Mutual Exclusion Statistics between Quasiparticles in the Fractional Quantum Hall Effect

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In this paper we propose a new assignment for mutual exclusion statistics between quasielectrons and quasiholes in the fractional quantum Hall effect. In addition to providing numerical evidence for this assignment, we show that the physical origin of this mutual statistics is a novel hard-core constraint due to correlation between the distinguishable vortexlike quasiparticles. [S0031-9007(96)01332-4]

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A two-dimensional electron system in a strong magnetic field exhibits the fractional quantum Hall effect (FQHE) [1,2] at certain "magic" fillings, say $\nu = 1/m$ (m an odd integer). One fascinating aspect of this effect is that quasiparticles are predicted to have exotic quantum numbers. For example, a quasihold (QH) or a quasielectron (QE) in these systems (called the Laughline $1/m$ fluids) has only a fraction, $\pm 1/m$, of the electron charge [3]. If two QH's (or QE's) are exchanged, their wave function acquires a fractional phase $\pm\pi/m$ [4,5].

Recently it has further been argued that these quasiparticles exhibit exotic quantum statistical [6] and thermodynamic [7] behavior, due to a novel rule for state counting: The total number of states with N_- QH's and N_+ QE's is given by

$$W = \binom{D_+ + N_+ - 1}{N_+}\binom{D_- + N_- - 1}{N_-}, \quad (1)$$

together with ($i, j = -$ for QH, $+$ for QE)

$$D_i = \frac{N_\phi}{m} - g_{ii}(N_i - 1) - \sum_{j\neq 1} g_{ij}N_j\,. \quad (2)$$

Here N_ϕ is the external magnetic flux in units of h/e, and N_ϕ/m is the Landau degeneracy for the quasiparticles due to their fractional charge. Though the formula (1) looks formally the same as that for bosons, the rule (2) is unusual, because the number D_i of available single-particle states for species i can be, as first suggested by Haldane [6], linearly dependent on the particle numbers. The coefficients g_{ij} describe statistical exclusion between particles in occupying single-particle states and, therefore, are called exclusion statistics. If $g_{ii} = 0$ or 1 and $g_{ij} = 0$ ($i \neq j$), the particle is a boson or fermion. The diagonal statistics g_{ii} in the Laughlin $1/m$ fluid has previously been determined [6,8–11] to be

$$g_{--} = 1/m, \qquad g_{++} = 2 - 1/m\,. \quad (3)$$

That g_{++} is $2 - 1/m$ rather than $-1/m$ is due to hard-core constraint between QE's [8,9]. A distinct feature of the new rule (2) is that it naturally allows for the possibility of *mutual statistical exclusion* ($g_{ij} \neq 0$) between *different* species ($i \neq j$). In this Letter we propose, and present numerical evidence to support, that indeed this happens to the FQHE quasiparticles with off-diagonal

$$g_{-+} = -(2 - 1/m), \qquad g_{+-} = 2 - 1/m\,, \quad (4)$$

and show that this is due to a hard-core constraint between QE and QH. Note the mutual exclusion statistics in FQHE are *antisymmetric*, in contrast to mutual exchange statistics which is always symmetric [12].

Let us first recall the arguments for the fractional diagonal statistics (3). For a Laughlin liquid, the electron number N_e and quasiparticle numbers N_- and N_+ satisfy

$$N_\phi = m(N_e - 1) + N_- - N_+\,. \quad (5)$$

If the quasiparticles, viewed as vortices [13] in the Laughlin liquid, are uncorrelated, the Hilbert-space dimension for a single vortex is determined by the number of fluid particles: $D_\pm = N_e + 1$. Eliminating N_e with the help of Eq. (5), one obtains an expression [14] of $D_\mp$ of the form of Eq. (2), which gives the statistics parameters [6]

$$g^0_{--} = -g^0_{++} = g^0_{+-} = -g^0_{-+} = 1/m\,. \quad (6)$$

This argument needs to be refined, if the vortices are correlated to each other; in such cases, one has instead

$$\begin{aligned} D_+ &= N_e + 1 - \alpha_{++}(N_+ - 1) - \alpha_{+-}N_-\,, \\ D_- &= N_e + 1 - \alpha_{--}(N_- - 1) - \alpha_{-+}N_+\,, \end{aligned} \quad (7)$$

with a linear dependence of D_i on the excitation numbers N_j. This will modify the statistics matrix to

$$g_{ij} = g^0_{ij} + \alpha_{ij}\,. \quad (8)$$

Previously, it has been shown [8–11] that due to a hard-core constraint for QE's (not for QH's), one should assign

$$\alpha_{++} = 2, \qquad \alpha_{--} = 0\,. \quad (9)$$

Thus, g_{++} should be modified to that given by Eq. (3).

