimensional electron gas on a (110) surface of GaAs, studied in Ref. 4, was also in the dirty limit, and the spin Hall effect observed there was attributed to extrinsic effects. On the other hand, p-doped two-dimensional (2D) samples used in the experiments of Ref. 3 were estimated to contain very few impurities and it was suggested that the observations reflect a spin Hall effect of the intrinsic type.

While experiments directly measure spin accumulation along the sample edges, theoretical contributions to the field of the spin Hall effect are mostly concerned with a different quantity which is easier to calculate, namely, the spin current. The latter is conventionally defined as the expectation value of the operator $\hat{j}_k^i = C\{\hat{s}_i, \hat{v}_k\}$, where $\hat{\mathbf{s}}$ and $\hat{\mathbf{v}}$ are, respectively, the operators of the carrier spin and velocity (which is, in general, also spin dependent), and the normalization constant C has been chosen differently by different authors. (In the present paper, we define $\hat{s}_i$ to have eigenvalues $\pm\hbar/2$, and we choose $C=\frac{1}{2}$, so that the spin current is equal to $\hbar/2$ times the difference in the particle currents for carriers with opposite spins.) Apart from the normalization, the definition of spin currents is largely a matter of convenience since it *is not* rigorously related to spin polarization (accumulation) by a conservation law, as opposed to, e.g., the electric charge conservation, namely, $\partial_t s_i + \nabla_k j_k^i \neq 0$. Nonconservation of spin currents (due to spin precession) makes the problem of its conversion into polarization highly nontrivial. Moreover, since the spin current is even with respect to time inversion, it is not necessarily absent even in equilibrium.[12] In the present paper we concentrate on the 2D electron and hole systems. In this case the components of spin current j_k^z polarized along the direction (z) perpendicular to the 2D plane appear due to nonequilibrium conditions only.

The paper[8] predicted a universal value for the intrinsic spin Hall current in a 2D electron gas, with spin-orbit coupling of the Rashba form, which, using the normalization of the present paper, is given by

$$j_y^z = -\frac{eE_x}{8\pi}, \tag{1}$$

independent of the strength of the spin-orbit (SO) interaction. (The prediction of Ref. 7 for *three-dimensional* hole systems, though dependent on the electron density, was also independent of the strength of the SO coupling). These results immediately posed a question of how impurities would affect

the spin Hall conductivity. The derivation of formula (1) neglected scattering, but also assumed a steady state. The latter, however, is impossible in an infinite system if the momentum relaxation is absent. Soon a number of papers appeared about the role of impurity scattering, which, initially, reached different conclusions. Some studies encountered the decreasing of the magnitude of the effect with increasing scattering,[13,14] while others found a universal value (1) even in the presence of scattering. However, it is now generally agreed that in the simplest model of a 2D electron gas (2DEG), with a spin-orbit coupling that is linear in the wave vector $\mathbf{k}$, the dc spin Hall conductivity will vanish, even in the case of arbitrarily small impurity concentrations. Inoue *et al.*[15] first reached this conclusion in the clean limit, $b\tau \to \infty$, where the spin-orbit splitting b exceeds the transport scattering rate τ^{-1}, while entertaining the possibility of a finite effect for finite values of $b\tau$.

The present authors[16] developed a theory based on a quantum kinetic equation to describe the spin-polarized transport in the two-dimensional electron systems for an *arbitrary* ratio of spin-orbit splitting and the scattering rate, as long as they are both small compared to the Fermi energy ($b, \tau^{-1} \ll E_F$). Applying this theory to the spin Hall effect in a 2DEG with ($\mathbf{k}$ linear) Rashba coupling and isotropic impurity scattering, we established the vanishing of spin Hall conductivity for arbitrary values of the product $b\tau$. However, the cancellation is complete only in the bulk. Near the contacts which inject unpolarized electrons, spin currents are nonzero and, hence, a spin polarization normal to a plane can accumulate near the corners of a sample.

The models used in Refs. 15 and 16 assumed short-range impurity potentials, with isotropic scattering. The result of the vanishing spin Hall conductivity in a 2DEG with Rashba coupling was subsequently confirmed by several studies,[17–19,23] and were shown to hold for an arbitrary angular dependence of the impurity scattering. Early numerical calculations[20,21] seemed to contradict the theoretical work but later ones supported the vanishing the of spin Hall current.[22] Results for a 2DEG with Rashba coupling can also be applied directly to a model for a 2DEG on a (001) surface with pure $\mathbf{k}$-linear Dresselhaus coupling, as the two problems are related by a rotation in spin space.

As was pointed out by Dimitrova,[23] the vanishing of dc spin currents j_k^z in an *infinite* 2D system with $\mathbf{k}$-linear spin-orbit coupling has a simple explanation in terms of the following operator argument. Consider a system with the spin-independent disorder and the "Rashba"-type spin-orbit coupling[24] $H_{SO}=\alpha(\sigma_x k_y - \sigma_y k_x)$, where $\boldsymbol{\sigma}$ represents the set of three Pauli spin matrices, and $\mathbf{k}$ is the canonical momentum. The equation of motion for the operator of spin polarization yields $(l=x,y): \partial_t \sigma_l = i[H_{SO}, \sigma_l]/\hbar = -4\alpha m j_l^z/\hbar^3$, where m is the electron effective mass. It follows from this identity that a nonzero expectation value of the spin current j_l^z would result in a time-dependent spin polarization along the corresponding direction l, which is impossible in a steady state. This operator argument applies also to a model with a combination of Rashba and $\mathbf{k}$-linear Dresselhaus coupling. In agreement with this argument, analytical calculations confirm the vanishing of weak-localization (of the first order in $1/E_F\tau$) corrections to the spin Hall conductivity.[25]

It should be emphasized that this vanishing of the spin Hall conductivity is characteristic of the 2D electron system with SO coupling *linear* in momentum. Cubic Dresselhaus terms in the spin-orbit Hamiltonian, which are $\propto \sigma_x k_x k_y^2 - \sigma_y k_y k_x^2$ for a 2D electron system on a (001) surface, should result in a finite spin Hall conductivity.[26]

For a 2D *hole* system in an asymmetric confining well, the Rashba effect gives rise to a spin-orbit field $\mathbf{b}(\mathbf{k})$ whose direction winds three times around a circle in the x-y plane as $\mathbf{k}$ moves once around the Fermi circle.[27,28] It turns out, in this case, that for short-range impurities there is no cancellation in the clean limit, and one then obtains the full value of the intrinsic spin Hall effect, at least in a simplified model which neglects cubic anisotropy.[29,30] Curiously, in contrast to the 2DEG, it is the vertex corrections that vanish in this case, analogous to the absence of the vertex corrections to the electrical conductivity for short-range impurities.[31]

Vertex corrections are also absent in an n-doped 3D system with Dresselhaus interaction, $H_{SO}=\lambda\sigma_x k_x (k_y^2 - k_z^2)$ +cycl. permut., and short-range impurity scattering, in this case due to the cubic crystal symmetry. In the absence of vertex corrections, the impurities lead to a smooth relaxational suppression of the spin Hall conductivity due to the broadening of the spectral function of carriers. In principle, one should obtain the full intrinsic value of the spin Hall conductivity in the clean limit, but 3D systems are always in the dirty limit, where the intrinsic spin Hall conductivity is reduced by a factor $\sim(b\tau)^2$. As mentioned above, the spin Hall effect observed in 3D n-type GaAs is most probably due to extrinsic scattering effects, omitted from the model discussed here.[2,9,10]

It is natural to ask whether the absence of vertex corrections for the 2D hole system is special to the assumption of short-range impurities, or whether it applies equally well to a model with an arbitrary angular dependence of the impurity scattering. In the present paper, we develop equations to calculate the spin Hall conductivity for an arbitrary form of the impurity potential, and for an arbitrary form of the spin-orbit field $\mathbf{b}(\mathbf{k})$ in a 2D system. A solution of these equations is simple in models which have an overall circular symmetry, and we obtain analytic results in the limit of small-angle scattering, resulting from a smooth disorder. This type of disorder is a good assumption for many 2D systems in which the impurity centers (dopants) reside relatively far from the heterostructure interface.

The result of our analysis is that the spin Hall conductivity, in general, does depend on the form of the impurity scattering, and except for the case of isotropic scattering, depends also on the energy dependence of the spin-orbit coupling and on the deviation from parabolicity of the energy dispersion for the carriers. In our circularly symmetric model, the magnitude of the spin-orbit field $\mathbf{b}(\mathbf{k})$ is independent of the direction of $\mathbf{k}$, while its direction winds N times around the unit circle in the x-y plane, at a uniform rate, as $\mathbf{k}$ moves around the Fermi circle. (N must be an odd integer due to time reversal symmetry.) The energy dependence of b and deviations of the energy dispersion ϵ_k from a parabolic form near the Fermi energy, are then characterized by two parameters $\tilde{N}$ and ζ, defined in Eq. (4) below, and for a given

form of the impurity scattering, the spin Hall conductivity is found to depend on N, $\tilde{N}$, and ζ.

In Sec. II, below, we define precisely the models we are considering and the relevant parameters. In Sec. III, the transport quantities are expressed in terms of the Wigner distribution function, for which Boltzmann-like equations of motion are presented in Sec. IV. In Sec. V, these equations are solved in the case of our circularly symmetric model. In Sec. VI, we obtain explicit formulas, Eqs. (57) and (61), for the two limiting cases of isotropic scattering and small-angle scattering by the impurities. The relation of our results to previous theoretical results and to recent experiments is discussed in Sec. VII. Some details of the derivations are presented in Appendix A, and an alternate derivation of the kinetic equations of motion, using Green's function methods, is presented in Appendix B.

II. MODEL

We consider a two-dimensional system of noninteracting electrons or holes, described by a one-body Hamiltonian of the form:

$$H = \epsilon_k - \tfrac{1}{2}\mathbf{b}(\mathbf{k})\cdot\boldsymbol{\sigma} + V(\mathbf{r}), \tag{2}$$

where V is the potential due to impurities, and $\mathbf{k}$ is the kinetic momentum operator. In the presence of a vector potential $\mathbf{A}$, we have $\mathbf{k}=\mathbf{p}-e\mathbf{A}/c$, where $\mathbf{p}\equiv -i\nabla$. Throughout this paper, e is the charge of the carriers, and we set $\hbar=1$ except where explicitly noted. We may include a uniform electric field $\mathbf{E}$ by letting $\mathbf{A}$ depend on time.

By time reversal symmetry, the spin-orbit field $\mathbf{b}$ must satisfy $\mathbf{b}(\mathbf{k})=-\mathbf{b}(-\mathbf{k})$. Note that vectors such as $\mathbf{k}$ and $\mathbf{r}$, which refer to spatial coordinates, have two Cartesian components, while vectors in the spin space, such as $\mathbf{b}$ have three components.

We have assumed for simplicity that ϵ_k, the energy dispersion in the absence of spin-orbit coupling, is isotropic and a monotonic function of k. The formulas derived below can be extended in a straightforward manner to more complicated band structures, but the formulas would then be much less transparent, and the resulting integral equations are more difficult to solve.

