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Noncommutative Geometry and D-Branes

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Abstract

We apply noncommutative geometry to a system of N parallel D-branes, which is interpreted as a quantum space. The Dirac operator defining the quantum differential calculus is identified to be the zero-momentum mode of the supercharge for strings connecting D-branes. As a result of the calculus, Connes' Yang-Mills action functional on the quantum space reproduces the dimensionally reduced $U(N)$ super Yang-Mills action as the low energy effective action for D-brane dynamics. Several features that may look *ad hoc* in a noncommutative geometric construction are shown to have very natural physical or geometric origin in the D-brane picture in superstring theory.

1 Introduction

D-branes [1] are extended dynamical objects in string theory, on which string endpoints can live (having the Dirichlet boundary condition). In recent developments, recognition [2, 3] of these nonperturbative degrees of freedom has played a central role in understanding string-dualities, M-theory unification, and small distance structure of space (or space-time) [4, 5]. One remarkable feature of D-branes is that when there are N parallel D-branes, their coordinates are lifted to $N \times N$ matrices [6], and their low-energy dynamics is described by dimensional reduction of ten-dimensional $U(N)$ super Yang-Mills gauge theory. This reminds us of noncommutative geometry [7], in which coordinates as local functions are allowed to be noncommuting. Indeed, there are striking similarities between the D-brane dynamics and the non-commutative geometric construction of the standard model [8]: the parallel D-branes versus the multi-sheet space-time, the inter-brane connections versus the Higgs fields, and so on. Moreover, noncommutative geometric features also appear in a recently conjectured light-cone formulation for eleven-dimensional M-theory [9]. We feel, as a warm-up for exploring the possible uses of noncommutative geometry in string theory including M-theory, it is instructive to first examine more closely the connection between noncommutative geometry and D-brane dynamics.

In string theory, one is used to start from bosonic degrees of freedom. For a D_p -brane in bosonic string theory, at low energies the dynamics is described by two kinds of fields living on the brane. Let us denote the coordinates on the D-brane by y^i , where $i = 0, 1, 2, \dots, p$. There is the $U(1)$ gauge field $A^i(y)$, $i = 0, 1, \dots, p$, coupled to the motion of string endpoints in the tangential directions on the brane. There is also the Higgs field $\Phi^a(y)$, $a = p + 1, p + 2, \dots, 25$, corresponding to vibrations of the D-brane (or the motion of string endpoints) in the normal directions. The effective action of a D-brane is the Dirac-Born-Infeld action [10], whose leading term in the gradient expansion is the usual Maxwell action. In superstring theory, the content of the fields living on a D-brane is enlarged to include Ψ , the fermionic super-partners of A^i and Φ^a .

When there are N parallel D-branes (with microscopic separations), all fields A^i , Φ^a and Ψ become anti-Hermitian $N \times N$ matrices [6], describing the effects of short open strings ending on different D-branes. At low energies and to the leading order in the gradient expansion, the effective action for such a system in superstring theory

should be the dimensionally-reduced $U(N)$ super Yang-Mills action [6]

$$S = \frac{1}{g} \int d^{p+1}y \text{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\Psi} \not{D} \Psi \right), \quad (1)$$

where $\mu, \nu = 0, 1, \dots, 9$, and Ψ is a Majorana-Weyl spinor in 10 dimensions. Both A_μ and Ψ are in the adjoint representation of $U(N)$. We will use i, j, k, \dots for indices of values $0, 1, \dots, p$, and a, b, c, \dots for indices of values $p+1, p+2, \dots, 9$. In this convention, $F_{\mu\nu}$ splits into

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j], \quad (2)$$

$$F_{ia} = \partial_i \Phi_a + [A_i, \Phi_a] \equiv \nabla_i \Phi_a, \quad (3)$$

$$F_{ab} = [\Phi_a, \Phi_b], \quad (4)$$

due to dimensional reduction from 10 to $p+1$. Similarly,

$$\not{D} \Psi = \gamma^i \nabla_i \Psi + \gamma^a [\Phi_a, \Psi]. \quad (5)$$

Explicitly, the low energy effective action for N D-branes is

$$S = \frac{1}{g} \int d^{p+1}y \text{Tr} \left(-\frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} \nabla_i \Phi_a \nabla^i \Phi^a - \frac{1}{4} [\Phi_a, \Phi_b] [\Phi^a, \Phi^b] + \frac{i}{2} \bar{\Psi} (\gamma^i \nabla_i \Psi + \gamma^a [\Phi_a, \Psi]) \right). \quad (6)$$

