

Fluid interpretation of Cardassian expansion

Paolo Gondolo

Case Western Reserve University, Department of Physics, 10900 Euclid Avenue, Cleveland, Ohio 44106-7079, USA

Katherine Freese

Michigan Center for Theoretical Physics, University of Michigan, Ann Arbor, Michigan 48109, USA

(Received 28 October 2002; published 24 September 2003)

A fluid interpretation of Cardassian expansion is developed. Here, the Friedmann equation takes the form $H^2 = g(\rho_M)$ where ρ_M contains only matter and radiation (no vacuum). The function $g(\rho_M)$ returns to the usual $8\pi\rho_M/(3m_{pl}^2)$ during the early history of the Universe, but takes a different form that drives an accelerated expansion after a redshift $z \sim 1$. One possible interpretation of this function (and of the right-hand side of Einstein's equations) is that it describes a fluid with total energy density $\rho_{tot} = (3m_{pl}^2/8\pi)g(\rho_M) = \rho_M + \rho_K$ containing not only matter density (mass times number density) but also interaction terms ρ_K . These interaction terms give rise to an effective negative pressure which drives cosmological acceleration. These interactions may be due to interacting dark matter, e.g. with a fifth force between particles $F \sim r^{\alpha-1}$. Such interactions may be intrinsically four dimensional or may result from higher dimensional physics. A fully relativistic fluid model is developed here, with conservation of energy, momentum, and particle number. A modified Poisson's equation is derived. A study of fluctuations in the early Universe is presented, although a fully relativistic treatment of the perturbations including gauge choice is as yet incomplete.

DOI: 10.1103/PhysRevD.68.063509

PACS number(s): 98.80.-k, 11.10.Lm, 11.27.+d, 95.35.+d

I. INTRODUCTION

Recent observations of type IA supernovae [1,2], as well as concordance with other observations, including the microwave background [3] and galaxy power spectra [4], indicate that the Universe is flat and accelerating. Many authors have explored possible explanations for the acceleration: a cosmological constant, time-dependent vacuum energy such as quintessence [5–10], and gravitational leakage into extra dimensions [11].

Recently, Freese and Lewis [12] (paper I) proposed an explanation for the acceleration which involves only matter and radiation, invoking no vacuum energy or cosmological constant whatsoever. In their model, called Cardassian, the Universe has a flat geometry as required by measurements of the cosmic background radiation [3] and yet consists only of matter and radiation. The Friedmann equation is modified from its usual form, $H^2 = (8\pi/3m_{pl}^2)\rho_M$, to

$$H^2 = g(\rho_M), \quad (1)$$

where $H = \dot{a}/a$ is the Hubble constant (as a function of time), a is the scale factor of the Universe, and the energy density ρ_M contains only ordinary matter and radiation. The function $g(\rho_M)$ reduces to $(8\pi/3m_{pl}^2)\rho_M$ in the early Universe, so that Eq. (1) reduces to the ordinary Friedmann equation during early epochs including primordial nucleosynthesis. Only at redshifts $z < \mathcal{O}(1)$ does the function $g(\rho_M)$ differ from the ordinary Friedmann-Robertson-Walker (FRW) case; during these late epochs, $g(\rho_M)$ gives rise to accelerated expansion. In paper I, the specific form of $g(\rho)$ that was considered was the power law Cardassian model:

$$H^2 = \frac{8\pi}{3m_{pl}^2}\rho_M + B\rho_M^n, \quad (2)$$

with

$$n < 2/3. \quad (3)$$

The second term only becomes important once $z < \mathcal{O}(1)$, at which point it dominates the equation and causes the Universe to accelerate. Other possible functions $g(\rho)$ [13] are discussed further below.

There remains the question of the fundamental origin of these modifications to the Friedmann equation. There is no unique four-dimensional or even higher-dimensional theory that gives Cardassian evolution. We consider two different motivations for these modifications:

(1) These functions may arise from fundamental theories of gravity in higher dimensions, as was discussed in [12]. Chung and Freese [14] showed that, generically, the Friedmann equations are modified as a consequence of embedding our Universe as a three-dimensional surface (3-brane) in higher dimensions.

(2) Alternatively these functions may arise in a purely four-dimensional theory in which the modified right-hand side of the Friedmann equation is due to an extra contribution to the total energy density. The right-hand side is treated as a single fluid, with an extra contribution to the energy-momentum tensor in (ordinary four dimensional) Einstein's equations.

The two motivations may or may not be linked, in that the fluid interpretation may be intrinsically four dimensional, or it may be an effective description of higher dimensional physics.