As a further simplification, we shall later specialize to models where $\mathbf{b}$ has a simple rotational symmetry; i.e., we assume

$$b_z = 0, \quad b_x + ib_y = b_0(k)e^{iN\theta}, \tag{3}$$

where $b_0(k)$ is a complex number, whose magnitude may vary with the magnitude of k, but whose phase is independent of $\mathbf{k}$. This will allow us to obtain analytic solutions to the equations. The final results will, of course, depend crucially on the winding number N. We shall see that results can also be affected by the energy dependence of b and by deviations of the energy dispersion ϵ_k from a parabolic form near the Fermi energy. We characterize these dependences by constants $\tilde{N}$ and ζ, with

$$\tilde{N} = \frac{d\ln|b_0|}{d\ln k}, \quad 1+\zeta = \frac{d\ln v}{d\ln k}, \tag{4}$$

for k equal to the Fermi momentum. [Here $v(k)=\partial\epsilon_k/\partial k$ is the electron velocity without spin-orbit corrections.] In other words, we assume that near the Fermi energy, $v\propto k^{1+\zeta}$, and

$$b_x + ib_y \propto k^{\tilde{N}}e^{iN\theta}. \tag{5}$$

For a 2D electron system on a (100) surface of a III-V semiconductor, the case $N=1$ corresponds to pure Rashba coupling, while $N=-1$ corresponds to a $\mathbf{k}$-linear Dresselhaus coupling. The case $N=3$ arises in a circularly symmetric model of a 2D hole system, which ignores warping due to the tetragonal symmetry. In the limit of small hole doping, the band structure is parabolic and $(b_x+ib_y)\propto(k_x+ik_y)^3$; hence $\zeta=0$ and $\tilde{N}=N=3$. At larger values of the doping, when the Fermi energy becomes comparable to the splitting between the light and heavy hole bands, the values of $\tilde{N}$ is reduced, and the value of ζ will be negative. At still higher doping, both light and heavy hole bands become occupied, and the situation is more complicated.

The Hamiltonian (2) assumes that there is no direct spin-orbit coupling associated with the impurity potential or with applied electric field. For example, it omits terms of the form $\lambda\boldsymbol{\sigma}\cdot(\mathbf{k}\times\nabla V)$, which are generally present in systems with spin-orbit coupling. This term leads to skew scattering by the impurities, giving an extrinsic contribution to the spin Hall conductivity. Since there is no direct general relation between the coupling constant λ and the magnitude of the intrinsic spin-orbit splitting b, it is at least logically consistent to consider a model with $\lambda=0$. Moreover, the skew-scattering cross section is of higher order in the impurity potential V than the ordinary transport scattering cross section, so it is normally dominated by places where the carrier enters a region of strong potential gradients, close to the impurity. In a remotely doped 2D hole system one might expect that the skew-scattering contribution to the spin Hall conductivity should be relatively small, at least if one can ignore the scattering by residual impurities close to the 2D system.

We concentrate on systems that are large compared to the mean free path. The "mesoscopic" spin Hall effect[32–34] in ballistic heterostructures is, therefore, beyond the scope of the present paper.

III. BOLTZMANN FORMULATION

We begin by defining a Wigner distribution function for the carriers, which is a 2×2 matrix in spin space, given by

$$n_{\alpha\beta}(\mathbf{k},\mathbf{r},\epsilon,t) = \frac{1}{2\pi}\int d^2\mathbf{s}\, d\tau\, e^{i\mathbf{k}\cdot\mathbf{s}-i\epsilon\tau} \times\left\langle \psi^\dagger_\beta\left(\mathbf{r}+\frac{\mathbf{s}}{2},t+\frac{\tau}{2}\right)\psi_\alpha\left(\mathbf{r}-\frac{\mathbf{s}}{2},t-\frac{\tau}{2}\right)\right\rangle. \tag{6}$$

In thermal equilibrium, if the spin-orbit coupling is absent

($\mathbf{b}=0$) and impurities are neglected, we have the standard result

$$n_{\alpha\beta}=f_0(\epsilon)\delta(\epsilon-\epsilon_k)\delta_{\alpha\beta}, \tag{7}$$

where f_0 is the Fermi function. For $b\neq 0$, there are two different wave vectors at a given energy ϵ and a given direction θ, corresponding to spin states parallel or antiparallel to $\mathbf{b}$. Introducing an index $\sigma=\pm 1$ to distinguish these two states, one finds in equilibrium, $n_{\alpha\beta}=n^{\rm eq}_{\alpha\beta}(k,\epsilon)$ with

$$n^{\rm eq}_{\alpha\beta}=\frac{1}{2}\sum_{\sigma=\pm 1}\left(\delta_{\alpha\beta}+\sigma\frac{\mathbf{b}\cdot\boldsymbol{\sigma}_{\alpha\beta}}{b}\right)f_0(\epsilon)\delta\left(\epsilon-\epsilon_k+\sigma\frac{b}{2}\right). \tag{8}$$

Impurity scattering broadens the δ functions in these expressions, but does not shift their centers or alter their amplitudes, provided that the scattering rate is small compared to the Fermi energy $E_{\rm F}$.

We define the distribution as a function of energy and the direction in $\mathbf{k}$ space by

$$\tilde{n}_{\alpha\beta}(\theta,\epsilon)\equiv\int_0^\infty\frac{k\,dk}{(2\pi)^2}n_{\alpha\beta}(\mathbf{k},\epsilon), \tag{9}$$

with $\mathbf{k}=k(\cos\theta,\sin\theta)$. (We suppress here the indices $\mathbf{r}$ and t, and we shall omit other indices as well when the meaning is clear.) Substituting (8) into (9), we find, in equilibrium, to first order in $\mathbf{b}/E_F$,

$$\tilde{n}_{\alpha\beta}=\tilde{n}^{\rm eq}_{\alpha\beta}(\theta,\epsilon)=\frac{k_\epsilon}{(2\pi)^2 v_\epsilon}f_0(\epsilon)\delta_{\alpha\beta}+f_0(\epsilon)\frac{\boldsymbol{\sigma}_{\alpha\beta}}{8\pi^2}\cdot\frac{\partial}{\partial\epsilon}\left(\frac{k_\epsilon\mathbf{b}(k_\epsilon)}{v_\epsilon}\right), \tag{10}$$

where $v_\epsilon=\partial\epsilon_k/\partial k$, and k_ϵ is the value of k such that ϵ_k is equal to the given value of ϵ. For ϵ at the Fermi energy, k_ϵ is the Fermi momentum in the absence of spin-orbit coupling, and v_ϵ is the corresponding Fermi velocity. The last term in (10) may be understood as arising from the difference in the densities of states for the two spin states $\sigma=\pm 1$, which may be written as $(2\pi)^{-2}(k_\epsilon+\delta k)/(v_\epsilon+\delta v)$, where $\delta k=(b\sigma/2v_\epsilon)$ is the spin-orbit correction to the particle momentum at a given energy and $\delta v=(d^2\epsilon_k/dk^2)\delta k+(\frac{1}{2})\sigma\partial b/\partial k$ is the correction to its velocity. (We used here a well-known expression for 2D density of states: $\nu_{2D}=(2\pi)^{-2}k_\epsilon/v_\epsilon$, see also Appendix A.) The shifts in momentum and velocity have the opposite sign for the two values of σ, and the final term in Eq. (10) is obtained by expanding to first order in these shifts.

Finally, we introduce functions $n(\theta,\epsilon)$ and $\boldsymbol{\Phi}(\theta,\epsilon)$, which describe the excess number and spin densities in the momentum direction θ and energy ϵ, defined by

$$\tilde{n}_{\alpha\beta}(\theta,\epsilon)-\tilde{n}^{\rm eq}_{\alpha\beta}(\theta,\epsilon)\equiv\frac{n(\theta,\epsilon)}{2}\delta_{\alpha\beta}+\boldsymbol{\Phi}(\theta,\epsilon)\cdot\boldsymbol{\sigma}_{\alpha\beta}. \tag{11}$$

If n and $\boldsymbol{\Phi}$ are known, we can readily compute the particle current and spin current, as well as the particle density and spin density at a spatial point $\mathbf{r}$. We define the spin density and spin current by

$$s^\mu(\mathbf{r})=\frac{1}{2}\sum_{\alpha,\beta}\langle\psi^+_\alpha(\mathbf{r})\sigma^\mu_{\alpha\beta}\psi_\beta(\mathbf{r})\rangle, \tag{12}$$

$$\mathbf{j}^\mu(\mathbf{r})=\frac{1}{4}\sum_{\alpha,\beta}\langle\psi^+_\alpha(\mathbf{r})\{\sigma^\mu,\mathbf{v}\}_{\alpha\beta}\psi_\beta(\mathbf{r})\rangle, \tag{13}$$

where $\mathbf{v}$ is the velocity operator,

$$\mathbf{v}=\frac{\partial\epsilon_k}{\partial\mathbf{k}}-\frac{\boldsymbol{\sigma}}{2}\cdot\frac{\partial\mathbf{b}(\mathbf{k})}{\partial\mathbf{k}}. \tag{14}$$

At any point $\mathbf{r}$, we then have for densities and currents

$$\langle n\rangle-n_0=\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}d\epsilon\,n(\theta,\epsilon), \tag{15}$$

$$s^\mu=\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}d\epsilon\,\Phi_\mu(\theta,\epsilon), \tag{16}$$

$$\mathbf{j}=\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}d\epsilon\,\mathbf{v}_\epsilon(\theta)n(\theta,\epsilon), \tag{17}$$

$$\mathbf{j}^\mu=\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}d\epsilon\left[\mathbf{v}_\epsilon(\theta)\Phi_\mu(\theta,\epsilon)+\tfrac{1}{2}\delta\mathbf{v}_\mu(\theta)n(\theta,\epsilon)\right]. \tag{18}$$

Here

$$\mathbf{v}_\epsilon(\theta)\equiv v_\epsilon(\cos\theta,\sin\theta), \tag{19}$$

is the "bare" velocity, and

$$\delta\mathbf{v}_\mu(\theta)\equiv-\frac{1}{2}\frac{\partial b_\mu(\mathbf{k})}{\partial\mathbf{k}} \tag{20}$$

is the spin-orbit contribution. In the equation for $\mathbf{j}$, we have dropped terms of order $|b|\Phi_\mu$. This is justified because we assume $|b|/E_{\rm F}\ll 1$, and in the situations we will consider, values of $|\boldsymbol{\Phi}|$ are smaller than the characteristic values of $n(\theta,\epsilon)$ by a factor of $|b|/E_{\rm F}$ or more.

IV. EQUATIONS OF MOTION

We will be interested in the linear response of the system to an infinitesimal applied electric field $\mathbf{E}$, which might, in general, depend on space as well as time. In developing the equations of motion, then, we need to only consider terms which are of first order in $\boldsymbol{\Phi}$ and n, or first order in E; we may ignore terms of order En or $E\Phi$.

In the absence of the spin-orbit coupling, the linearized equation of motion for $n(\theta,\epsilon,\mathbf{r},t)$ may be written as

$$\frac{\partial n}{\partial t}=\left.\frac{\partial n}{\partial t}\right|_v+\left.\frac{\partial n}{\partial t}\right|_E+\left.\frac{\partial n}{\partial t}\right|_{\rm scat}, \tag{21}$$

where

$$\left.\frac{\partial n}{\partial t}\right|_v=-v_\epsilon\hat{k}\frac{\partial}{\partial\mathbf{r}}n, \tag{22}$$

is the standard advection term,

$$\left.\frac{\partial n}{\partial t}\right|_E=\frac{2eEk\cos\theta}{(2\pi)^2}\left(-\frac{\partial f_0(\epsilon)}{\partial\epsilon}\right), \tag{23}$$

describes acceleration of particles by the external field, and

$$\left.\frac{\partial n(\theta)}{\partial t}\right|_{\text{scat}} = \int_0^{2\pi} d\theta' K(\theta-\theta')[n(\theta')-n(\theta)], \quad (24)$$

is the collision integral. The scattering kernel K is given in the Born approximation, by

$$K(\theta-\theta') = W(q)k_\epsilon/(2\pi v_\epsilon),$$

$$q \equiv 2k_\epsilon \sin(|\theta-\theta'|/2),$$

$$W(q) \equiv \langle |V(\mathbf{q})|^2 \rangle. \quad (25)$$

Here $V(\mathbf{q})$ is the Fourier component of the scattering potential at wave vector $\mathbf{q}$, and we have assumed that its mean-square value is independent of the direction in space. In (23), we have assumed that the electric field $\mathbf{E}$ is oriented along the x axis. The equations assume that spatial variations are slow on the scale of the Fermi wavelength, and they may need to be supplemented by boundary conditions derived from microscopic equations, in the case of a sharp interface or sample edge.