On the other side, Connes' noncommutative geometry [7] generalizes differential calculus and geometry to spaces, called “quantum spaces”, on which the algebra of functions (including coordinates) is noncommutative. In this generalization, ordinary smooth manifolds may allow new noncommutative differential calculi. Noncommutative geometric ideas have been used to reformulate the action for the standard model [8] and the $SU(5)$ grand unified theory [11]. This is done by starting from a certain noncommutative algebra acting on the fermion fields and then introducing appropriate Dirac operator to formulate Connes' action functional (including both Yang-Mills and fermion parts) on a multi-sheet space-time, with the inter-sheet distances directly related to the vacuum expectation values of the Higgs fields. In ref. [12], some supersymmetric Yang-Mills actions are reformulated as Connes' action functional on certain quantum spaces. In this note, we will show that the D-brane action (6) can be rewritten as Yang-Mills-Connes action functional on a quantum space representing D-branes.

In Sec. 2 we will first review basics of quantum differential calculus and Yang-Mills gauge theory on a quantum space, and then in Sec. 3 we will consider a certain class

of quantum spaces in detail, which is used in this paper to model a system of N D-branes. Subsequently we deduce the Dirac operator that defines the desired quantum calculus from the string supercharge in Sec. 4 and find in Sec. 5 that the corresponding Yang-Mills-Connes functional for this calculus is equivalent to the action (6). In Sec. 6 we comment on the relation of T-duality with the choice of the Dirac operator. In the concluding section, we summarize in retrospect several features, which look *ad hoc* in a generic noncommutative geometric construction but become very natural when put in the context of D-branes in string theory.

2 Yang-Mills-Connes Functional

A quantum space is described by a $*$ -algebra¹ of functions \mathcal{A} on the quantum space. A differential calculus on a quantum space is an extension of \mathcal{A} to a graded $*$ -algebra $\Omega^*(\mathcal{A}) = \bigoplus_{n=0} \Omega^{(n)}(\mathcal{A})$, where $\Omega^{(0)}(\mathcal{A}) = \mathcal{A}$ and $\Omega^{(n)}(\mathcal{A})$ are right \mathcal{A} -modules.² An element in $\Omega^{(n)}(\mathcal{A})$ is called a differential form of degree n or an n -form. The differential algebra $\Omega^*(\mathcal{A})$ also needs to be equipped with the exterior derivative d . The exterior derivative is a map $d : \Omega^{(n)}(\mathcal{A}) \rightarrow \Omega^{(n+1)}(\mathcal{A})$ satisfying the graded Leibniz rule $d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{n_1}\omega_1(d\omega_2)$ for $\omega_1 \in \Omega^{(n_1)}(\mathcal{A})$ and $\omega_2 \in \Omega^*(\mathcal{A})$ and the nilpotency condition $d^2 = 0$. Typically an element in $\Omega^{(n)}(\mathcal{A})$ can be written as $\xi^{\mu_1} \dots \xi^{\mu_n} a_{\mu_1 \dots \mu_n}$ for some $a_{\mu_1 \dots \mu_n} \in \mathcal{A}$, where $\{\xi^\mu\}$ is a basis of one-forms in $\Omega^{(1)}(\mathcal{A})$.

In Connes' formulation of noncommutative geometry [7], all information about a quantum space is encoded in the spectral triple $(\mathcal{A}, D, \mathcal{H})$, where D is an anti-self-adjoint operator (called the Dirac operator) acting on \mathcal{H} , which is a Hilbert space with a $*$ -representation π of \mathcal{A} , namely, the $*$ -anti-involution is realized as the Hermitian conjugation: $\pi(a^*) = \pi(a)^\dagger$, $\forall a \in \mathcal{A}$, where the Hermitian conjugation is denoted by \dagger . Using D one can define a noncommutative differential calculus $\Omega^*(\mathcal{A})$ and extend π to a representation $\hat{\pi}$ of $\Omega^*(\mathcal{A})$ [13]. This procedure is shown by an example in Sec. 3.

Other essential ingredients of a quantum space are the inner product and integration over $\Omega^*(\mathcal{A})$. Let $\langle \mathcal{O} \rangle_{\mathcal{H}}$ denote the regularized average of an operator \mathcal{O} on the Hilbert space \mathcal{H} , e.g.

$$\langle \mathcal{O} \rangle_{\mathcal{H}} = \lim_{\Lambda \rightarrow \infty} \frac{\text{Tr}_{\mathcal{H}}(\mathcal{O} \exp\{-|D|^2/\Lambda^2\})}{\text{Tr}_{\mathcal{H}}(\exp\{-|D|^2/\Lambda^2\})}, \quad (7)$$

where Λ is the cutoff for the spectrum of the Dirac operator. The inner product on

¹ An algebra with a map $*$: $a \rightarrow a^* \in \mathcal{A}$ which is an anti-involution, i.e., $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for $a, b \in \mathcal{A}$, is called a $*$ -algebra.