Recently we have numerically determined the off-diagonal $\alpha_{\pm\mp}$ for small systems of interacting electrons as a sphere, with a magnetic monopole at its center [15] providing a total flux $N_\phi = 2S$.

The simplest case with coexisting QE and QH is the magnetic roton band with $N_- = N_+ = 1$, just above the $\nu = 1/m$ ground state. In Fig. 1, we present the energy spectrum of $N_e = 6$ electrons with $2S = 15$, which according to Eq. (5) corresponds to $m = 3$ and $N_- = N_+$.

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The unique ground state (with $N_- = N_+ = 0$) is seen well separated by a gap from the excited states. The low-lying excited states above it form a visible second band and are thought of as containing a pair of QE and QH. To identify states of such quasiparticle composition [16], one first considers the subspace spanned by wave functions of the form $S^+(\alpha_0, \beta_0)S^-(\alpha_1, \beta_1)\Psi_m$, where

$$\Psi_m = \prod_{i<j}(u_i v_j - v_i u_j)^m \qquad (10)$$

is the Laughlin wave function on the sphere, (u, v) and (α, β) are the spinor variables describing electron and quasiparticle coordinates, and the operators

$$S^+(\alpha, \beta) = \prod_{j=1}^{N}\left(\beta^* \frac{\partial}{\partial u_j} - \alpha^* \frac{\partial}{\partial v_j}\right), \qquad S^-(\alpha, \beta) = \prod_{j=1}^{N}(\beta u_j - \alpha v_j), \qquad (11)$$

are, respectively, the QE and QH creation operators. By diagonalizing the Hamiltonian in this subspace and by inspecting both angular momentum and energy, one is tempted to identify the states of a QE-QH pair to be those with a long bar in the exact spectrum of Fig. 1. Notice that the $L = 0$ state in this subspace actually is the ground state, so it should not be counted as a true QE-QH pair state. Moreover, the lowest states with $L = 1$ obtained this way are clearly in the continuum above the well-separated second band, suggesting they do not really belong to the true subspace of one QE-QH pair. Thus, one should exclude these four states (with $L = 0, 1$) from the QE-QH pair subspace; the remaining states stay between two dotted lines in Fig. 1. If we change N_e, while keeping $\nu = 1/3$ fixed, numerical data always show the missing of four states in the magnetic roton band. This is an indication of mutual exclusion between QE and QH. Indeed, the number of states in the QE-QH subspace is $(N_e + 1 - \alpha_{+-})(N_e + 1 - \alpha_{-+})$. With $\alpha_{+-} = \alpha_{-+} = 0$, there should be $(N_e + 1)^2$ states in the second band. Four states missing implies

$$\alpha_{+-} = -\alpha_{-+} = \pm 2 . \qquad (12)$$

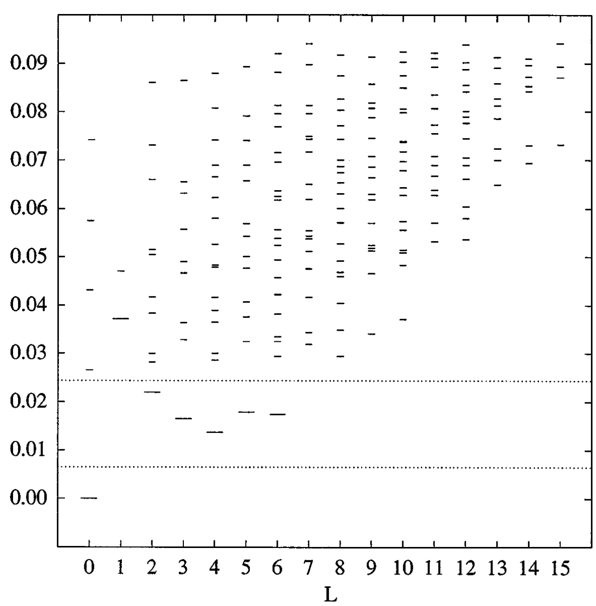

FIG. 1. Energy spectrum of six electrons at $\nu = 1/3$. The long bars represent the exact states whose angular momentum and energy are compatible with states in the uncorrelated QE-QH scheme. However, the states which are beyond the dotted lines actually do not belong to the real QE-QH subspace.