In this paper we restrict our discussion to four dimensions, and treat the right-hand side of Einstein's equations as a single fluid. We consider models with an extra energy density associated with matter that contributes in such a way as to drive acceleration. This extra energy density may be in-

trinsically four dimensional or may serve as an effective description of higher dimensional physics. We take the total energy density of the matter

$$\rho_{\text{tot}} = \frac{3m_{pl}^2}{8\pi} g(\rho) = \rho_M + \rho_K \quad (4)$$

(plus possible internal thermal energy which is unimportant on cosmological scales) to contain not only the ordinary mass density ρ_M (mass times number density) but also an additional contribution ρ_K . For example, in Eq. (2),

$$\rho_K = \frac{3m_{pl}^2}{8\pi} B \rho_M^n. \quad (5)$$

Given this total energy density, we can now compute the accompanying pressure, and find that the Cardassian contribution has a *negative pressure*, $p_K < 0$. This negative pressure is responsible for the Universe's acceleration. In fact, one can obtain any negative equation of state $w_K = p_K/\rho_K < 0$, including $w_K < -1$.

The fluid approach has several advantages: (i) it is fully relativistic, (ii) it allows for the conservation of energy and momentum as well as of particle number, (iii) it admits a sensible weak-field limit which leads to a modified Poisson's equation, and (iv) it permits the study of fluctuations in the early Universe, the study of effects on the cosmic microwave background anisotropies, and other observables.

The primary purpose of this paper is to examine this fluid approach. However, we briefly speculate on a possible origin for this extra term ρ_K in the energy density. It may arise from (dark) matter self-interactions that contribute a negative pressure, for example, through a long-range confining force which may be of gravitational origin or may be a fifth force. This self-interacting dark matter is different from any such component considered in the past, in that it has a negative rather than a positive pressure. We speculate on a form of the force between particles that may be responsible for such an interaction, $F \sim r^{\alpha-1}$, although this Newtonian form must of course be modified on horizon scales. This description of a self-interacting dark fluid may be an effective description of a more fundamental theory. The fluid approach does not rely on the validity of such an interpretation of self-interacting dark matter, e.g., the interactions may be an effective description of higher dimensional physics.

We begin by reviewing the idea of Cardassian expansion in Sec. II. We present a general fluid formulation in Sec. III, and then give specific examples in Sec. IV. In Sec. V we address the growth of density perturbations, and in Sec. VI we speculate on the possible origin of an interaction energy with negative pressure.

II. REVIEW OF CARDASSIAN MODELS

The general form of a Cardassian model was described in Eq. (1), in which a general function of matter density replaces the ordinary energy density in the Friedmann equation. The simplest version, the power law Cardassian model of Eq. (2), can equivalently be written as

$$H^2 = \frac{8\pi G}{3} \rho_M \left[1 + \left(\frac{\rho_M}{\rho_{\text{Card}}} \right)^{n-1} \right]. \quad (6)$$

For $n < 2/3$, the new term is negligible initially, and only comes to dominate at redshift $z \sim 1$ (once $\rho_M \sim \rho_{\text{Card}}$); once it dominates, it causes the Universe to accelerate. We can consider the contribution of ordinary matter, with

$$\rho_M = \rho_{M,0} (a/a_0)^{-3} \quad (7)$$

to this new term. Here, subscript 0 refers to today. Once the new term dominates the right-hand side of the equation, we have accelerated expansion. When the new term is so large that the ordinary first term can be neglected, the solution to Eq. (2) is

$$a \propto t^{2/3n} \quad (8)$$

so that the expansion is superluminal (accelerated) for $n < 2/3$.

The Cardassian model also has the attractive feature that matter alone is sufficient to provide a flat geometry. The numerical value of the critical mass density for which the Universe is flat can be modified. For example, in paper I it was shown that in the model of Eq. (2), the value of the critical mass density can be 0.3 of the usual value. Hence the matter density can have exactly this new critical value and satisfy all the observational constraints, such as those given by the baryon cluster fraction and the galaxy power spectrum.

In a ‘‘generalized Cardassian model,’’ other functions $g(\rho_M)$ of the matter (or radiation) density on the right-hand side of the Friedmann equation can also drive an accelerated expansion in the recent past of the Universe without affecting its early history [13]. Several of these alternative functions will be discussed below [see Eqs. (48) and (52)].

III. BASIC EQUATIONS

A. Perfect fluid

We use the ordinary four-dimensional Einstein's equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (9)$$

On the right-hand side, we take as our ansatz that the energy-momentum tensor is made only of matter and radiation, and has the perfect fluid form,

$$T^{\mu\nu} = p g^{\mu\nu} + (p + \rho) u^\mu u^\nu, \quad (10)$$

where p is the pressure, ρ is the *total* energy density of the matter and radiation, and u^μ is the fluid four-velocity. Here the *total* energy density for matter includes not only the mass density ρ_M (mass times number density) but also any interactions or additional terms [corresponding to the new terms on the right-hand side of Eq. (1)]. In general, p and ρ are functions of the mass density ρ_M and of thermodynamic variables. We assume that at recent times matter is non-relativistic, i.e. the typical speeds involved are much smaller than the speed of light, but we do not assume that $p \ll \rho$.