² That is, if $\omega \in \Omega^{(n)}(\mathcal{A})$ then $\omega a \in \Omega^{(n)}(\mathcal{A})$ for all $a \in \mathcal{A}$.

$\Omega^{(n)}(\mathcal{A})$ is defined by [13]³

$$\langle \omega_1 | \omega_2 \rangle_\Omega = \langle \hat{\pi}(\omega_1)^\dagger \hat{\pi}(\omega_2) \rangle_{\mathcal{H}} \quad (8)$$

for $\omega_1, \omega_2 \in \Omega^{(n)}(\mathcal{A})$. The integration of $\omega \in \Omega^*(\mathcal{A})$ is defined by [13]

$$\int \omega = \langle \hat{\pi}(\omega) \rangle_{\mathcal{H}}, \quad (9)$$

where $\hat{\pi}$ is a representation of $\Omega^*(\mathcal{A})$ on \mathcal{H} .

The distance between two states in the Hilbert space is defined by [7]

$$\text{dist}(\psi_1, \psi_2) = \sup \left\{ |\langle \psi_1 | \hat{\pi}(a) \psi_1 \rangle - \langle \psi_2 | \hat{\pi}(a) \psi_2 \rangle| : a \in \mathcal{A}, \|\hat{\pi}(da)\|^2 \leq 1 \right\}, \quad (10)$$

where $\|\cdot\|$ is the operator norm and $\langle \cdot | \cdot \rangle$ is the inner product on the Hilbert space.

A gauge field theory fits into this framework easily. The group of unitary elements $\mathcal{U} = \{u : uu^* = u^*u = 1, u \in \mathcal{A}\}$ in \mathcal{A} acts on \mathcal{H} as a group of transformations, which is identified with the gauge group. For example, if \mathcal{A} is the algebra of $N \times N$ matrices of complex functions on a manifold, the group \mathcal{U} is the gauge group $U(N)$ on the manifold. The gauge field is a one-form $A = \xi^\mu A_\mu \in \Omega^{(1)}(\mathcal{A})$ for some $A_\mu \in \mathcal{A}$. It transforms under $u \in \mathcal{U}$ as $A \rightarrow A^u = uAu^* + u(du^*)$, which implies that $\hat{\pi}(A) \rightarrow \hat{\pi}(A^u) = U\hat{\pi}(A)U^\dagger + U[D, U^\dagger]$, where $U = \hat{\pi}(u)$ (and $U^\dagger = \hat{\pi}(u^*)$). The modified Dirac operator $\tilde{D} = D + \hat{\pi}(A)$ is covariant. The gauge-covariant field strength is defined as usual by $F = dA + A^2$. The only new ingredient so far in this straightforward generalization is the quantum differential calculus behind each expression above. The Yang-Mills-Connes action functional defined by [7]

$$S = \langle F | F \rangle_\Omega + \langle \Psi | \tilde{D} \Psi \rangle, \quad (11)$$

where $\langle \cdot | \cdot \rangle$ is the inner product on \mathcal{H} , is another natural but non-trivial ingredient of the noncommutative generalization.

For a non-Abelian gauge field $A = dx^\mu A_\mu$ on a classical manifold, the distance defined by (10) with $\tilde{D} = \not{D} + \not{A}$ between two vectors in the fibers located at two points on the manifold is the length of the shortest path on the manifold which connects the two vectors by parallel transport [14].

³ The inner product and integration can also be defined in terms of the Dixmier trace [7]; what we use here is more familiar to physicists.

3 A Class of Quantum Calculi

A basic idea in applying noncommutative geometry to field theory is that the geometry of the relevant quantum space is determined by matter or, more precisely, by fermion fields. Thus one is led to consider a special class of quantum calculi, where the $*$ -algebra \mathcal{A} is a noncommutative algebra acting on fermion fields, and the Dirac operator acting on the fermions is of the form

$$D = \gamma^\mu \otimes D_\mu, \quad (12)$$

where the γ^μ 's are usual γ -matrices. The Hilbert space \mathcal{H} is one for fermions, of the form $\mathcal{H} = S \otimes \mathcal{H}_0$, where S is a representation of the Clifford algebra (e.g. $S = \mathbf{C}^{32}$ for a Dirac spinor in 10 dimensions), and \mathcal{H}_0 is the Hilbert space in which the algebra \mathcal{A} acts with a representation π_0 . The representation of $a \in \mathcal{A}$ on \mathcal{H} is $\pi(a) = 1 \otimes \pi_0(a)$. From now on we will suppress the symbol of tensor product \otimes .