The equations given above for $\partial n/\partial t$ remain valid in the presence of spin-orbit coupling, to first order in b/E_{F}. The equations of motion for $\mathbf{\Phi}$, however, are affected in an essential way by a nonzero b. We find here

$$\frac{\partial \mathbf{\Phi}}{\partial t} = \left.\frac{\partial \mathbf{\Phi}}{\partial t}\right|_v + \left.\frac{\partial \mathbf{\Phi}}{\partial t}\right|_b + \left.\frac{\partial \mathbf{\Phi}}{\partial t}\right|_E + \left.\frac{\partial \mathbf{\Phi}}{\partial t}\right|_{\text{scat}}, \quad (26)$$

$$\left.\frac{\partial \mathbf{\Phi}}{\partial t}\right|_v = -v_\epsilon \hat{k}\frac{\partial}{\partial \mathbf{r}}\mathbf{\Phi}, \quad (27)$$

$$\left.\frac{\partial \mathbf{\Phi}}{\partial t}\right|_b = -\mathbf{b}\times\mathbf{\Phi} + n\left(\mathbf{b}\times\frac{\partial \mathbf{b}}{\partial k}\right)\frac{1}{4v_\epsilon}, \quad (28)$$

$$\left.\frac{\partial \mathbf{\Phi}}{\partial t}\right|_E = \frac{Ee}{8\pi^2 v_\epsilon}\left(-\frac{\partial f_0}{\partial \epsilon}\right)\frac{\partial}{\partial \theta}(\mathbf{b}\sin\theta), \quad (29)$$

$$\left.\frac{\partial \mathbf{\Phi}(\theta)}{\partial t}\right|_{\text{scat}} = \int_0^{2\pi} d\theta' K(\theta-\theta')[\mathbf{\Phi}(\theta')-\mathbf{\Phi}(\theta)] + \int_0^{2\pi} d\theta'[\mathbf{M}(\theta,\theta')n(\theta') - \mathbf{M}(\theta',\theta)n(\theta)]. \quad (30)$$

The spin-orbit contribution to the scattering kernel can be divided into two different parts:

$$\mathbf{M} = \mathbf{M}^d + \mathbf{M}^w, \quad (31)$$

where

$$\mathbf{M}^d(\theta,\theta') = \frac{v_\epsilon K(\theta-\theta')}{4k_\epsilon}\frac{\partial}{\partial \epsilon}\left[\frac{k_\epsilon \mathbf{b}(k_\epsilon,\theta)}{v_\epsilon}\right], \quad (32)$$

originates from the spin-orbit correction to the density of states, and

$$\mathbf{M}^w(\theta,\theta') = \frac{[\mathbf{b}(\theta)+\mathbf{b}(\theta')]}{4k_\epsilon v_\epsilon}\tan\left(\frac{\theta-\theta'}{2}\right)\frac{\partial K(\theta-\theta')}{\partial \theta} \quad (33)$$

is due to the spin dependence of the momentum transfer entering the scattering matrix element. In the case of small-angle scattering, Eqs. (32) and (33), which are correct to first order in $\mathbf{b}$, require not only that $b/E_F \ll 1$, but also that $b < v_F\bar{q}$, where $\bar{q}$ is the typical scattering momentum. In other words, we do not consider an extremely smooth potential for which $\bar{q} < b/v_F$.

Equations (26)–(33) contain a precession of electron spin due to the spin-orbit interaction, acceleration of electrons by an electric field, and impurity scattering. It therefore describes the polarization of electron spins due to spin Hall effect and relaxation of this polarization via the Dyakonov-Perel mechanism. A more detailed derivation of these equations will be given in Appendixes A and B. We give here just a brief indication of the origin of the various terms. We then shall explore some consequences of these equations in simple cases, which will shed additional light on their meaning and will provide some nontrivial consistency checks.

A. Origins

The contribution given by (28) may be understood as arising from the precession of the spin, induced by the spin-orbit field $\mathbf{b}$. The second term on the right-hand side of this equation is zero in any case where the direction of $\mathbf{b}$ is determined by the direction of $\mathbf{k}$, independent of the magnitude of k, and we shall omit it in the following. The electric field term (29) reflects the action of $\mathbf{E}$ on the equilibrium distribution (8), as explained in Appendix A.

The scattering kernel $\mathbf{M}$ entering (30) gives a contribution to $\partial\mathbf{\Phi}/\partial t$ proportional to n, which is present even when $\mathbf{\Phi}=0$. The term $\mathbf{M}^d$, given by (32), arises because there are two different Fermi radii at a given energy ϵ and a given direction θ, corresponding to spin states parallel or antiparallel to $\mathbf{b}$, and the densities of final states are generally different for particles scattered into these two spin states. The second term $\mathbf{M}^w$, given by (33) arises when the scattering matrix element depends on q, because the momentum transfer actually depends not only on $\theta-\theta'$, but also on the initial and final Fermi radii, and hence on the component of the spin in the directions of $\mathbf{b}(\theta)$ and $\mathbf{b}(\theta')$.

B. Application to simple examples

First consider a situation where $E=0$, and $n(\theta,\epsilon)$ is independent of θ. The equations of motion for $\mathbf{\Phi}$ then have a time-independent solution where

$$\mathbf{\Phi} = \frac{nv_\epsilon}{4k_\epsilon}\frac{\partial}{\partial \epsilon}\left(\frac{k_\epsilon \mathbf{b}}{v_\epsilon}\right). \quad (34)$$

A change in the electron density independent of θ is just what one would find if one makes a small change in the chemical potential, by an amount $\delta\mu$ such that

$$n = -\delta\mu \frac{v_\epsilon k_\epsilon}{2\pi^2}\left(\frac{\partial f_0}{\partial \epsilon}\right). \tag{35}$$

The change in the equilibrium distribution produced by a shift $\delta\mu$ can be obtained from (10) by replacing $f_0(\epsilon)$ by $f_0(\epsilon-\delta\mu)$. We see that this would change the polarization by an amount precisely equivalent to (34), as is required.

As another example, we may consider a situation where there is no impurity scattering as well as no electric field. We may then construct spatially varying time-independent solutions of the equations of motion of the form

$$\mathbf{\Phi}(\mathbf{r},\theta,\epsilon) = a\hat{b} + c[\hat{g}\cos(\delta\mathbf{k}\cdot\mathbf{r}) + (\hat{k}\times\hat{g})\sin(\delta\mathbf{k}\cdot\mathbf{r})], \tag{36}$$

where $\hat{b}$ is a unit vector parallel to $\mathbf{b}(k_\epsilon,\theta)$, $\hat{g}$ is a unit vector perpendicular to $\hat{b}$, a and c are arbitrary constants, which may depend on θ and ϵ, and

$$\delta\mathbf{k} = \frac{|b|}{2v_\epsilon}(\cos\theta,\sin\theta). \tag{37}$$

This steady state corresponds to a situation where we have populated eigenstates of the Hamiltonian with energy ϵ and wave vector direction θ, whose spin at $\mathbf{r}=0$ is polarized in the direction $a\hat{b}+c\hat{g}$. The spin polarization precesses with a spatial frequency $\delta\mathbf{k}$ as one moves away from the origin.

Finally, we may consider a situation where impurities exist only in a small region about the origin, whose radius is small compared to $1/\delta k$ but large on the scale of the Fermi wavelength. We can then construct a scattering wave solution of the Hamiltonian, for which there is an incident plane wave of definite energy ϵ and wave vector $\mathbf{k}$, with polarization parallel or antiparallel to $\mathbf{b}(\mathbf{k})$. The scattered wave radiating from the origin will have spin parallel to the spin of the incident wave for $\mathbf{r}$ close to the origin, but will have a spin direction that depends on $\mathbf{r}$ away from the origin. If one moves out along a constant direction $\hat{r}$, the spin direction will precess about the direction $\mathbf{b}(\mathbf{k}')$, where $\mathbf{k}'$ is a vector of magnitude k_ϵ and direction $\hat{r}$. The polarization $\mathbf{\Phi}(\mathbf{r},\theta,\epsilon)$ produced by such a scattering wave solution will give a time-independent solution of the equations of motion derived above, at least within the Born approximation.

V. CASE OF CIRCULAR SYMMETRY

We now specialize to the case where $\mathbf{b}(\mathbf{k})$ has the simple dependence on θ given by (3). We wish to find the linear response to a uniform electric field, in an infinite homogeneous system.

When the angular dependence of $\mathbf{b}$ is given by (3), the equations can be solved by Fourier transform in θ, i.e., by expanding $\mathbf{\Phi}(\theta)$ in circular harmonics:

$$\Phi_z = \sum_{m=-\infty}^{\infty} \Phi_m^z e^{im\theta},$$

$$\Phi^+ \equiv \Phi_x + i\Phi_y = \sum_m \Phi_m e^{im\theta},$$

$$K(\theta-\theta') = \sum_m K_m e^{im(\theta-\theta')}. \tag{38}$$

Since Φ_z is real, we have $\Phi_m^z=(\Phi_{-m}^z)^*$. Also, since K is real and an even function of $\theta-\theta'$, we see that K_m is real and $K_m=K_{-m}$. Equations (28) and (29) can then be written as

$$\left.\frac{d\Phi_m^z}{dt}\right|_b = \frac{-b_0^*\Phi_{m+N} + b_0\Phi_{N-m}^*}{2i},$$

$$\left.\frac{d\Phi_m}{dt}\right|_b = ib_0\Phi_{m-N}^z,$$

$$\left.\frac{d\Phi_m^z}{dt}\right|_E = 0,$$

$$\left.\frac{d\Phi_m}{dt}\right|_E = -\frac{Ee\eta_0 m}{2}[\delta_{m,(N+1)} - \delta_{m,(N-1)}]\frac{\partial f_0}{\partial\epsilon}, \tag{39}$$

$$\eta_0 \equiv \frac{b_0}{8\pi^2 v_\epsilon}. \tag{40}$$

The scattering contribution depends on $n(\theta,\epsilon)$. For a dc electric field, one obtains the standard result,

$$n(\theta,\epsilon) = E\tau\frac{ek_\epsilon}{2\pi^2}\left(-\frac{\partial f_0}{\partial\epsilon}\right)\cos\theta, \tag{41}$$

where τ is the transport scattering lifetime, given by

$$\frac{1}{\tau} = 2\int_{-\pi}^{\pi} d\theta\, K(\theta)\sin^2\left[\frac{\theta}{2}\right]. \tag{42}$$

We then find

$$\left.\frac{d\Phi_m^z}{dt}\right|_{\text{scat}} = -2\pi\Phi_m^z(K_0-K_m), \tag{43}$$

$$\left.\frac{d\Phi_m}{dt}\right|_{\text{scat}} = -2\pi\Phi_m(K_0-K_m) + \frac{eE}{2}[\gamma_{(+)}\delta_{m,(N+1)} + \gamma_{(-)}\delta_{m,(N-1)}]\left(-\frac{\partial f_0}{\partial\epsilon}\right), \tag{44}$$

where

$$\gamma_{(\pm)} = \gamma_0' + \gamma_0^W \pm \lambda^W, \tag{45}$$

$$\gamma_0' = 2\pi\eta_0\tau(K_1-K_N)(\tilde{N}-\zeta), \tag{46}$$

$$\gamma_0^W = 2\pi\eta_0\tau\left[\frac{\tilde{K}_{N+1}+\tilde{K}_{N-1}}{2} - \tilde{K}_N + \tilde{K}_1 - \tilde{K}_0\right], \tag{47}$$

$$\lambda^W = 2\pi\eta_0\tau\left[\frac{\tilde{K}_{N+1}-\tilde{K}_{N-1}}{2}\right], \tag{48}$$

with

$$\tilde{K}(\theta-\theta') \equiv \tan\left(\frac{\theta-\theta'}{2}\right)\frac{\partial K(\theta-\theta')}{\partial\theta} \equiv \sum_m \tilde{K}_m e^{i(\theta-\theta')}. \tag{49}$$

