Soluble Ising model in $2+1/N$ dimensions and XY model in $1+1/N$ dimensions

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The XY model is exactly soluble on a comblike structure, by means of a modified Jordan-Wigner transformation. This method is used to reduce the transfer matrix of the Ising model, on planes which are joined to each other at one edge, to a soluble canonical form.

We outline an exactly soluble $(2+1/N)$-dimensional Ising model.[1] As the details are rather involved, we illustrate by deriving the solution of a similarly soluble XY model in $1+1/N$ dimensions, which may also serve as a possible test of conjectures on the two-dimensional XY model.[2]

In the Ising model, consider N_2 planes of $N_1 \times N_1$ spins each, connected by *intra*planar nearest-neighbor bonds $-J_1$, with *inter*planar bonds $-J_2$ connecting one edge of each plane to the similar edge of its nearest neighbors. The topology is that of a book: the planes are the pages, held together at the spine (i.e., the binding), as shown in Fig. 1. Spins at the spine have five neighbors each, whereas the others have only four. The neighborhood of the spine is therefore closer to three-dimensional than the rest, and a study of the thermodynamic functions which relate to this neighborhood may teach us something concerning the transition from two to three dimensions.

By applying periodic boundary conditions on the vertical direction, one can derive the partition function as the largest eigenvalue of a transfer matrix[3] on the lattice illustrated in Fig. 2. With J_2 set to zero, the individual Ising planes are reduced to the vertical lines in this figure, and the transfer matrix on each can be brought into canonical form[4] by means of the Jordan-Wigner transformation from spin operators to fermion operators, thence to be readily diagonalized.[5] For $J_2 \neq 0$ the strictly linear enumeration required in this procedure is no longer feasible, but we have found an alternative. The Jordan-Wigner transformation is written as follows:

$$S_n^+ = c_n^* \exp \pi i \sum_{m>n} c_m^* c_m \ , \tag{1}$$

with the direction of the enumeration indicated by arrows in Fig. 2. S_n^- is given by the Hermitian conjugate of the above, and $S_n^z = c_n^* c_n - \frac{1}{2}$. Two operators c_n and c_m that are connected by an arrow *anticom-*

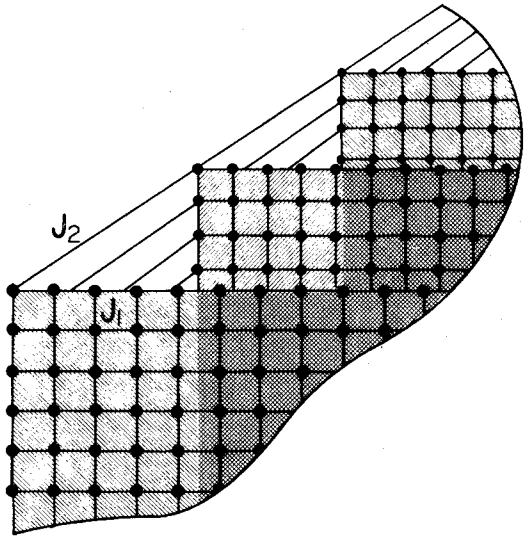

FIG. 1. Slightly more than two-dimensional lattice. Intraplanar bonds are $-J_1$, interplanar are $-J_2$, between nearest-neighbor spins.

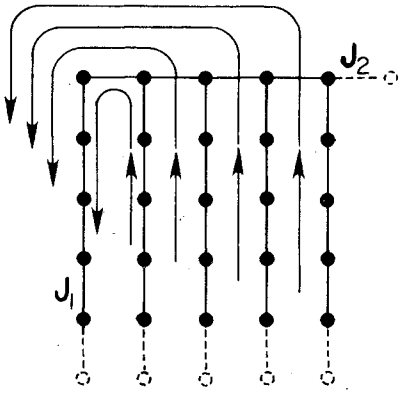

FIG. 2. Comblike structure of a typical transfer-matrix plane. Arrows indicate the direction of Jordan-Wigner enumerations which lead to a canonical form (see Ref. 4).