To resolve the sign ambiguity, we study larger systems. We choose $N_e = 6$ as before, but add one or two extra flux quanta (i.e., $2S = 16$ or 17). Hereafter the two systems will be referred to as $\frac{1}{3}+$ and $\frac{1}{3}++$, respectively. Their energy spectra are shown in Figs. 2(a) and 2(b). According to Eq. (5), the systems have, respectively, $N_- = N_+ + 1$ and $N_- = N_+ + 2$, so the minimum number of QH's is 1 and 2, respectively. In Fig. 2(a), the lowest energy band consists of a single multiple with $L = 3$, corresponding to states with a single QH; in Fig. 2(b), four L multiplets of lowest energies form the lowest band, corresponding to states with 2 QH's. The state counting for these bands agrees with the statistics $\alpha-- = 0$ or $g_{--} = 1/3$. In order to study mutual statistics, one needs to examine higher-energy states which contain one more QE and QH. To properly identify these states, we invoke microscopic Laughlin wave functions with appropriate quasiparticle composition, i.e., to consider the subspace spanned by $S^+(\alpha_0, \beta_0)S^-(\alpha_1, \beta_1)S^-(\alpha_2, \beta_2)\Psi_m$ for the $\frac{1}{3}+$ system and $S^+(\alpha_0, \beta_0)S^-(\alpha_1, \beta_1)S^-(\alpha_3, \beta_3)\Psi_m$ for the $\frac{1}{3}++$ system. We first project the quasiparticle-coordinate dependence down to the "lowest Landau level (LLL)" (with electrons as sources of quantized flux for the quasiparticles), resulting in a basis of many-electron wave functions in this subspace [17]. Then we diagonalize the Hamiltonian in this basis for the two systems, respectively, and calculate the overlaps of states thus obtained with the corresponding exact states, whose energies are marked again by long bars in Fig. 2.

Let us first examine the $\frac{1}{3}+$ system more carefully. The above construction amounts to having $\alpha_{+-} = \alpha_{+-} = 0$, and gives 196 states in 20 multiplets, corresponding to those with a long bar in Fig. 2(a). In Table I, we see that 16 multiplets of them have fairly large overlaps with the exact eigenstates, while 4 multiplets (with $L = 1, 2, 3, 4$) with higher energies have small overlaps. The latter four multiplets, together with the lowest-energy states at $L = 3$ that actually correspond to a single QH, are expected not to belong to the true ($N_- = 2, N_+ = 1$) subspace. In total 31 states should be excluded: We are left with only 165 states, exactly what is predicted by the formulas (1) and (7), with $\alpha_{--} = 0$ and

$$\alpha_{+-} = -2, \qquad \alpha_{-+} = 2 . \qquad (13)$$

For the $\frac{1}{3}++$ system, based on the same procedure and reasoning as in the last two paragraphs, we have identified a total of 133 exact states, marked by a long bar outside