As our ansatz, we take the total energy density of the fluid during the matter dominated era to arise from the sum of three terms:

$$\rho = \rho_M + \rho_{\text{internal}} + \rho_K, \quad (11)$$

$$p = p_M + p_K, \quad (12)$$

where

$$\rho_M = mn_M \quad (13)$$

is the ordinary matter density of some particle of mass m and number density n_M , ρ_{internal} and p_M are the ordinary internal energy density and pressure of matter (for example, for an ideal monoatomic gas at temperature T_M , $\rho_{\text{internal}} = \frac{3}{2}n_M T_M$ and $p_M = n_M T_M$), and ρ_K and p_K are extra (Cardassian) contributions to energy and pressure. With regard to the gross properties of the Universe (such as its expansion), we can ignore ρ_{internal} compared to ρ_M and p_M compared to p_K for nonrelativistic matter; however, in the context of galaxies, ρ_{internal} and p_M can be important.

Equation (11) is quite general. Even in those cases where the total energy density is not a simple linear sum of terms (see Secs. II B and II C), one can always write Eq. (11) in this fashion.

In Cardassian models we assume there is no vacuum contribution to the energy density. We also assume that the new (Cardassian) contributions ρ_K and p_K are only functions of ρ_M , i.e., the Cardassian fluid is barotropic. For example, in paper I, we took

$$\rho_K = b \rho_M^n, \quad (14)$$

where

$$b = \frac{3m_{pl}^2}{8\pi} B \quad (15)$$

with B as in Eq. (2).

Note that, throughout the rest of this paper, we will focus on the matter-dominated era and hence concentrate on the matter contribution to the fluid (rather than the radiation).

With the ansatz of Eqs. (11) and (12), Eq. (1) can be rewritten

$$H^2 = 8\pi G \rho / 3. \quad (16)$$

Hence we recover the ordinary FRW equations, but with a modified energy density on the right-hand side.

One can think of this total energy density as including the effects of an interaction term: perhaps this term is simply an effective term on large scales (possibly arising from extra dimensions) describing the expansion of the Universe; or perhaps this term is due to the interaction energy of the dark matter. A possible origin of an interaction energy with a negative pressure is described in Sec. VI. We iterate again that the fluid approach is only one of the many ways that Cardassian expansion of Eq. (1) could result.

B. Conservation laws

The Bianchi identities guarantee the conservation of energy and momentum,

$$T^{\mu\nu}_{;\nu} = 0. \quad (17)$$

One can follow the evolution of the energy density along each fluid world line using the general relativistic fluid flow equations. In a comoving frame, energy-momentum conservation gives the (fully relativistic) energy conservation and Euler equations [15,16]

$$\dot{\rho} = -u^\mu_{;\mu}(\rho + p), \quad (18)$$

$$\dot{u}_\mu = -\frac{h^\nu_{\mu} p_{;\nu}}{\rho + p}, \quad (19)$$

where the dot denotes a derivative with respect to comoving time and the tensor $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ projects onto comoving hypersurfaces.

We impose in addition that mass (or equivalently particle number) is conserved,

$$(\rho_M u^\mu)_{;\mu} = 0. \quad (20)$$

This will give us the usual dependence of the matter mass density on the scale factor of the Universe.

C. Thermodynamics

General thermodynamic relations connect p and ρ . In particular, consider the first law of thermodynamics, $Td(sV) = d(\rho V) + p dV$. In an adiabatic expansion, $d(sV) = 0$. Together with mass conservation, $d(\rho_M V) = 0$, this equation leads to

$$p = \rho_M \left(\frac{\partial \rho}{\partial \rho_M} \right)_s - \rho, \quad (21)$$

which allows p to be computed from an expression for ρ . It also allows the speed of sound to be written as

$$c_s^2 = \left(\frac{\partial p}{\partial \rho} \right)_s = \rho_M \left(\frac{\partial^2 \rho}{\partial \rho_M^2} \right)_s \bigg/ \left(\frac{\partial \rho}{\partial \rho_M} \right)_s. \quad (22)$$

We can define $w = p/\rho$, which in general will not be constant.

Using the ansatz given in Eqs. (11) and (12) in Eq. (21), we can now relate the Cardassian contributions to energy and pressure via

$$p_K = \rho_M \left(\frac{\partial \rho_K}{\partial \rho_M} \right)_s - \rho_K. \quad (23)$$

Again we can define $w_K = p_K/\rho_K$, which in general will differ from $w = p/\rho$ and will not be a constant.