It is seen by inspection that the solution of these equations has all components zero except for $\Phi_{N\pm1}$ and $\Phi^z_1=(\Phi^z_{-1})^*$, which are therefore determined by a set of three coupled linear equations. For a dc applied field, where $d\Phi/dt=d\Phi^z/dt=0$, these equations may be written in the form

$$2i\Phi^z_1 = (-b_0^*\Phi_{N+1}+b_0\Phi^*_{N-1})\tau, \tag{50}$$

$$\tau^{-1}k_{N+1}\Phi_{N+1} - ib_0\Phi^z_1 = \frac{Ee}{2}\left(\frac{-\partial f_0}{\partial\epsilon}\right)\alpha_{(+)}, \tag{51}$$

$$\tau^{-1}k_{N-1}\Phi^*_{N-1} + ib_0^*\Phi^z_1 = -\frac{Ee}{2}\left(\frac{-\partial f_0}{\partial\epsilon}\right)\alpha^*_{(-)}, \tag{52}$$

where

$$k_m \equiv 2\pi(K_0-K_m)\tau, \quad \alpha_{(+)} \equiv [+\gamma_{(+)}+\eta_0(N+1)],$$

$$\alpha^*_{(-)} \equiv [-\gamma^*_{(-)}+\eta_0^*(N-1)]. \tag{53}$$

Note that k_m are dimensionless constants that depend on the shape of the angular dependence of the impurity scattering, but not on its overall strength.

Solution of these equations gives

$$2i\Phi^z_1 = Ee\left(\frac{\partial f_0}{\partial\epsilon}\right)\left(\frac{k_{N+1}b_0\alpha^*_{(-)}+k_{N-1}b_0^*\alpha_{(+)}}{2D}\right), \tag{54}$$

$$D \equiv (\tau^{-2}/2)[|b_0|^2\tau^2(k_{N+1}+k_{N-1})+2k_{N+1}k_{N-1}]. \tag{55}$$

Note that the right-hand side of (54) is real, so that Φ^z_1 is pure imaginary.

Integrating (54) over the energy ϵ, we find the spin currents $j^z_x=0$ and $j^z_y \equiv -\sigma_{\rm sH}E$, with a spin Hall conductivity given by

$$\sigma_{\rm sH} = \pi e v_F\left(\frac{k_{N+1}b_0\alpha^*_{(-)}+k_{N-1}b_0^*\alpha_{(+)}}{2D}\right). \tag{56}$$

In the clean limit, $|b|\tau\gg1$, one can replace the denominator D in Eq. (56) by $|b_0|^2(k_{N+1}+k_{N-1})/2$. Since $\alpha_{(\pm)}\propto|b_0|$, we see that $\sigma_{\rm sH}$ is independent of τ and independent of the magnitude of $|b|$ in this limit. By contrast, in the dirty limit, when $|b|\tau\ll1$, we see that $\sigma_{\rm sH}\propto|b\tau|^2$.

Note that the spin Hall conductivity cancels at $N=1$ for an arbitrary scattering kernel $K(\theta-\theta')$. Indeed, according to Eq. (46), γ^d_0 is identically zero for $N=1$, while γ^w_0 and λ^w given by Eqs. (46) and (47) are equal. Thus, both $\gamma^{(-)}$, and $\alpha_{(-)}$ are zero, therefore the first term in Eq. (56) vanishes. Also, $k_0=0$ by definition [see Eq. (53)], and the spin Hall conductivity is identically zero. This result is in agreement with a general argument given in Ref. 23.

VI. ISOTROPIC SCATTERING AND SMALL-ANGLE SCATTERING

For the case of isotropic impurity scattering, we have $K_0=(2\pi\tau)^{-1}$, and $K_m=0$ for $m\neq0$. We also have $\tilde{K}=0$, $\gamma^d_0=0$, and $\gamma_{(\pm)}=0$. Then, for $|N|\neq1$, we find in the clean limit,

$$\sigma_{\rm sH} = \frac{eN}{8\pi}. \tag{57}$$

This is just the "universal intrinsic value" of the spin Hall conductivity for this model, as has been found previously by other authors, written for our present normalization. The result will be reduced by a factor of $(1+|b_0|^2\tau^2)^{-1}$ if $|b_0|\tau$ is not small.

In the limit of small-angle scattering, we have

$$\tau^{-1} = \int_{-\pi}^{\pi} K(\theta)\frac{\theta^2}{2}d\theta,$$

$$K_0-K_m = \frac{m^2}{2\pi\tau},$$

$$\tilde{K}(\theta) = \frac{\theta}{2}\frac{\partial}{\partial\theta}K(\theta). \tag{58}$$

[This form of K_m reflects the well-known proportionality of the collision integral to $\partial^2 f(\theta)/\partial\theta^2$ in the small-angle scattering limit.] To compute $\tilde{K}_m$, one has to integrate by parts and expand the integrand to the second order in θ. The result is

$$\tilde{K}_m = -\frac{K_0}{2}+\frac{3m^2}{4\pi\tau}. \tag{59}$$

We then find

$$\gamma^d_0 = \eta_0(N^2-1)(\tilde{N}-\zeta),$$

$$\gamma^w_0 = 3\eta_0,$$

$$\lambda^w = 3N\eta_0. \tag{60}$$

Using Eqs. (45), (53), (55), and (56), we thus obtain, in the limit of the small-angle scattering and $|b_0|\tau\gg1$,

$$\sigma_{\rm sH} = -\frac{eN}{4\pi}\left(\frac{N^2-1}{N^2+1}\right)(\tilde{N}-\zeta-2). \tag{61}$$

As expected, this result gives $\sigma_{\rm sH}=0$ for $|N|=1$. For $|N|\neq1$, the result for small-angle scattering can be larger or smaller than (57), and can even have an opposite sign, depending on the values of $\tilde{N}$ and ζ.

For a 2D hole system in GaAs, at low doping, with $N=\tilde{N}=3$ and $\zeta=0$, we obtain, for small-angle scattering,

$$\sigma_{\rm sH} = -\frac{3}{5}\frac{e}{\pi}, \tag{62}$$

which has a different sign and is approximately twice as large as the result (57) for isotropic scattering.

In a remotely doped 2D electron or hole system, the charged impurities that compensate the carriers are located

far from the 2D system, so they contribute only long-wavelength Fourier components to the disorder potential, and produce only small-angle scattering. As the set back distance is made large, their contribution to the transport scattering rate decreases, so that eventually the latter may be dominated by a small concentration of residual impurities close to the layer, whose scattering amplitudes may be nearly isotropic. Thus, it is natural to consider a model where both types of impurities are important, so the scattering kernel $K(\theta)$ contains both a narrow peak at $\theta \approx 0$, and an isotropic θ-independent part. We assume that a fraction p of the transport scattering rate arises from small-angle scatterers and a fraction $1-p$ arises from large-angle scatterers, i.e.,

$$\frac{p}{1-p} = \frac{n_s}{n_i}\frac{\sigma_s^{\mathrm{tr}}}{\sigma_i^{\mathrm{tr}}}, \tag{63}$$

where n_i and n_s are the densities of isotropic and small-angle scatterers and the respective transport cross sections are σ_i^{tr} and σ_s^{tr}. We then Fourier transform $K(\theta)$ and find that the coefficients k_m entering (56) are given by $k_m = 1-p+pm^2$, for $m \neq 0$. [From (42) and (53), it may be seen that for arbitrary $K(\theta)$, the transport scattering rate is determined by $K_0 - K_1$, so that one always has $k_1 = 1$.] The constants γ_0^d, γ_0^w, and λ^w, given by Eqs. (46)–(48) have no contribution from the isotropic part, and, are therefore, each reduced from the values given in (60) by a factor of p. The quantity η_0 is independent of p. Substituting these values in Eqs. (55) and (56), one finds, in the clean limit, for $N \neq 1$:

$$\sigma_{\mathrm{sH}} = \frac{e}{8\pi}\frac{N}{1+pN^2} \times \{(1+3p)(1-p) - p(N^2-1) \times [3p - 1 + 2p(\tilde{N} - \zeta - 3)]\}. \tag{64}$$

Equation (64) reproduces the "universal intrinsic value" (57) for $p=0$, and the small-angle scattering limit (61) for $p=1$. Note, however, that it does not simply interpolate between the two values: Eq. (64) is not necessarily a monotonic function of p.

In general, the parameters $\tilde{N}$ and ζ are related to each other. However, they enter Eq. (64) only via the combination $\tilde{N} - \zeta$. In order to illustrate the behavior of our model for different values of $\tilde{N} - \zeta$, we plotted $\sigma_{\mathrm{sH}}(p)$ at different values of ζ, keeping $\tilde{N}$ fixed (see Fig. 1). Note that spin Hall conductivity can even change its sign for $\tilde{N} - \zeta > 2$. Also, σ_{sH} can exceed the universal intrinsic value $3e/8\pi$.

The model with a combination of isotropic and small-angle scattering can also be considered as an approximation to a situation where there is a single type of impurity with a scattering kernel that is neither isotropic nor strongly peaked at small angles.

VII. RELATION TO EXPERIMENTS AND PREVIOUS THEORETICAL RESULTS

In agreement with previous theoretical work, we have found that there is no spin Hall conductivity in the case $N = \pm 1$, applicable to a 2D electron system with pure Rashba or

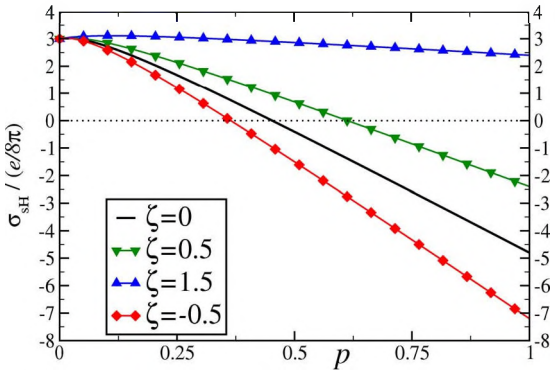

FIG. 1. (Color online) Spin Hall conductivity σ_{sH}, in units of $e/8\pi$, for various values of the band curvature parameter ζ, as a function of p, the fraction of transport scattering rate due to small-angle scatterers. All curves are for parameters $N = \tilde{N} = 3$. [See Eqs. (3) and (4) for definitions.] The clean limit ($b_0\tau \gg 1$) is assumed. The case $\zeta = 0$ corresponds to a circularly symmetric model of a 2D hole gas in the limit of low carrier density.

k-linear Dresselhaus coupling, for arbitrary form of the impurity scattering, while the spin Hall conductivity has the full intrinsic value for the case $N=3$, which models a 2D hole system, when the impurity scattering is isotropic. For angle-dependent scattering, with $N=3$, we find that the spin Hall conductivity is altered, so that its precise value, and even its sign, can depend on such details as the ratio of small-angle to large-angle scattering, and the energy dependences of the spin-orbit coupling and of the hole velocity. For the 2D hole systems that actually occur in GaAs, an accurate computation should also take into account warping of the Fermi surface due to tetragonal anisotropy, which has been omitted from our model.