In the universal differential calculus $\Omega^* \mathcal{A}$ [7], a differential one-form $\rho \in \Omega^{(1)} \mathcal{A}$ is a formal expression $\rho = \sum_\alpha a_\alpha db_\alpha$, where a_α, b_α are elements in \mathcal{A} . To simplify the notation, we will omit the index α in the following.

With the help of the Dirac operator, the representation π of \mathcal{A} on \mathcal{H} is extended to $\Omega^* \mathcal{A}$ [7] by defining, for $\rho \in \Omega^{(1)} \mathcal{A}$,⁴

$$\pi(\rho) = \sum a[D, b]. \quad (13)$$

By (12), it is $\pi(\rho) = \gamma^\mu \sum a[D_\mu, b] = \gamma^\mu \rho_\mu$. The representation of a two-form $\omega = \sum adbdc$ is thus

$$\pi(\omega) = \sum a[D, b][D, c], \quad (14)$$

and similarly for forms of higher degrees.

In particular, the representation of $d\rho = \sum dadb$ is

$$\begin{aligned} \pi(d\rho) &= \gamma^{\mu\nu}([D_\mu, \rho_\nu] - \frac{1}{2} \sum a[[D_\mu, D_\nu], b]) \\ &\quad + g^{\mu\nu}([D_\mu, \rho_\nu] - \sum a[D_\mu, [D_\nu, b]]), \end{aligned} \quad (15)$$

where $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and $g^{\mu\nu} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}$ is the metric.

The general differential calculus $\Omega^*(\mathcal{A})$ [7] is defined by the quotient

$$\Omega^*(\mathcal{A}) = \Omega^* \mathcal{A} / J, \quad (16)$$

where $J = \ker \pi + d(\ker \pi)$. This means that two differential forms ω_1 and ω_2 of the same degree will be considered the same if $(\omega_1 - \omega_2) \in J$.

⁴We will simply write a to stand for $\pi(a)$ for $a \in \mathcal{A}$ in the following.

To find $\Omega^{(2)}(\mathcal{A})$, the differential calculus of degree two, we consider a one-form $\rho \in \ker \pi$. According to (15), if the zero-curvature condition for the unperturbed Dirac operator

$$[D_\mu, D_\nu] = 0 \quad (17)$$

is satisfied, then

$$\pi(d\rho) = - \sum a[g^{\mu\nu} D_\mu D_\nu, b] \in \pi(\mathcal{A}). \quad (18)$$

For a non-Abelian gauge theory on a classical manifold, D_μ can be ∂_μ plus a pure gauge to satisfy (17).

Denote the degree-two component of J by $J^{(2)}$, then $\pi(J^{(2)})$ is composed of all elements (18) for all $\rho = \sum adb \in \ker \pi$. We will focus on the cases for which D_μ satisfies (17) and

$$\pi(J^{(2)}) = \pi(\mathcal{A}). \quad (19)$$

The representation π defined above is, in general, not a good representation of $\Omega^*(\mathcal{A})$ because one and the same differential form may admit many equivalent expressions of the form $\sum adbdc \dots$, so that the representation is not unique.

A good representation is given by $\hat{\pi} = P_J \circ \pi$ [13] where P_J is the projection perpendicular to $\pi(J)$. By (15) and (19), it follows that for a two-form ω ,

$$\hat{\pi}(\omega) = \gamma^{\mu\nu} \omega_{\mu\nu} \quad (20)$$

for some $\omega_{\mu\nu} \in \mathcal{A}$. In particular, by (17),

$$\hat{\pi}(d\rho) = \frac{1}{2} \gamma^{\mu\nu} ([D_\mu, \rho_\nu] - [D_\nu, \rho_\mu]). \quad (21)$$

It can be shown that the conditions (17) and (19) also imply that for a three-form ω ,

$$\hat{\pi}(\omega) = \gamma^{\mu\nu\kappa} \omega_{\mu\nu\kappa} \quad (22)$$

for some $\omega_{\mu\nu\kappa} \in \mathcal{A}$, and similarly for higher degrees.