mute, although two such operators lying on distinct arrows *commute*. Although the set of operators (c and c^*) is complete it is not a totally anticommuting set. For the geometry of Fig. 2 and the particular enumeration shown by the arrows this lack of total anticommutativity does not matter, for spins which are connected by interplanar bonds $-J_2$ all lie on a common arrow. We now discuss the transfer matrix. The new factor, associated with the $-J_2$ bonds, is

$$V(J_2) = \exp\beta J_2 \sum_{(ij)} S_i^x S_j^x$$
$$= \exp\beta J_2 \sum_{(ij)} (c_i - c_i^*)(c_j + c_j^*) \quad , \tag{2}$$

where (i,j) are nearest neighbors on the spine. We can express the free energy F as the optimal solution of the following eigenvalue problem:

$$V\Psi = e^{-\beta(F-F_0)}\Psi \tag{3}$$

in which F_0 is the free energy for $J_2 = 0$ and V is given by

$$V = V_0^{1/2} V(J_2) V_0^{1/2} \quad . \tag{4}$$

Fourier transformation of Eq. (2) yields

$$V(J_2) = \exp\left(\frac{\beta J_2}{N_1}\right) \sum_{qkk'} e^{iq}(a_{qk}u_k - a_{-qk}^* u_k^*)$$
$$\times (a_{-qk'}v_{k'} + a_{qk'}^* v_{k'}^*) \quad , \tag{5}$$

where the a's are the (horizontal) Fourier transforms of the a's, and the u's and v's are appropriate amplitudes at the spine. Similarly,

$$V_0 = \exp - \sum_{qk} a_{qk}^* a_{qk} \epsilon(k) \quad , \tag{6}$$

where the $\epsilon(k)$ are the eigenvalues of the $J_2 = 0$ canonical form.[6] Thus, Eqs. (3)–(6) express the reduction of the stated problem to quadrature.[7] The calculation is lengthy and will be reported elsewhere. But there exists a more transparent problem, also of some intrinsic interest, which is more readily solved on the lattice of Fig. 2 and to which we now turn our attention.

We are, of course, referring to the XY model which, after its original introduction and solution in one dimension almost 20 years ago,[8] has come to be studied in higher dimensions both for its own sake, and as a simulation of the hard sphere bose fluid. In two as in one dimension there is no long-range order[9] although there exists evidence for a phase transition and much interesting structure.[10] The bond operators are $(S_i^+ S_j^- + \text{H.c.})$ for nearest neighbors, multiplied by $-J_1$ for vertical bonds and $-J_2$ for bonds on the spine. We shall also introduce a transverse magnetic field term hS_i^z at each site, with h taking the place of a chemical potential in the bose-fluid analog. We fix the sign of $J_1 > 0$ for the sake of definiteness, but allow J_2 and h to have either sign.

Following the procedure already outlined for the transfer matrix, we transform the Hamiltonian of the XY model into canonical form

$$H = H_0 + H_2 \tag{7}$$

in which H_2, the perturbing Hamiltonian containing the effects of the J_2 bonds, takes the form

$$H_2 = -4J_2 \sum_q \cos q \left(\frac{1}{N_1}\right) \sum_{k,k'} \sin k \sin k' a_{qk}^* a_{qk'} \tag{8}$$

and the unperturbed Hamiltonian is just

$$H_0 = \sum_{qk} a_{qk}^* a_{qk} \epsilon(k) \quad ,$$

with

$$\epsilon(k) = -2J_1 \cos k + h \quad . \tag{9}$$

Because H can now be written as $\sum H_q$, the eigenstates factor. Each H_q can be diagonalized by standard scattering theory, but there is a simpler way. We recognize each H_q as the Hamiltonian of a linear chain of N_1 spins each interacting with its neighbor by a bond $-J_1$, and each excepting the first in a magnetic (transverse) field h. The first, spinal, spin is in an effective transverse field $h' = h - 2J_2 \cos q$.