Our results imply that spin-Hall conductivity for $N \neq 1$ is not universal. Quantization of σ_{sH} in units of $e/8\pi$ is broken by small-angle scattering processes even in the clean limit ($b_0\tau \gg 1$), when the spin-orbit field is strong. In contrast, charge Hall conductivity in the strong field limit is insensitive to the details of disorder scattering. This difference probably reflects the nontopological origin of spin Hall conductivity. This conclusion is indirectly supported by the result of numerical Laughlin gauge flux experiments performed in Ref. 38 for $N=1$.

In a recent paper, Liu and Lei[39] have performed numerical calculations of the spin Hall conductivity for a two-dimensional hole model, with scattering from impurities set back 50 nm from the layer. Their model has a quadratic energy spectrum, and $\mathbf{k}^3$ spin-orbit coupling, corresponding to our model with $N = \tilde{N} = 3$, $\zeta = 0$, and has densities varying from 10^{10} to 10^{12} cm^{-2}. (The densities in the published version of Ref. 39 were misstated by a factor of 100.) Their numerical results for σ_{sH} range from slightly smaller than the intrinsic value, at their lowest densities, to roughly twice the intrinsic value at high densities, in all cases quite different from the prediction of Eq. (62). For high densities, it appears that their model is in the regime where the spin-orbit splitting b/v_F is larger than the momentum transfer in a scattering event, so that our formulas would not apply. (The nu-

merical results actually coincide with a prediction for this regime by Khaetskii[41]) However, we do not have a complete understanding of the numerical results at lower densities.

Although we do not know the precise value to be expected for the spin Hall conductivity in the 2D hole samples studied by Wunderlich *et al.*,[3] it seems plausible that the magnitude should be similar to the universal intrinsic value for $N=3$. In a recent work, Nomura *et al.*[35] have analyzed the geometry of the experiment in Ref. 3 and conclude that the amount of polarized light obtained is consistent in magnitude with what might be expected to arise from the predicted intrinsic spin Hall conductivity. Nevertheless, more work seems necessary before one can be confident that one has a complete understanding of the origin of the the observed polarization in these experiments. As has been noted by various authors, spin polarization near a boundary depends on the boundary conditions, and may not be simply related to spin Hall currents away from the boundary. As an example, for the 2D *electron* system with pure Rashba coupling, where $\sigma_{sH}=0$, one may still find spin polarization in the z direction near the lateral boundaries of a sample if spin-flip processes are strong at the boundary,[36] or if electrons can cross the boundary into a contact or a region with different spin-orbit coupling.[16,37] In any case, it is expected that the polarization near a boundary will be insensitive to processes that occur further from the boundary than a few times the length scale for spin relaxation, and this length scale is quite short for holes in GaAs. It seems unlikely that one will be able to obtain a *quantitative* measure of the bulk spin Hall conductivity in such samples based on observations of spin accumulation near a boundary.

In an earlier theoretical work by two of the present authors,[40] a set of Boltzmann-like kinetic equations was derived for a matrix distribution, denoted $\hat{f}_{\mathbf{p}}(\mathbf{x},t)$, which contained spin information but was integrated over the energy ϵ. The model was restricted to the case of **k**-linear spin-orbit coupling, and impurities were not included. The focus of Ref. 40 was on the time evolution of distributions that are nonuniform in space, and the linear response to a time-dependent perturbation in the absence of impurity scattering. The formalism developed there cannot be directly applied to a spin-dependent transport in the presence of impurities, which is the focus of the present work.

ACKNOWLEDGMENTS

This work was supported in part by NSF Grants Nos. PHY 01-17795 and DMR 02-33773. The research of A.V.S. is supported by U.S. DOE under Contract No. DEAC 02-98 CH 10886. We are very thankful to A. Khaetskii, who derived independently spin-Hall conductivity for small angle scattering,[41] and discussions with whom helped to correct errors in the previous versions of our papers.[42,43] The authors are also grateful to S. Y. Liu for calling their attention to the existence of Ref. 39 and for discussions of the model used in that paper. Discussions with E. I. Rashba are also gratefully acknowledged.

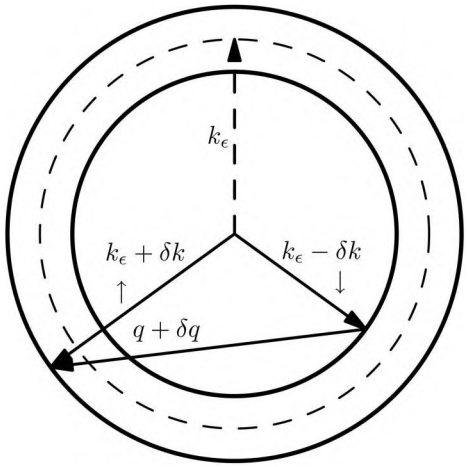

FIG. 2. Spin-orbit splitting splits the surface of a constant energy into two concentric circles with different radii $k_\epsilon \pm \delta k$, corresponding to a different spin orientation (relative to spin-orbit fields). As a result, momentum transferred in a scattering process $q+\delta q$ depends on the electron spin orientation.

APPENDIX A: DERIVATION OF SCATTERING KERNEL AND THE ELECTRIC FIELD TERM

1. Scattering kernel

We give here a derivation of the two terms $\mathbf{M}^d$ and $\mathbf{M}^w$ in the scattering kernel (31), using an approach based on Fermi's golden rule.

The term $\mathbf{M}^d$ arises from the spin-orbit corrections to the density of final states. Consider an unpolarized electron of energy ϵ, traveling in a direction θ', incident on an impurity located at a point $\mathbf{r}_0$. (The incident electron may be considered to be an incoherent 50-50 mixture of spin up and spin down states along any convenient axis of quantization.) Suppose the electron is scattered into a direction θ. At the angle θ, conservation of energy $\epsilon=\epsilon_k+\sigma b_k$ permits two possible outgoing momenta $k_\epsilon+\delta k$, with $\delta k=b(\theta)\sigma/2v_\epsilon$, where $\sigma=\pm 1$ denotes a spin parallel or antiparallel to $\mathbf{b}(\theta)$. (See Fig. 2.) As in the discussion following (10), the corresponding densities of states, per unit $\delta\theta$, are given by $(2\pi)^{-1}(k_\epsilon+\delta k)/(v_\epsilon+\delta v)$, and the difference between $\sigma=\pm 1$ is given by $\delta\tilde{\nu}=(2\pi)^{-1}(\partial/\partial\epsilon)(k_\epsilon b/v_\epsilon)$. If we ignore possible differences in matrix elements for the two spin states, the difference in transition rates is given by $W(q)\delta\tilde{\nu}/2$, where $W(q)$ is defined in (25). This leads to a net polarization of the scattered spin, giving a contribution to $\partial\mathbf{\Phi}(\theta)/\partial t$ equal to $\mathbf{M}^d(\theta,\theta')\delta n(\theta')$, where $\mathbf{M}^d$ is given by (32). Since the spin is conserved during the scattering event, we will also have a contribution to $\partial\mathbf{\Phi}(\theta')/\partial t$ equal to $-\mathbf{M}^d(\theta,\theta')\delta n(\theta')$.

The term $\mathbf{M}^w$ arises from spin-dependent momentum shifts in the matrix elements. The momentum transferred during the scattering process depends on the spin orientation of both the initial and final state. Letting $\sigma'=\pm 1$ distinguish between initial spin parallel or antiparallel to $\mathbf{b}(\theta')$, we see that the actual momentum transfer is $q+\delta q$, where q is given by Eq. (25) and

$$\delta q = \left(\tfrac{1}{4}\right)[b(\theta)\sigma + b(\theta')\sigma'](\partial q/\partial\epsilon) \tag{A1}$$

is due to spin-orbit interaction. As we are assuming that $b/E_F \ll 1$, and we only need the scattering kernel correct to linear order in b, we may consider the effects of $b(\theta)$ and $b(\theta')$ separately, and add the results at the end. Suppose that $b(\theta)=0$. Then $\delta q=[b(\theta')\sigma'/4](\partial q/\partial\epsilon)$. The change in the squared matrix element for the scattering process is given by $\delta W=(\partial W/\partial q)\delta q$, which leads to different scattering rates for $\sigma'=\pm 1$. The out-scattering process thus creates a net polarization of the electrons remaining at angle θ', and gives a contribution to $\partial\mathbf{\Phi}(\theta')/\partial t)$ equal to

$$-\frac{\mathbf{b}(\theta')}{4k_\epsilon v_\epsilon}\tan\left(\frac{\theta-\theta'}{2}\right)\frac{\partial K(\theta-\theta')}{\partial\theta}\delta n(\theta'). \tag{A2}$$

Since the spin is conserved in the scattering process, we must have a contribution to $\partial\mathbf{\Phi}(\theta)/\partial t$ which is the negative of (A2).

Next consider the case where $b(\theta')=0$. Now δq depends on σ, so the scattering rate depends on which of the two final states is involved. The difference in matrix elements leads to a contribution to the polarization at θ given by

$$\frac{\partial\mathbf{\Phi}(\theta)}{\partial t}=\frac{\mathbf{b}(\theta)}{4k_\epsilon v_\epsilon}\tan\left(\frac{\theta-\theta'}{2}\right)\frac{\partial K(\theta-\theta')}{\partial\theta}\delta n(\theta'). \tag{A3}$$

Again, since the spin is conserved in the scattering process, we must have a contribution to $\partial\mathbf{\Phi}(\theta')/\partial t$ which is the negative of (A3). The sum of these four contributions give the contributions proportional to $\mathbf{M}^w(\theta,\theta')\delta n(\theta')$ in Eq. (30).

Although we have used Fermi's golden rule and the Born approximation in deriving these results, we expect that Eqs. (32) and (33) should hold more generally, within the model given by the Hamiltonian (2), as long as the impurities are sufficiently dilute.

2. Electric field term, Eq. (29)

The direct effect of a uniform electric field term in the Hamiltonian, in an infinitesimal time interval δt, is to displace the momentum distribution by an amount $\delta\mathbf{k}=\mathbf{E}e\,\delta t$, while keeping fixed the orientations of the spins. To see the effect of this, it is convenient to integrate over energy and the magnitude of the momentum, and define a spin density at an angle θ by

$$\mathbf{m}(\theta)=\int_0^\infty\frac{k\,dk}{8\pi^2}\int d\epsilon[\boldsymbol{\sigma}_{\alpha\beta}n_{\beta\alpha}(\mathbf{k},\epsilon)], \tag{A4}$$

with $\mathbf{k}$ oriented in the direction θ. In thermal equilibrium, $\mathbf{m}$ is given by

$$\mathbf{m}^{\mathrm{eq}}(\theta)=\mathbf{b}(\theta)k_F/(8\pi^2 v_F), \tag{A5}$$

under the assumption that b and the temperature T are small compared to E_F.

The electric field term conserves the total spin, but gives rise to a flow of $\mathbf{m}$ around the unit circle at a velocity $\dot\theta=-Eek_F^{-1}\sin\theta$, for $\mathbf{E}$ along the x axis. Then, to lowest order in the electric field, we have

$$\left.\frac{\partial\mathbf{m}(\theta)}{\partial t}\right|_E=-\frac{\partial}{\partial\theta}(\dot\theta\mathbf{m}^{\mathrm{eq}})=\frac{Ee}{8\pi^2 v_F}\frac{\partial}{\partial\theta}(\mathbf{b}\sin\theta). \tag{A6}$$

By definition, $[\mathbf{m}(\theta)-\mathbf{m}^{\mathrm{eq}}(\theta)]=\int d\epsilon\,\mathbf{\Phi}(\theta,\epsilon)$. Moreover, it is clear that $\mathbf{\Phi}=0$ for energies far from the Fermi energy, and for any fixed value of θ, we must have $\partial\mathbf{\Phi}/\partial t|_E\propto-\partial f_0/\partial\epsilon$. Equation (29) then follows from (A6).