Let ξ^μ denote the basis of one-forms which is represented by γ -matrices: $\pi(\xi^\mu) = \gamma^\mu$. Then it follows that the calculus $\Omega^*(\mathcal{A})$ is generated by elements in \mathcal{A} and one-forms ξ^μ , where the ξ^μ 's anticommute with each other and commute with elements in \mathcal{A} as in the classical case. The only possible source of noncommutativity is \mathcal{A} .

4 Dirac Operator and Supercharge

The low-energy dynamics of N parallel D-branes is described by a field theory on the D-brane world volume. So we may try to reformulate the D-brane action (6) in terms of

the quantum calculus discussed in last section. The key is to find an appropriate Dirac operator for the fermion fields, which are massless spinor states of strings connecting the D-branes. The Dirac operator should be motivated from string theory: Indeed we find a natural candidate to be the supercharge operator for strings connecting D-branes, truncated in the subspace of massless spinor states.

It was shown by Witten [15] that the quantized zero-momentum modes in a supersymmetric non-linear σ -model in $1 + 1$ dimensions can be identified with the de Rham complex of the target space. The bosonic fields X^μ are the coordinates on the target space. The fermionic fields ψ^μ are Majorana spinors on the world sheet, which splits into two Majorana-Weyl spinors ψ_+^μ, ψ_-^μ in $1 + 1$ dimensions. By canonical quantization, ψ_+^μ and ψ_-^μ satisfy two anticommuting sets of Clifford algebra: $\{\psi_A^\mu, \psi_B^\nu\} = g^{\mu\nu} \delta_{AB}$, where $A, B = +, -$ and $g^{\mu\nu}$ is the metric of the target space. The supercharge Q on the world sheet for zero-momentum modes is also a Majorana spinor and has two Weyl components Q_+ and Q_- :

$$Q_\pm = \psi_\pm^\mu P_\mu, \quad (23)$$

where the momentum P^μ acts on functions of X^μ as a derivative. Let $Q = \frac{1}{2}(Q_+ + iQ_-)$ and $Q^* = \frac{1}{2}(Q_+ - iQ_-)$. It is remarkable that the supercharges Q and Q^* realize the exterior derivative d and its adjoint d^* [15]. $\psi^\mu = \psi_+^\mu + i\psi_-^\mu$ and $\psi^{\mu*} = \psi_+^\mu - i\psi_-^\mu$ correspond to differential one-forms and inner derivatives, respectively. Hermitian conjugation realizes Poincaré duality.

All these are also true for closed strings. For an open string with Neumann boundary conditions, however, certain modification is necessary, because the zero-momentum modes of the right-moving and left-moving sectors of ψ^μ are identified. Then there is only one set of Clifford algebra, and the supercharge of zero-momentum modes has only one independent component

$$Q_0 = \psi_0^\mu P_\mu. \quad (24)$$

After canonical quantization, ψ_0^μ 's become γ -matrices acting on massless spinor states, which are (after GSO projection) Majorana-Weyl spinors in the supermultiplet of a Yang-Mills theory in 10 dimensions [16]. Being the Dirac operator for these spinors, the supercharge Q_0 realizes the exterior derivative in the target space in the sense of Connes.

Later it has also been argued by Witten [17] that the generalized Dirac operator in the full superstring theory (including nonzero-momentum sector), the so-called Dirac-Ramond operator, is the supercharge on the world sheet for the following three reasons:

(1) Its zero-momentum limit is the usual Dirac operator. (2) It annihilates physical states. (3) It anticommutes with an analogue of the chirality operator: $(-1)^F$.

Motivated by these observations, we try to deduce in the following the Dirac operator for the quantum space representing D-branes from the supercharge on the world sheet of strings ending on D-branes.

It is well known [1] that for open strings ending on a Dp -brane, we have for X^a , $a = p + 1, \dots, 9$, Dirichlet boundary conditions, which originate from T -duality for Neumann boundary conditions. Since T -duality simply reverses the relative sign on the left-moving and right-moving modes for both X^μ and ψ^μ , the appropriate boundary conditions for ψ^μ on a string with Dirichlet boundary conditions are still [3]

$$\psi_+^\mu(0, \tau) = \psi_-^\mu(0, \tau), \quad \psi_+^\mu(\pi, \tau) = \pm \psi_-^\mu(\pi, \tau), \quad (25)$$