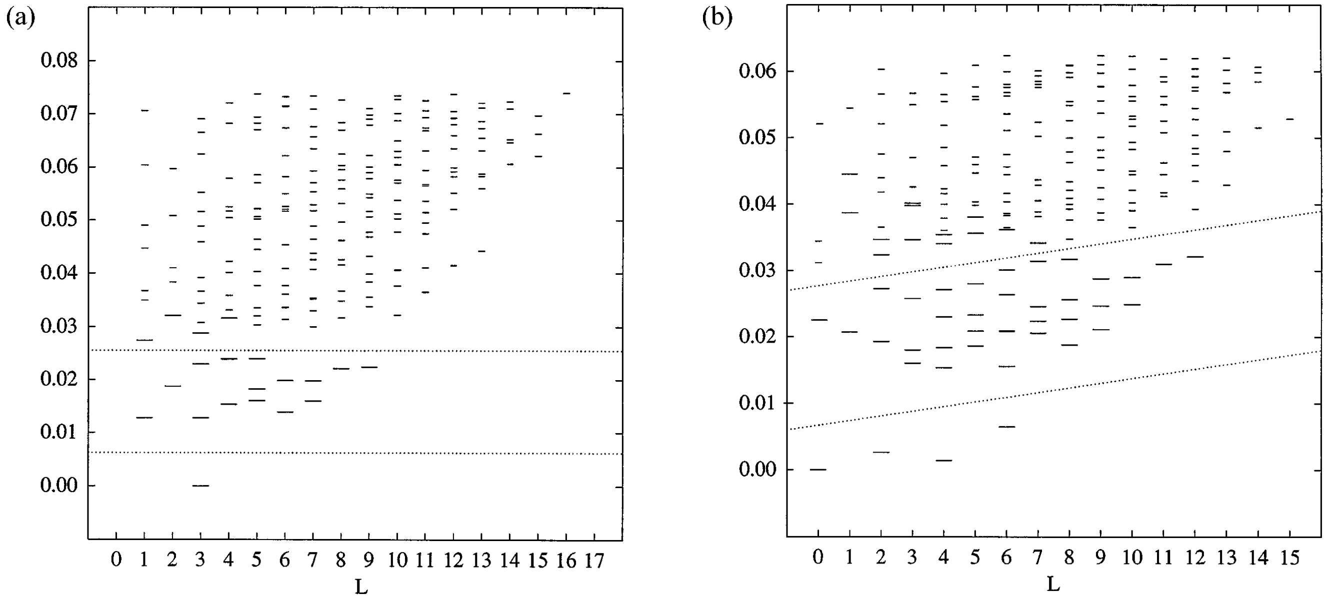

FIG. 2. Same as Fig. 1 for (a) the $\frac{1}{3}+$ and (b) the $\frac{1}{3}++$ system.

the two dashed lines in Fig. 2(b), which all have small overlaps with the Laughlin quasiparticle wave functions corresponding to $N_- = 3$ and $N_+ = 1$. They should not belong to the true $(N_- = 3, N_+ = 1)$ subspace, while those with a long bar between the two dashed lines have large wave function overlaps and thus do belong to it. The total number of states of the latter is $588 - 133 = 455$, again exactly what is predicted by the formulas (1) and (7), with $\alpha_{--} = 0$ and $\alpha_{+-} = -\alpha_{-+} = -2$.

Thus, as far as state counting is concerned, the off-diagonal parameters (13) are verified by our numerical data. From Eqs. (6) and (8), we have derived the mutual statistics parameters (4).

To understand the origin of this statistical exclusion between QE and QH, we need a highly nontrivial mechanism as it increases the Hilbert-space dimension for a single QE as more QH's are added, while it decreases the Hilbert-space dimension for a single QH as more QE's are added. It is amusing to see that a simple hard-core constraint between QE and QH can achieve just that.

Recall that in the subspace having one QE and several QH's, the wave functions that describe statistically independent QE and QH's are those obtained by applying $S^+(\alpha_0, \beta_0)$ and $S^-(\alpha_i, \beta_i)$ to the Laughlin ground state Ψ_m. To construct a many-electron basis (not necessarily orthogonal) in this subspace, one may integrate over quasiparticle coordinates as follows [10]:

$$\int d\Omega(\alpha_0, \beta_0)\phi^{+}_{2S_+,k_0}(\alpha_0, \beta_0)S^+(\alpha_0, \beta_0)$$

$$\int \prod_q d\Omega(\alpha_q, \beta_q)\phi^{-}_{2S_-,k_q}(\alpha_q, \beta_q)S^-(\alpha_q, \beta_q)\Psi_m\,, \quad (14)$$

where $(\alpha, \beta) = (\cos(\theta/2)e^{i\phi/2}, \sin(\theta/2)e^{-i\phi/2})$, and $d\Omega(\alpha, \beta) = \sin\theta d\theta d\phi$; $\phi^{+}_{2S_+,k}(\alpha, \beta)$ and $\phi^{-}_{2S_-,k}(\alpha, \beta)$ are the single-quasiparticle wave function in the LLL with

TABLE I. Overlaps between the exact states, with energy increasing from top to bottom, and the corresponding states in the uncorrelated QE-QH scheme for (a) the $\frac{1}{3}+$ and (b) the $\frac{1}{3}++$ system.