APPENDIX B: ALTERNATE DERIVATION USING GREEN'S FUNCTIONS

The kinetic equations (26)–(33) contains nondiagonal elements of the spin density matrix and is therefore a quantum kinetic equation, rather than a conventional Boltzmann equation, which contains only occupation numbers. In this section we show that this equation can be derived from the microscopic Hamiltonian (2) by Green's function methods. In our derivation, we shall mainly follow the route outlined in Ref. 44. As usual, we introduce one-particle retarded and advanced Green's function $\hat G_R$ and $\hat G_A$, and the Keldysh function $\hat G_K$ describing the nonequilibrium state of the system. (Note that all Green's functions are operators in spin space.) Treating the disorder potential as a perturbation, one may write the Dyson equation for these functions in a matrix notation:

$$(\hat G_0^{-1}-\hat{\underline{\Sigma}})\hat{\underline{G}}=1,\quad \hat{\underline{G}}=\begin{pmatrix}\hat G^R & \hat G^K\\ 0 & \hat G^A\end{pmatrix}. \tag{B1}$$

Here the lower bar denotes the matrix in Keldysh space, and $\hat G_0^{-1}=i\partial_t-\hat H$. In this work, we neglect localization effects, and consider only diagrams with noncrossing impurity lines. This gives the standard approximation for the self-energy part

$$\hat{\underline{\Sigma}}(\mathbf{x},t,\mathbf{x}',t')=\tilde W(\mathbf{x}-\mathbf{x}')\hat{\underline{G}}(\mathbf{x},t,\mathbf{x}',t'), \tag{B2}$$

where $\tilde W(\mathbf{x}-\mathbf{x}')=\langle V(\mathbf{x})V(\mathbf{x}')\rangle$ is the disorder correlation function.

For a time-independent disorder and in the absence of electron-electron and electron phonon interactions, the functions $\hat G_R$ and $\hat G_A$ are independent of the Keldysh function $\hat G_K$. In fact, they are Hermitian conjugate to each other:

$$\hat G_A(E,\mathbf{k})=\hat G_R^\dagger(E,\mathbf{k}), \tag{B3}$$

and it is sufficient to find only one of them. Since the functions $\hat G_R$ and $\hat G_A$ are independent of the nonequilibrium state, they depend only on $\mathbf{x}-\mathbf{x}'$ in a translationally invariant system. Going to the Fourier representation, one finds for $\hat G_R$:

$$\hat G_R(E,\mathbf{k})=\frac{1}{E-\epsilon_{\mathbf{k}}+\frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k})-\hat\Sigma_R(E,\mathbf{k})}, \tag{B4}$$

where ϵ_k is the hole dispersion without spin orbits, and

$$\hat{\Sigma}_R(E,\mathbf{k}) = \int \frac{d^2\mathbf{k}'}{(2\pi)^2} W(\mathbf{k}-\mathbf{k}')\hat{G}_R(E,\mathbf{k}'), \tag{B5}$$

where $W(\mathbf{k})$ is the Fourier transform of the disorder correlation function $\tilde{W}(\mathbf{x}-\mathbf{x}')$.

In general, the self-energy part (B5) is an operator in spin space. Its Hermitian part describing disorder-induced renormalization of the electron spectrum and spin-orbit coupling will be ignored in the following. The anti-Hermitian part of $\hat{\Sigma}_R(E,\mathbf{k})$ which we denote $-i/2\hat{\tau}$ describes the decay of one-particle state because of elastic scattering. It is convenient to project the function $G_R(\mathbf{k}',E)$ onto spin eigenstates:

$$\hat{G}_R(E,\mathbf{k}') = \sum_{\beta=\pm} \frac{1}{E - E_\beta(\mathbf{k}') + \dfrac{i}{2\tau_\beta(E,\mathbf{k}')}} \times \left(1 + \frac{\beta\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}')}{b(\mathbf{k}')}\right), \tag{B6}$$

where

$$E_\beta(\mathbf{k}) = \epsilon_\mathbf{k} - \frac{\beta}{2} b(\mathbf{k}) \tag{B7}$$

are corresponding eigenvalues, and $\tau_\beta(E,\mathbf{k})$ are lifetimes. Since the imaginary part of $\hat{G}_R(E,\mathbf{k}')$ is peaked near $E = E_\beta(\mathbf{k}')$, one can replace it by a delta-function to get the self-energy in the leading order in impurity concentration. Thus, we find

$$\frac{1}{\hat{\tau}(E,\mathbf{k})} = 2\pi \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \sum_\beta \left(1 + \frac{\beta\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}')}{b(\mathbf{k}')}\right) \times W(\mathbf{k}-\mathbf{k}')\delta(E - E_\beta(\mathbf{k}')). \tag{B8}$$

In this work, we restrict ourselves by the first order corrections in spin-orbit coupling. Also, we will be primarily interested in the decay time on the "mass shell:"

$$E = \epsilon_\mathbf{k} - \frac{\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k})}{2}. \tag{B9}$$

Expanding Eq. (B8) to the first order in $\mathbf{b}(\mathbf{k}')$, one finds, on the mass shell,

$$\frac{1}{\hat{\tau}_\mathbf{k}} = 2\pi \int \frac{d^2\mathbf{k}'}{(2\pi)^2} W(\mathbf{k}-\mathbf{k}')[\delta(\epsilon_\mathbf{k} - \epsilon_{\mathbf{k}'}) - \boldsymbol{\sigma}\cdot[\mathbf{b}(\mathbf{k}) - \mathbf{b}(\mathbf{k}')]\delta'(\epsilon_\mathbf{k} - \epsilon_{\mathbf{k}'})]. \tag{B10}$$

Thus, in general, scattering rates are different for spin majority and spin minority bands.

Having found the retarded function $\hat{G}_R(E,\mathbf{k})$, one can now rewrite the remaining equation for $\hat{G}_K$ to make it uniform in $\hat{G}_K$ (see Ref. 44 for details):

$$\hat{G}_R^{-1}\hat{G}_K - \hat{G}_K\hat{G}_A^{-1} = \hat{G}_R\hat{\Sigma}_K - \hat{\Sigma}_K\hat{G}_A, \tag{B11}$$

where

$$\hat{\Sigma}_K(\mathbf{x},\mathbf{x}',t,t') = \tilde{W}(\mathbf{x}-\mathbf{x}')\hat{G}_K(\mathbf{x},\mathbf{x}',t,t'), \tag{B12}$$

and operators $\hat{G}_R^{-1}$ and $\hat{G}_A^{-1}$ act on the left and the right argument of $\hat{G}_K$, respectively. We now apply Wigner transform:

$$\hat{G}^K(t,\mathbf{x};t',\mathbf{x}') = \int \frac{dE}{2\pi}\frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-iE(t-t')} \times \hat{g}_{\mathbf{k}E}\left(\frac{t+t'}{2}, \frac{\mathbf{x}+\mathbf{x}'}{2}\right), \tag{B13}$$

and the left-hand side of Eq. (B11) becomes

$$(\hat{G}_R^{-1}\hat{G}_K - \hat{G}_K\hat{G}_A^{-1}) = \frac{\partial\hat{g}_{\mathbf{k}E}}{\partial t} + \frac{1}{2}\left\{\frac{\mathbf{k}}{m} - \frac{1}{2}\frac{\partial(\mathbf{b}\cdot\boldsymbol{\sigma})}{\partial\mathbf{k}}, \boldsymbol{\nabla}\hat{g}_{\mathbf{k}E}\right\} - \frac{i}{2}[\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}), \hat{g}_{\mathbf{k}E}] + \frac{1}{2}\left\{\frac{1}{\hat{\tau}_\mathbf{k}}, \hat{g}_{\mathbf{k}E}\right\}, \tag{B14}$$

where the scattering rate operator $1/\hat{\tau}_\mathbf{k}$ is given by Eq. (B10). The right-hand side of Eq. (B11) takes the form

$$(\hat{G}_R\hat{\Sigma}_K - \hat{\Sigma}_K\hat{G}_A) = \int \frac{d^2\mathbf{k}'}{(2\pi)^2} W(\mathbf{k}-\mathbf{k}') \times [\hat{G}_R(E,\mathbf{k})\hat{g}_{\mathbf{k}'E}(\mathbf{x},t) - \hat{g}_{\mathbf{k}'E}(\mathbf{x},t)\hat{G}_A(E,\mathbf{k})] \tag{B15}$$

(here we assumed that $\mathbf{x}$ and t dependence of $\hat{g}_{\mathbf{k}E}$ is smooth on the scale of Fermi wavelength).

Calculating transport quantities, we are normally interested in Keldysh Green's function at coinciding temporal points, $t=t'$, which in the Wigner representation corresponds to energy-integrated functions. However, Eqs. (B11)–(B15) contain the Green's function for arbitrary energies. To express all quantities in terms of functions at $t=t'$, we use the following ansatz for $\hat{g}_{\mathbf{k}E}$:

$$\hat{g}_{\mathbf{k}E} = \hat{G}_R(E,\mathbf{k})\hat{h}_\mathbf{k} - \hat{h}_\mathbf{k}\hat{G}_A(E,\mathbf{k}), \tag{B16}$$

where the function $\hat{h}_\mathbf{k}$ depends only on momentum. Integrating Eq. (B16) over E, one finds

$$\int \frac{dE}{2\pi}\hat{g}_{\mathbf{k}E} = i\hat{h}_\mathbf{k}. \tag{B17}$$

Therefore, the function $\hat{h}_\mathbf{k}(\mathbf{x},t)$ can be used to compute transport quantities. In fact, the function $\hat{f}_\mathbf{k} = 1 - 2\hat{h}_\mathbf{k}$ can be interpreted as a distribution function, with diagonal elements of $\hat{f}_\mathbf{k}$ being the occupation numbers.

Now we can derive the equation for the function $\hat{h}_\mathbf{k}(\mathbf{x},t)$ by integrating left- and right-hand sides of quantum kinetic equations (B14) and (B15). Since the left-hand side depends on E only via $\hat{G}_K(E)$, the integration is trivial. The terms on the right-hand side [Eq. (B15)] contain E explicitly, and should be carefully integrated. Substituting the definition (B16) into the Eq. (B15), one gets four different terms. Two of them contain the product of two retarded (or two advanced) Green's functions, and vanish after integration over

E. The two remaining terms are anti-Hermitian conjugate to each other. One of them has the form

$$R_1 = \int \frac{dE}{2\pi} \hat{G}_R(E,\mathbf{k}) \hat{h}_{\mathbf{k}'} \hat{G}_A(E,\mathbf{k}'). \tag{B18}$$

Projecting Green's function onto eigenstates with the help of Eq. (B6) and integrating over E, one finds

$$R_1 = i \sum_{\beta,\beta'=\pm} \left(1 + \beta \frac{\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k})}{b(\mathbf{k})}\right) \times \frac{\hat{h}'_{\mathbf{k}}}{E_\beta(\mathbf{k}) - E_{\beta'}(\mathbf{k}') + \frac{i}{2}\left(\frac{1}{\tau_\beta(\mathbf{k})} + \frac{1}{\tau_{\beta'}(\mathbf{k}')}\right)} \times \left(1 + \beta' \frac{\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}')}{b(\mathbf{k}')}\right), \tag{B19}$$

where the spin-majority and spin-minority energy bands are given by Eq. (B7). Assuming the scattering rate $1/\tau_\sigma$ in the denominator to be small, we now rewrite this term using the Sokhotsky formula:

$$\frac{1}{x+i0} = \mathrm{P}\frac{1}{x} - i\pi\delta(x), \tag{B20}$$

where P denotes the principal value. Subtracting the Hermitian conjugate and expanding to the first order in spin-orbit field $\mathbf{b}(\mathbf{k})$, one finds, for the right-hand side of quantum kinetic equation

$$R_1 - R_1^\dagger = 2\pi \hat{h}_{\mathbf{k}'} \delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) - \pi\{\hat{h}_{\mathbf{k}'}, \boldsymbol{\sigma}\cdot[\mathbf{b}(\mathbf{k}) - \mathbf{b}(\mathbf{k}')]\} \times \delta'(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) - i\mathrm{P}\frac{[\hat{h}_{\mathbf{k}'}, \boldsymbol{\sigma}\cdot[\mathbf{b}(\mathbf{k}) + \mathbf{b}(\mathbf{k}')]]}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2}. \tag{B21}$$