(except changes in sign for dualized directions) of the same type as for open strings with Neumann boundary conditions. As mentioned above, the supercharge (24) acts on the states of massless spinors. Upon identifying these massless string states with the field Ψ in the super Yang-Mills theory on a 9-brane, the string supercharge reduces (or truncates) to the Dirac operator in 10 dimensional spacetime. For a D-brane of lower dimensions, the momentum operators in directions normal to the brane vanish due to the Dirichlet boundary conditions, hence the supercharge becomes a Dirac operator on the $(p + 1)$ dimensional world volume of the Dp -brane. However, to fully describe the dynamics of massless fields on a D-brane, one has to include the effects of the tadpole diagram for closed strings created from the D-brane. Although it was mentioned before that the supercharge has two components Q, Q^* on a closed string, the boundary conditions on the brane identify ψ^μ and $\psi^{\mu*}$ up to a sign [2] (after all, a closed string tadpole diagram can be viewed as an open string disk diagram), so half of the supersymmetry is broken. Hence the D-brane is a BPS state [2] and there is only one independent component of the supercharge on a closed string created from a D-brane. The momentum of the closed string is shifted by the gauge field ϕ^a normal to the brane, which originates from a pure gauge transformation $\Lambda^a = \phi^a x^a$ in the dual picture. Including contributions from both open and closed strings, the Dirac operator obtained by truncating the supercharge on strings ending on a D-brane is

$$D = \gamma^i \partial_i + \gamma^a \phi_a. \quad (26)$$

5 D-branes as Quantum Space

Now we are able to formulate precisely how to interpret the system of N parallel D-branes (with microscopic separations) as a quantum space. We take a $p+1$ dimensional coordinate system on one of the branes as the world volume coordinates, and treat the structure arising from the strings connecting D-branes as the “internal” structure that defines a quantum space. (Closed string tubes connecting two branes can be viewed as loops of open strings ending on different branes.) Each D-brane has a label r ($r = 1, 2, \dots, N$), and an open (oriented) string from D-brane r to D-brane s , and the states on such string may be labeled by an ordered pair, (r, s) , of indices. (These indices are also called Chan-Paton labels, since such an open string is dual to an open string with usual Chan-Paton labels (r, s) [3].) In particular, the massless spinor states of the open string connecting D-brane r and D-brane s result in fermion fields living on the D-brane world volume, which therefore also carry the Chan-Paton labels (r, s) . Thus the fermion field Ψ is an $N \times N$ matrix with entries being Majorana-Weyl spinors in 10 dimensions, which are naturally anti-Hermitian, belonging to the adjoint representation of $U(N)$: exchanging the labels r and s leads to inverting the orientation of the string. The Hilbert space on which the Dirac operator acts is taken to be the space of $N \times N$ matrices of Dirac spinors, which is larger than the configuration space of Ψ , since the Dirac operator always reverses the chirality.

To define a quantum space representing the D-branes, in addition to \mathcal{H} we need to specify the other two elements in the spectral triple $(\mathcal{A}, D, \mathcal{H})$. Recall that the algebra \mathcal{A} defines the gauge group as the group \mathcal{U} of unitary elements of \mathcal{A} . Hence we take \mathcal{A} to be $M_N(\mathbf{C}) \otimes L^2(\mathbf{R}^{p+1})$, the algebra of $N \times N$ matrices of square-integrable functions on the $p+1$ dimensional D-brane world volume,⁵ so that \mathcal{U} is the $U(N)$ gauge group. The representation π of \mathcal{A} on the fermion Hilbert space \mathcal{H} is simply the matrix multiplication. The Dirac operator is chosen to be a natural generalization of the operator (26) to the multi-brane case. By introducing a Wilson line $A^\alpha = \phi_\alpha \equiv \text{diag}(\phi_1^\alpha, \dots, \phi_N^\alpha)$ in the T-dual picture, the Dirac operator resulting from the (truncated) supercharge operator is found to be

$$D = \gamma^i \partial_i + \gamma^\alpha \phi_\alpha, \quad (27)$$

where the ϕ_α 's are $N \times N$ matrices, automatically satisfying

$$[\phi_a, \phi_b] = 0, \quad \forall a, b = p+1, \dots, 9. \quad (28)$$

⁵It is also possible to define \mathcal{A} to be the algebra of the $U(N)$ gauge group represented in its adjoint representation on \mathcal{H} .

The first term in eq. (27) is the classical Dirac operator on the $(p + 1)$ dimensional D-brane world volume. Viewing a p -brane as dimensional reduction of a 9-brane, one can think of the second term as the remnant of the dimensionally reduced $(9 - p)$ directions along which the partial derivatives ∂_a vanish but the pure gauge terms survive. (Recall the statement following (18).)