Such a decoupled chain is, of course, straightforwardly solved and we merely report the findings. The spinal amplitude, $(2/N_1)^{1/2} \sin k$ when $J_2 = 0$, is now $(2/N_1)^{1/2} \sin\theta_{qk}$, where

$$\theta_{q,k} = \tan^{-1}\left(\frac{J_1 \sin k}{J_1 \cos k - 2J_2 \cos q}\right) \quad . \tag{10}$$

Bound states appear at any q such that $|2J_2 \cos q| > J_1$ (recall $J_1 > 0$). We denote the set of such bound states b.s.q. The bound-state energies are

$$\epsilon_{\text{b.s.}}(q) = h - 2\left(2J_2 \cos q + \frac{J_1^2}{2J_2 \cos q}\right) \tag{11}$$

and their spinal amplitudes are

$$A(q) = \left(\frac{(2J_2 \cos q)^2 - J_1^2}{(2J_2 \cos q)^2}\right)^{1/2} \quad . \tag{12}$$

Consequently, we can evaluate any quantity of interest on the spine, as, for example, the thermal expectation value of H_2. We obtain

$$\langle H_2 \rangle_{\text{T.A.}} = -4J_2 \sum_q \cos q \frac{1}{N_1} \sum_k \sin^2\theta_{qk} f(\beta\epsilon(k)) - 2J_2 \sum_{\text{b.s.}q} (\cos q) \left(\frac{(2J_2 \cos q)^2 - J_1^2}{(2J_2 \cos q)^2}\right) f(\beta\epsilon_{\text{b.s.}}(q)) \quad , \tag{13}$$

where f is the usual Fermi distribution function. The free energy associated with the introduction of the J_2 bonds can be obtained from the above, by the usual integration over coupling constant; in the process, it should not be forgotten that the set b.s.q depends on J_2.

Being a $1/N$ effect, the appearance or disappearance of bound states at or near the spine cannot affect the thermodynamic properties of the lattice as a whole. It is, nevertheless, a locally important feature and therefore local functions such as $\langle H_2 \rangle$ depend sensitively on whether $J_2 \gtrless \frac{1}{2}J_1$. The numerical results will be published in due course.

It is not unreasonable to expect that there exist other lattices which, after a proper assignment of arrows, share with the comblike structure of Fig. 2 the possibility of reducing spin problems to canonical form. But it would appear that multiple paths connecting two spins must, alas, be excluded.

ACKNOWLEDGMENTS

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[1]There is 1 cross linkage for every N spins, hence the terminology (obviously inspired by the recent ϵ expansions).

[2]J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).

[3]H. A. Kramers and G. H. Wannier, Phys. Rev. 60, 252 (1941).

[4]I.e., into a quadratic form, in a set of anticommuting operators.

[5]See T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. 36, 856 (1964); where the agreement of this method with the solutions of L. Onsager, Phys. Rev. 65, 117 (1944); C. N. Yang, Phys. Rev. 85, 808 (1952), and others is demonstrated.

[6]See T. D. Schultz, D. C. Mattis, and E. H. Lieb, Ref. 5; or D. Mattis, *The Theory of Magnetism* (Harper, New York, 1965), p. 266.

[7]Because of the absence of the (horizontal) momentum parameter q from the eigenvalues and amplitudes, the problem factors into N_2 separate problems, one for each value of q. Each is then exactly diagonalizable by standard techniques of scattering theory. *Vide infra, XY* model.

[8]E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (New York) 16, 407 (1961); S. Katsura, Phys. Rev. 127, 1508 (1962).

[9]H. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).

[10]See Ref. 2 or V. L. Berezinskii, Sov. Phys. JETP 32, 493 (1971).