The first term in Eq. (B21) is a conventional scattering term, while the second is the spin-orbit correction to the scattering rate. The third term is due to the renormalization of one-particle spectrum by the disorder, and should be ignored in the following, since we already neglected similar terms in the real part of self-energy. Now we can use Eqs. (B21) and (B14) to write the equation for $\hat{h}_{\mathbf{k}}$:

$$\frac{\partial \hat{h}_{\mathbf{k}}}{\partial t} + \frac{1}{2}\left\{\frac{\mathbf{k}}{m} - \frac{1}{2}\frac{\partial(\mathbf{b}\cdot\boldsymbol{\sigma})}{\partial \mathbf{k}}, \boldsymbol{\nabla}\hat{h}_{\mathbf{k}}\right\} - \frac{i}{2}[\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}), \hat{h}_{\mathbf{k}}] = 2\pi \int \frac{d^2\mathbf{k}'}{(2\pi)^2} W(\mathbf{k}-\mathbf{k}') \left((\hat{h}_{\mathbf{k}'} - \hat{h}_{\mathbf{k}})\delta(\epsilon'_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) - \frac{1}{2}\{\hat{h}_{\mathbf{k}'} - \hat{h}_{\mathbf{k}}, \boldsymbol{\sigma}\cdot[\mathbf{b}(\mathbf{k}) - \mathbf{b}(\mathbf{k}')]\}\delta'(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) \right) \tag{B22}$$

(we grouped together terms proportional to W).

To this point, we did not take into account the effects of the external electric field. To incorporate the electric field into the kinetic equation, consider a very smooth external scalar potential $U(\mathbf{r})$. Strictly speaking, this potential breaks translational invariance, and the calculations we have done so far become invalid. However, for the small field the potential $U(\mathbf{r})$ is smooth on the spin-orbit precession length $\hbar v_\epsilon / b$ and does not lead to transitions between spin subbands. Therefore, $U(\mathbf{r})$ can be considered as a local shift of the electron energy. The term $\hat{G}_R^{-1}\hat{G}_K - \hat{G}_K\hat{G}_A^{-1}$ now contains $U(\mathbf{r}) - U(\mathbf{r}')$, which becomes, after the Wigner transform, $\nabla_{\mathbf{r}} U \nabla_{\mathbf{k}} \hat{g}_{\mathbf{k}E}$. Thus, the electric field term in the kinetic equation takes the standard form. Rewriting kinetic equations in terms of function $\hat{f}_{\mathbf{k}} = 1 - 2\hat{h}_{\mathbf{k}}$, one finds

$$\frac{\partial \hat{f}_{\mathbf{k}}}{\partial t} + \frac{1}{2}\left\{\frac{\mathbf{k}}{m} - \frac{1}{2}\frac{\partial(\mathbf{b}\cdot\boldsymbol{\sigma})}{\partial \mathbf{k}}, \boldsymbol{\nabla}\hat{f}_{\mathbf{k}}\right\} - \frac{i}{2}[\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}), \hat{f}_{\mathbf{k}}] + e\mathbf{E}\frac{\partial \hat{f}_{\mathbf{k}}}{\partial \mathbf{k}} = 2\pi \int \frac{d^2\mathbf{k}'}{(2\pi)^2} W(\mathbf{k}-\mathbf{k}') \left((\hat{f}_{\mathbf{k}'} - \hat{f}_{\mathbf{k}})\delta(\epsilon'_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) - \frac{1}{2}\{\hat{f}_{\mathbf{k}'} - \hat{f}_{\mathbf{k}}, \boldsymbol{\sigma}\cdot[\mathbf{b}(\mathbf{k}) - \mathbf{b}(\mathbf{k}')]\}\delta'(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}) \right). \tag{B23}$$

An equivalent kinetic equation, written in the helicity basis, was used in Ref. 17 for the case of the winding number $N=1$, and was later extended for $N \neq 1$ in Ref. 41. An alternative derivation of Eq. (B23) is to start with the kinetic equation of Ref. 17, rewrite it in the original basis, and expand to the first order in the spin-orbit field.

In principle, the kinetic equation in the form (B23) allows one to compute spin transport in a 2D hole gas. Note, however, that in general, spin-orbit terms on the right-hand side mix states with different values of $\epsilon_{\mathbf{k}}$ because of the presence of the derivative of delta-function. Thus, one has to consider all electron states with different energies simultaneously. This mixing occurs because the quantity $\epsilon_{\mathbf{k}}$ is not conserved in the scattering process. Instead, the true electron energy

$$\epsilon = \epsilon_{\mathbf{k}} - \frac{\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k})}{2} \tag{B24}$$

is conserved. We are now going to use this integral of motion to separate the contribution of electrons with different energies, and make the energy conservation explicit. We shall make the change of variables, expressing the electron momentum in terms of energy ϵ and momentum direction θ. Note, however, that the electron energy ϵ contains a spin-dependent part, and therefore replacing momentum length by ϵ is not a simple algebraic transformation: one has to ensure the proper ordering of spin operators in the resulting expression. To figure out this operator order, consider the effect of spin-dependent gauge transformation

$$\Psi'(\mathbf{r}) = \hat{U}_\sigma \Psi(\mathbf{r}) = \exp[i(\mathbf{q}\cdot\mathbf{r})(\boldsymbol{\sigma}\cdot\mathbf{b})]\Psi(\mathbf{r}) \tag{B25}$$

on the one-particle density matrix $\hat{\rho}_{\mathbf{r},\mathbf{r}'} = \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}')$. In the first order in $\mathbf{b}$, one finds

$$\hat{\rho}'_{\mathbf{r},\mathbf{r}'} = \hat{U}^{\dagger}_{\sigma}\hat{\rho}_{\mathbf{r},\mathbf{r}'}\hat{U}_{\sigma} \approx \hat{\rho}_{\mathbf{r},\mathbf{r}} - \frac{i\mathbf{q}\cdot(\mathbf{r}+\mathbf{r}')}{2}[\hat{\rho}_{\mathbf{r},\mathbf{r}'},\boldsymbol{\sigma}\cdot\mathbf{b}] - \frac{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}{2}\{\hat{\rho}_{\mathbf{r},\mathbf{r}'},\boldsymbol{\sigma}\cdot\mathbf{b}\}. \quad \text{(B26)}$$

By doing Wigner transforms with respect to $\mathbf{r}$ and $\mathbf{r}'$, one arrives at

$$\hat{\rho}'(\mathbf{R},\mathbf{p}) = \hat{\rho}(\mathbf{R},\mathbf{p}) - i[\hat{\rho}(\mathbf{R},\mathbf{p}),(\mathbf{q}\cdot\mathbf{R})(\boldsymbol{\sigma}\cdot\mathbf{b})] + \{(\mathbf{q}\cdot\boldsymbol{\nabla}_{\mathbf{p}})\hat{\rho}(\mathbf{R},\mathbf{p}),(\boldsymbol{\sigma}\cdot\mathbf{b})\}. \quad \text{(B27)}$$

The second term in Eq. (B27) describes spin rotation due to transformation $\hat{U}_{\sigma}$, and the third term describes the spin-dependent momentum shift. Note that the spin operator in the last term acts symmetrically on $\hat{\rho}$. This symmetry is related to the antisymmetric action of momentum operator on $\Psi(\mathbf{r})$ and $\Psi^{\dagger}(\mathbf{r}')$ in $\rho_{\mathbf{r},\mathbf{r}'}$. One can undo the spin rotation described by the second term in Eq. (B27), and use the third term as a prescription of how the spin-dependent momentum shift should be performed:

$$\hat{\rho}[\mathbf{R},\mathbf{p}+\mathbf{q}(\boldsymbol{\sigma}\cdot\mathbf{b})] = \hat{\rho}(\mathbf{R},\mathbf{p}) + \tfrac{1}{2}\{(\mathbf{q}\cdot\boldsymbol{\nabla}_{\mathbf{p}})\hat{\rho}(\mathbf{R},\mathbf{p}),(\boldsymbol{\sigma}\cdot\mathbf{b})\}. \quad \text{(B28)}$$

Now we can replace the momentum in Eq. (B23) by

$$\mathbf{k} = k_{\epsilon}\hat{k} + \frac{\boldsymbol{\sigma}\cdot\mathbf{b}}{2v_{\epsilon}}\hat{k}, \quad \text{(B29)}$$

where $\hat{k}=(\cos\theta,\sin\theta)$ is the unit vector in the momentum direction, and k_{ϵ} and $v_{\epsilon}=\partial\epsilon_k/\partial k$ are defined in Sec. III. To rewrite the gradient term in this representation, one can simply replace the momentum according to Eq. (B29):

$$\left.\frac{\partial\hat{f}}{\partial t}\right|_{v} = -\frac{1}{2}\left\{v_{\epsilon}\hat{k} + \frac{\partial v_{\epsilon}}{\partial\epsilon}\frac{\boldsymbol{\sigma}\cdot\mathbf{b}}{2}\hat{k} - \frac{v_{\epsilon}\hat{k}}{2}\frac{\partial[\mathbf{b}(\mathbf{k})\cdot\boldsymbol{\sigma}]}{\partial\epsilon},\boldsymbol{\nabla}_{\mathbf{r}}\hat{f}\right\}. \quad \text{(B30)}$$

The spin-orbit rotation term already contains spin-orbit fields and should not be modified at all, because corrections to it would produce the terms of order of $(b/E_F)^2$ which are neglected in this derivation:

$$\left.\frac{\partial\hat{f}}{\partial t}\right|_{b} = -\frac{i}{2}[\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}),\hat{f}]. \quad \text{(B31)}$$

To find the electric field contribution, note that for small field it should be computed only for an equilibrium distribution. In the equilibrium, the occupation numbers $\hat{f}_{\mathbf{k}}$ depend only on the total energy. Thus,

$$\left.\frac{\partial\hat{f}}{\partial t}\right|_{E} = eE\frac{\partial f_0}{\partial\epsilon}\frac{\partial\epsilon}{\partial k_x} = eE\frac{\partial f_0}{\partial\epsilon}\left[v_{\epsilon}(k_{\epsilon})\cos\theta + \frac{\boldsymbol{\sigma}\cdot\mathbf{b}(k_{\epsilon})}{2}\times\frac{\partial v_{\epsilon}}{\partial\epsilon}\cos\theta - \frac{v_{\epsilon}}{2}\frac{\partial(\mathbf{b}\cdot\boldsymbol{\sigma})}{\partial\epsilon}\cos\theta + \frac{1}{2}\frac{\partial(\mathbf{b}\cdot\boldsymbol{\sigma})}{\partial\theta}\frac{\sin\theta}{k_{\epsilon}}\right]. \quad \text{(B32)}$$

[the second term here takes into account the momentum shift given by Eq. (B29)].