The quantum calculus considered in Sec. 3 is applicable to the present case. Now let us show that the Yang-Mills-Connes functional (11) with the Dirac operator (27) reproduces the super Yang-Mills action (6) describing the dynamics of N D-branes.

The Dirac operator (27) satisfies (17) because of (28), and for generic ϕ^a it also satisfies (19), thus according to the discussions in Sec. 3, for generic ϕ^a the calculus $\Omega^*(\mathcal{A})$ on D-branes is generated by \mathcal{A} and dx^μ .⁶ The only noncommutativity resides in \mathcal{A} , the algebra of matrices $M_N(\mathbf{C}) \otimes L^2(\mathbf{R}^{p+1})$. The one-forms dx^μ are represented by γ -matrices: $\hat{\pi}(dx^{\mu_1} \cdots dx^{\mu_k}) = \gamma^{\mu_1 \cdots \mu_k}$ and so they anticommute with each other and commute with elements in \mathcal{A} .

The gauge field is a one-form

$$A = dx^i A_i + dx^a A_a, \quad (29)$$

where A_i and A_a are required to be anti-Hermitian. It modifies the Dirac operator to

$$\tilde{D} = \gamma^i \nabla_i + \gamma^a \Phi_a, \quad (30)$$

where $\nabla_i = \partial_i + A_i$ and

$$\Phi_a = \phi_a + A_a. \quad (31)$$

The field strength,

$$F = \frac{1}{2} dx^\mu dx^\nu F_{\mu\nu} = dA + A^2, \quad (32)$$

is given by

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j], \quad (33)$$

$$F_{ia} = \partial_i \Phi_a + [A_i, \Phi_a] \equiv \nabla_i \Phi_a, \quad (34)$$

$$F_{ab} = [\Phi_a, \Phi_b] - [\phi_a, \phi_b] = [\Phi_a, \Phi_b], \quad (35)$$

where we have used (28).

⁶ Strictly speaking, our notation for dx^a ($a = p + 1, \dots, 9$) is inappropriate because while dx^i ($i = 0, \dots, p$) is exact, dx^a is not exact but a closed one-form $\sum_\alpha a_\alpha db_\alpha$ for a_α, b_α being some matrices in \mathcal{A} .

The first term in the Yang-Mills functional (11) involves the trace of the Hilbert space in (7), which is composed of three kinds of traces. The first is the trace of γ -matrices, which gives rise to the contraction of the components of two $F_{\mu\nu}$. The second is the trace over square-integrable functions on \mathbf{R}^{p+1} , which turns into the integration over the world volume of the Dp -brane. The trace of $N \times N$ matrices remains explicit as in (6). It is then straightforward to see that by constraining physical states to be anti-Hermitian $N \times N$ Majorana-Weyl spinors Ψ after Wick rotation, the Yang-Mills functional (11) for this quantum space is equivalent to the effective action (6) for N D-branes. Obviously the same formulation can be applied to other cases, for example, the $N = 2$ super Yang-Mills theory in four dimensions as dimensionally reduced from $N = 1$ super Yang-Mills theory in six dimensions.

6 T-duality and Dirac operator

Using T-duality, we can define another Dirac operator by taking the T-dual of the supercharge on an open string with Neumann boundary conditions. Introducing a Wilson line in the dual picture: $A^a = \text{diag}(\phi_1^a, \dots, \phi_N^a)$ in some compactified dimensions of radius R^a , the open string with Chan-Paton labels (r, s) is T-dual to an open string stretching between two D-branes at positions $x_r^a = \phi_r^a$ and $x_s^a = \phi_s^a$ (or a simultaneous translation of them) [3]. The momentum p^a in the compactified direction is shifted by the gauge field:

$$p^a = \frac{n^a}{R^a} + (\phi_s^a - \phi_r^a), \quad (36)$$

where n is the quantum number for momentum p^a in the dual picture and becomes the winding number in a compactified dimension of radius $R^a = \alpha'/R^a$. As we are focusing on low energy modes, we set $n^a = 0$.

To describe all string states at the same time, it is natural to put the p^a 's for all possible string configurations into an antisymmetric matrix

$$P_{rs}^a = \phi_s^a - \phi_r^a, \quad (37)$$

which is in the Lie algebra of $SO(N) \subset U(N)$. The Dirac operator as the “total” supercharge (24) therefore becomes

$$D_{dual} = \gamma^i \partial_i + \gamma^a P_a, \quad (38)$$

where $i = 0, \dots, p$, and $a = p + 1, \dots, 9$.