$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$	$L=7$	$L=8$	$L=9$
0.9963	0.9935	0.9949	0.9885	0.9956	0.9898	0.9813	0.9848	0.9603
0.1705	0.0542	0.9932	0.9288	0.9759	0.9830	0.9902	···	···
···	···	0.9469	0.0915	0.9416	···	···	···	···
···	···	0.1450	···	···	···	···	···	···

$L=0$	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$	$L=7$	$L=8$	$L=9$	$L=10$	$L=11$	$L=12$
0.9975	0.9999	0.9961	0.9988	0.9985	0.9955	0.9959	0.9899	0.9896	0.9847	0.9724	0.9814	0.9565
0.9783	0.9955	0.9945	0.9982	0.9951	0.9924	0.9957	0.9948	0.9869	0.9846	0.9838	···	···
···	0.9891	0.9106	0.9889	0.9953	0.9947	0.9291	0.9782	0.9706	0.9823	···	···	···
···	···	0.1698	0.7001	0.9842	0.9637	0.9438	0.9248	0.9295	···	···	···	···
···	···	0.1217	0.9093	0.9286	0.4448	0.9462	0.0862	···	···	···	···	···
···	···	···	0.3458	0.1773	0.0569	0.9384	···	···	···	···	···	···
···	···	···	···	0.1018	···	0.0644	···	···	···	···	···	···

total flux $2S_+$ and $-2S_-$, respectively, with k an integer between 1 and $2S_\pm + 1$ labeling the LLL states. One obtains explicitly

$$\delta_{2S_+,N_e}\delta_{2S_-,N_e} g_{k_0}^+ \prod_q g_{k_q}^- \Psi_m\,, \tag{15}$$

where

$$g_k^+ = (-1)^{k-1}[N_e!/(N_e - k + 1)!\,(k-1)!]^{-1/2} \times \sum_{1\le l_1<l_2\cdots<l_{N_e-k+1}\le N_e} \frac{\partial}{\partial v_{l_1}}\frac{\partial}{\partial v_{l_2}}\cdots\frac{\partial}{\partial v_{l_{N_e-k+1}}} \times \prod_{l(\neq l_i)} \frac{\partial}{\partial u_l}\,, \tag{16}$$

$$g_k^- = [N_e!/(N_e - k + 1)!\,(k-1)!]^{-1/2} \times \sum_{1\le l_1<l_2\cdots<l_{k-1}\le N_e} v_{l_1}v_{l_2}\cdots v_{l_{k-1}} \prod_{l(\neq l_i)} u_l\,. \tag{17}$$

The number of these basis functions gives the dimension of the subspace in agreement with Eq. (6).

In the integral (14), a QE can be on top of a QH. This can be avoided, in the spirit of Ref. [8], by inserting a hard-core Jastrow factor between QE and QH [9],

$$\prod_q (\alpha_0^*\beta_q^* - \alpha_q^*\beta_0^*)^l, \tag{18}$$

with l being a positive integer, into the integrand. Expanding this Jastrow factor, one can carry out the integration over quasiparticle coordinates and obtain the following basis functions:

$$\delta_{2S_+,N_e+lN_-}\delta_{2S_-,N_e-l}(-1)^{\sum k_q}\frac{\prod C_{2S_-,k_q}^-}{C_{2S_+,k_0}^+} \times \sum_{m_1,m_2,\ldots,m_n}\prod_q \binom{l}{m_q}(-1)^{\sum m_q} g_{k_0-\sum m_q}^+ \times \prod_q \frac{g_{k_g+m_q}^-}{(C_{N_e,k_q+m_q}^+)^2}\,\Psi_m\,, \tag{19}$$

where

$$C_{N,k}^+ = (-1)^{N-k+1}C_{N,k}^- = \left[\frac{N+1}{4\pi}\,\frac{N!}{(N-k+1)!\,(k-1)!}\right]^{1/2}.$$

These new basis functions, which are linear superpositions of the old basis functions $g_{k_0}^+\prod_q g_{k_q}^-\Psi_m$, are nonvanishing only when

$$2S_+ = N_e + lN_-, \qquad 2S_- = N_e - l\,. \tag{20}$$

Here $D_\pm = 2S_\pm + 1$ gives the degeneracy for species i. If we set $l = 2$, the dimension of the subspace spanned by these basis functions is precisely that required by $\alpha_{+-} = -\alpha_{-+} = -2$. This shows that the physics behind the mutual statistics (4) is the hard-core constraint between QE and QH, i.e., the insertion of $\prod_q(\alpha_0^*\beta_q^* - \alpha_q^*\beta_0^*)^2$. [The conjugate $\prod_q(\alpha_0\beta_q - \alpha_q\beta_0)^2$ would not work.]

In fact, we have found that the hard-core modified wave functions provide a very good description of the exact states between the two dashed lines in Fig. 2. (The details will be published elsewhere.) This strongly supports the correctness of our hard-core constraint and, as a result, that of the mutual statistics given in Eq. (13) or Eq. (4). Implications of the mutual statistics on the thermodynamic properties of FQHE quasiparticles will be published [18].

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