In the scattering term, the delta-function and its derivative combine into $\delta(\epsilon-\epsilon')$, due to the energy conservation. In zeroth order, the scattering contribution is

$$\left.\frac{\partial\hat{f}}{\partial t}\right|_{0} = \int\frac{k_{\epsilon}d\theta'}{2\pi v_{\epsilon}}W(\mathbf{k}-\mathbf{k}')[\hat{f}(\theta')-\hat{f}(\theta)], \quad \text{(B33)}$$

where both $\mathbf{k}$ and $\mathbf{k}'$ are taken on the surface $\epsilon_k=\epsilon$. Spin-orbit corrections to Eq. (B33) arise from either the volume element $d^2\mathbf{k}'$ or the matrix element $W(\mathbf{k}-\mathbf{k}')$. The matrix element contribution is due to spin-orbit-induced changes of $\mathbf{k}$ and $\mathbf{k}'$:

$$\left.\frac{\partial\hat{f}}{\partial t}\right|_{w} = -\int\frac{k_{\epsilon}d\theta'}{8\pi v_{\epsilon}^2}\times\left\{\boldsymbol{\sigma}\cdot\left(\frac{\partial W}{\partial k}\mathbf{b}(\mathbf{k}) + \frac{\partial W}{\partial k'}\mathbf{b}(\mathbf{k}')\right),\hat{f}(\epsilon,\theta')-\hat{f}(\epsilon,\theta)\right\}. \quad \text{(B34)}$$

Transforming this expression as explained in Appendix A, one recovers the scattering kernel $\mathbf{M}_w$.

The volume element is computed as follows:

$$d^2\mathbf{k} = k\,dk\,d\theta = \left(k_{\epsilon}+\frac{\boldsymbol{\sigma}\cdot\mathbf{b}}{2v_{\epsilon}}\right)\frac{dk}{d\epsilon}d\theta\,d\epsilon. \quad \text{(B35)}$$

The derivative $dk/d\epsilon$ should be computed to the first order in spin-orbit field, as explained in comments to Eq. (10), and the volume element becomes

$$d^2\mathbf{k} = d\epsilon\,d\theta\left(\frac{k_{\epsilon}}{v_{\epsilon}}+\frac{1}{2}\frac{\partial}{\partial\epsilon}\frac{k_{\epsilon}\mathbf{b}_{\epsilon}\cdot\boldsymbol{\sigma}}{v_{\epsilon}}\right) \quad \text{(B36)}$$

(as usual, its matrix part should be applied symmetrically). This gives additional contributions to the collision integral:

$$\left.\frac{\partial\hat{f}}{\partial t}\right|_{d} = -\int\frac{d\theta'}{4\pi}W(\mathbf{k}-\mathbf{k}')\times\left\{\frac{\partial}{\partial\epsilon}\frac{k_{\epsilon}\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}_{\epsilon})}{2v_{\epsilon}},\hat{f}(\epsilon,\theta')-\hat{f}(\epsilon,\theta)\right\}. \quad \text{(B37)}$$

The evolution of the density matrix ρ in $\epsilon-\theta$ representation is then given by

$$\frac{\partial h}{\partial t} = \left.\frac{\partial\hat{f}}{\partial t}\right|_{g} + \left.\frac{\partial\hat{f}}{\partial t}\right|_{b} + \left.\frac{\partial\hat{f}}{\partial t}\right|_{E} + \left.\frac{\partial\hat{f}}{\partial t}\right|_{0} + \left.\frac{\partial\hat{f}}{\partial t}\right|_{d} + \left.\frac{\partial\hat{f}}{\partial t}\right|_{w}, \quad \text{(B38)}$$

where all terms are given by Eqs. (B30)–(B34) and (B37).

To simplify the calculation of physical quantities, one can remove the spin dependence from the volume element. To do that, we include the volume element in the distribution function via the following transformation:

$$\hat{n}(\epsilon,\theta) = \frac{k_{\epsilon}}{v_{\epsilon}}\hat{f}(\epsilon,\theta) + \frac{1}{4}\left\{\hat{f}(\epsilon,\theta),\frac{\partial}{\partial\epsilon}\frac{k_{\epsilon}\boldsymbol{\sigma}\cdot\mathbf{b}(\mathbf{k}_{\epsilon})}{v_{\epsilon}}\right\}. \quad \text{(B39)}$$

One can see that in the first order in spin-orbit coupling

$$\tfrac{1}{2}\{\hat{f}(\epsilon,\theta),d^2\mathbf{k}\}=\hat{n}(\epsilon,\theta)d\epsilon\, d\theta. \tag{B40}$$

The function $\hat{n}(\epsilon,\theta)$ can be now identified with the distribution used throughout the paper, up to the factor $1/(2\pi)^2$. Substituting Eqs. (B39) and (11) into Eqs. (B32), (B34), and (B37), one can reproduce Eqs. (29), (33), and (32), respectively. Indeed, the spin-orbit correction to the electric field term is given by the sum of contributions from Eqs. (B32) and (B39):

$$\left.\frac{\partial\hat{n}}{\partial t}\right|_{E.so}=-eE\frac{\partial f_0}{\partial\epsilon}\frac{\boldsymbol{\sigma}}{2}\cdot\left\{v_\epsilon\cos\theta\frac{\partial}{\partial\epsilon}\frac{k_\epsilon\mathbf{b}_\epsilon}{v_\epsilon}+\frac{k_\epsilon\mathbf{b}_\epsilon}{v_\epsilon}\cos\theta\frac{\partial v_\epsilon}{\partial\epsilon}-k_\epsilon\cos\theta\frac{\partial\mathbf{b}_\epsilon}{\partial\epsilon}+\frac{\sin\theta}{v_\epsilon}\frac{\partial\mathbf{b}}{\partial\theta}\right\}. \tag{B41}$$

Collecting the terms proportional to $\cos\theta$, one finds

$$\left.\frac{\partial\hat{n}}{\partial t}\right|_{E.so}=-eE\frac{\partial f_0}{\partial\epsilon}\frac{\boldsymbol{\sigma}}{2}\left\{\cos\theta\frac{\mathbf{b}_\epsilon}{v_\epsilon}+\frac{\sin\theta}{v_\epsilon}\frac{\partial\mathbf{b}}{\partial\theta}\right\}, \tag{B42}$$

thus reproducing Eq. (29). The matrix element correction to the collision integral (B34) reproduces Eq. (33). The density of states correction (B37), however, is modified by a similar contribution that arises from the transformation of the collision integral (B33). In the latter, one has to express $\hat{f}(\epsilon,\theta')$ in terms of $\hat{n}(\epsilon,\theta)$ and *then* transform the resulting contribution according to Eq. (B39) to get $\partial\hat{n}/\partial t$. As a result, four spin-orbit terms arise. Then, the term in Eq. (B37) containing $\hat{n}(\theta')$ is canceled by the corresponding contribution to Eq. (B33). A similar cancellation occurs for the $\hat{n}(\theta)$ term. Thus, one recovers Eq. (32). This concludes the derivation of the kinetic equation (26) by Green's function method.

[1] M. I. Dyakonov and V. I. Perel, Phys. Lett. **35A**, 459 (1971).
[2] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Science **306**, 1910 (2004).
[3] J. Wunderlich, B. Kaestner, J. Sinova, and T. Jungwirth, Phys. Rev. Lett. **94**, 047204 (2005).
[4] V. Sih, R. C. Meyers, Y. R. Kato, W. H. Lau, A. C. Gossard, and D. D. Awschalom, Nature (London) **1**, 31 (2005).
[5] J. E. Hirsch, Phys. Rev. Lett. **83**, 1834 (1999).
[6] S. Zhang, Phys. Rev. Lett. **85**, 393 (2000).
[7] S. Murakami, N. Nagaosa, and S. C. Zhang, Science **301**, 1348 (2003).
[8] J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett. **92**, 126603 (2004).
[9] H. A. Engel, B. I. Halperin, and E. I. Rashba, Phys. Rev. Lett. **95**, 166605 (2005).
[10] W.-K. Tse and S. Das Sarma, cond-mat/0507149 (unpublished).
[11] B. A. Bernevig and S. C. Zhang, cond-mat/0412550 (unpublished).
[12] E. I. Rashba, Phys. Rev. B **68**, 241315(R) (2003).
[13] J. Schliemann and D. Loss, Phys. Rev. B **69**, 165315 (2004).
[14] A. A. Burkov and L. Balents, Phys. Rev. B **69**, 245312 (2004).
[15] J. I. Inoue, G. E. W. Bauer, and L. W. Molenkamp, Phys. Rev. B **70**, 041303(R) (2004).
[16] E. G. Mishchenko, A. V. Shytov, and B. I. Halperin, Phys. Rev. Lett. **93**, 226602 (2004).
[17] A. Khaetskii, cond-mat/0408136 (unpublished).
[18] R. Raimondi and P. Schwab, Phys. Rev. B **71**, 033311 (2005).
[19] E. I. Rashba, cond-mat/0404723 (unpublished).
[20] N. Sugimoto, S. Onoda, S. Murakami, and N. Nagaosa, cond-mat/0503475 (unpublished).
[21] K. Nomura, J. Sinova, T. Jungwirth, Q. Niu, and A. H. MacDonald, Phys. Rev. B **71**, 041304(R) (2005).
[22] K. Nomura, J. Sinova, N. A. Sinitsyn, and A. H. MacDonald, Phys. Rev. B **72**, 165316 (2005).
[23] O. V. Dimitrova, Phys. Rev. B **71**, 245327 (2005).
[24] F. T. Vas'ko, JETP Lett. **30**, 540 (1979); Yu. A. Bychkov and E. I. Rashba, J. Phys. C **17**, 6039 (1984).
[25] O. Chalaev and D. Loss, Phys. Rev. B **71**, 245318 (2005).
[26] A. G. Mal'shukov and K. A. Chao, Phys. Rev. B **71**, 121308(R) (2005).
[27] R. Winkler, Phys. Rev. B **62**, 4245 (2000).
[28] J. Schliemann and D. Loss, Phys. Rev. B **71**, 085308 (2005).
[29] S. Murakami, Phys. Rev. B **69**, 241202(R) (2004).
[30] B. A. Bernevig and S. C. Zhang, Phys. Rev. Lett. **95**, 016801 (2005).
[31] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, NJ, 1963).
[32] B. K. Nikolić, L. P. Zârbo, and S. Souma, Phys. Rev. B **72**, 075361 (2005).
[33] E. M. Hankiewicz, L. W. Molenkamp, T. Jungwirth, and J. Sinova, Phys. Rev. B **70**, 241301(R) (2004).
[34] B. K. Nikolic, S. Souma, L. P. Zarbo, and J. Sinova, Phys. Rev. Lett. **95**, 046601 (2005).
[35] K. Nomura, J. Wunderlich, J. Sinova, B. Kaestner, A. H. MacDonald, and T. Jungwirth, Phys. Rev. B **72**, 245330 (2005).
[36] E. I. Rashba, cond-mat/0507007 (unpublished).
[37] I. Adagideli and G. E. W. Bauer, Phys. Rev. Lett. **95**, 256602 (2005).
[38] D. N. Sheng, L. Sheng, Z. Y. Weng, and F. D. M. Haldane, Phys. Rev. B **72**, 153307 (2005).
[39] S. Y. Liu and X. L. Lei, Phys. Rev. B **72**, 155314 (2005); **72**, 249901(E) (2005).
[40] E. G. Mishchenko and B. I. Halperin, Phys. Rev. B **68**, 045317 (2003).
[41] A. Khaetskii, cond-mat/0510815 v2 (unpublished).
[42] A. Khaetskii, cond-mat/0510815 v1 (unpublished).
[43] A. V. Shytov, E. G. Mishchenko, H. A. Engel, and B. I. Halperin, cond-mat/0509702 (unpublished).
[44] J. Rammer and H. Smith, Rev. Mod. Phys. **58**, 323 (1986).