Consider the case of two D8-branes ($N = 2$). The matrix P^a ($a = 9$) is

$$\begin{pmatrix} 0 & (x_1^a - x_2^a) \\ (x_2^a - x_1^a) & 0 \end{pmatrix}. \quad (39)$$

When the distance $|x_1^a - x_2^a|$ between two branes is large, the gauge group is $U(1)^2$. Hence the algebra of functions \mathcal{A} is taken to be diagonal 2×2 matrices for this case. This is precisely the two-sheet model Connes considered [7] and the distance (10) for this case is $|(x_1^a - x_2^a)|^{-1}$, the inverse of the actual distance. This is not surprising because we are using the Dirac operator obtained from the supercharge in the dual picture and T-duality inverses the length. It is interesting to note that if we take the inverse of every element of P^a to “correct” this inversion in length: i.e. use

$$P'_{rs} = (x_s^a - x_r^a)^{-1} \quad (40)$$

to replace P^a in the Dirac operator for N D8-branes, then the new Dirac operator will define the geometry of N -sheets separated by the actual distances $|(x_r^a - x_s^a)|$. The algebra \mathcal{A} in this case is the algebra of diagonal $N \times N$ matrices, as appropriate for the $U(1)^N$ gauge symmetry.

7 Discussions

In this paper, we have interpreted the system of N parallel D-branes as a quantum space in the sense of noncommutative geometry. The associated Yang-Mills-Connes action functional on this quantum space is shown to reproduce the dimensionally reduced $U(N)$ super Yang-Mills action as the low energy effective action for D-brane dynamics. To conclude, in this section we note in retrospect that several features that would look *ad hoc* in a noncommutative geometric construction actually have very natural physical or geometric interpretation in the D-brane picture in string theory.

First, the source of noncommutativity resides in the matrix algebra \mathcal{A} , which arises naturally due to the Chan-Paton labels of the fermion fields, which in turn originate from the strings ending on different D-branes. In other words, parallel D-branes provide a physical realization of “multi-sheet space-time” and a geometric origin for the gauge group $U(N)$. One may wonder whether our universe could really be such a system of D-branes or, equivalently, have spacetime of a discrete Kaluza-Klein structure.

Second, the choice of the Dirac operator (27) is dictated by the D-brane picture, where the addition of the second term is due to the fluctuations in the position of the



D-branes. In particular, the commutativity (28) that makes the condition (17) satisfied is not an ansatz as in usual noncommutative geometric reformulation of super Yang-Mills action [12]; it is deduced here from T-duality of the D-branes: the inter-brane separations is dual to a Wilson line for pure gauge configuration [3, 6].

Third, in the Yang-Mills-Connes action functional (6), the ϕ_a that is introduced in the unperturbed Dirac operator (27) appears only in the combination $\Phi_a = A_a + \phi_a$. In the D-brane picture this is a reflection of the fact that ϕ_a stands for classical inter-brane separation, while A_a its quantum fluctuations, as is consistent with ϕ_a being diagonal and constant and with the commutativity constraint (28). In accordance to T-duality, in string theory it is the total Φ_a (together with A_i) that stands for the D-brane “coordinates” (divided by α' , the string tension) lifted to a matrix [6]. We note that such interpretation is not available in usual noncommutative geometric construction.

Finally, in general the Yang-Mills-Connes action functional is not necessarily supersymmetric. However, in the present case, our Yang-Mills-Connes action functional (6) happens to be supersymmetric. This is closely related to the fact that we start with a very special fermion field content in a special dimensionality (dimensional reduction of a Majorana-Weyl spinor in ten dimensions), which is inherited from superstring theory.

From the above discussions, we see that there is a close relationship and deep internal consistency between noncommutative geometry (at least on discrete Kaluza-Klein space-time) and D-brane dynamics at low energies. An interesting question arises: whether or not this close relationship of D-brane dynamics with noncommutative geometry can be extended to a deeper level? (Either to the full D-brane dynamics which should be described by a supersymmetric and non-abelian generalization of the Dirac-Born-Infeld action, or to superstring theory or even M-theory.) This seems to call for a generalization of noncommutative geometry to superstrings or M-theory that incorporates D-branes.

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Note Added: When we are completing the paper, we learn that in a recent preprint of M. Douglas, hep-th/9610041, a comparison between the D-brane action and non-commutative geometric construction of the standard model action is briefly discussed (without much detail).

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