D. Newtonian limit

Now we obtain the basic equations in the Newtonian limit. In Minkowski space, we write $u^\alpha = \gamma(1, \vec{v})$ and the metric $\eta^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$, where $\gamma = 1/\sqrt{1-v^2}$ and \vec{v} is the 3-dimensional fluid velocity. Then, $T_{\mu\nu} = \text{diag}(-\rho, p, p, p)$.

We can obtain the gravitational field equations. In a weak static field produced by a nonrelativistic mass density, the time-time component of the metric tensor is approximately given by $g_{00} = -(1 + 2\phi)$. Here ϕ is the Newtonian potential for the gravitational field $\vec{g} = -\vec{\nabla}\phi$. The Ricci scalar $R = G_{00} = 2\nabla^2\phi$. Poisson's equation follows from $R = 8\pi GT^\mu{}_\mu$ as

$$\nabla^2\phi = 4\pi G(\rho + 3p). \tag{24}$$

We also have

$$\vec{\nabla} \times \vec{g} = 0. \tag{25}$$

In the same weak field limit, and for a fluid moving nonrelativistically, the energy conservation and Euler's equations can be found following Ref. [17], but without assuming $p \ll \rho$. From $T^{0\beta}{}_{;\beta} = 0$ we find

$$\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot [(\rho + p)\vec{v}] = 0. \tag{26}$$

From $T^{i\beta}{}_{;\beta} = 0$, we find Euler's equation,

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho + p} - \vec{\nabla}\phi. \tag{27}$$

Notice that we do not assume $p \ll \rho$ in the right-hand side of Eqs. (24)–(27).

With the additional constraint of particle number conservation, we have the ordinary continuity equation for matter,

$$\frac{\partial\rho_M}{\partial t} + \vec{\nabla} \cdot (\rho_M\vec{v}) = 0. \tag{28}$$

Using this equation, and in those situations where the internal energy and pressure can be neglected (e.g. when considering the overall expansion of the Universe), we can rewrite Eq. (26) in the following way:

$$\vec{v} \cdot \vec{\nabla} p_K + \left(\rho_K + p_K - \rho_M \frac{\partial\rho_K}{\partial\rho_M} \right) \vec{\nabla} \cdot \vec{v} = 0. \tag{29}$$

Note that Eq. (29) in the homogeneous background of our Universe reproduces Eq. (23), here in the Newtonian limit. Hence our basic nonrelativistic equations are Eqs. (24)–(28).

E. Friedmann-Robertson-Walker models

As mentioned above, Friedmann-Robertson-Walker cosmological models take the usual form [17]. The scale factor of the Universe a obeys the equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{30}$$

and the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{31}$$

where $k = 0, \pm 1$ fixes the curvature of the spatial sections and $H = \dot{a}/a$ is the Hubble parameter. As mentioned previously, we consider only a flat Universe with $k = 0$, as motivated by microwave background data [3]. Of course, the total energy density here contains new terms and is given by Eq. (11).

Cardassian expansion requires that the Universe be accelerating today. From Eq. (30), at matter densities of the order of the matter density in the Universe today, $\rho_M \approx \rho_{M0}$, we require $\rho_0 = \rho(\rho_{M0})$ and $p_0 = p(\rho_{M0})$ to satisfy

$$\rho_0 + 3p_0 < 0 \tag{32}$$

so as to have an accelerating Universe. With p_K given in Eq. (23), one can see that acceleration results if $\rho_M - 2\rho_K + 3\rho_M(\partial\rho_K/\partial\rho_M) < 0$. In the limit where $\rho_K \gg \rho_M$, one can see that acceleration results as long as ρ_K goes to zero faster than $\rho_M^{2/3}$ (for $\rho_K = b\rho_M^n$, this requirement becomes $n < 2/3$ as stated previously).

The energy conservation equation in an FRW model is

$$\dot{\rho} = -3H(\rho + p). \tag{33}$$

Particle number conservation gives

$$\dot{\rho}_M = -3H\rho_M, \tag{34}$$

from which we obtain the usual scaling $\rho_M \sim a^{-3}$ that was used in Eq. (7).

IV. EXAMPLES

In this section we discuss three different examples of barotropic Cardassian models, where the new contribution to the energy density is a function only of the mass density (mass times number density), $\rho_K \equiv \rho_K(\rho_M)$. In another paper we examine the consequences for supernova data of these three models [18].

A. Original power law Cardassian model

In the original Cardassian model of Ref. [12],

$$\rho_K = b\rho_M^n = \rho_M \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n}, \tag{35}$$

with $n < 2/3$ as in Eqs. (2) and (6). The pressure associated with this model in the fluid approach follows from Eq. (23) as

$$p_K = -(1-n)\rho_M \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n}. \tag{36}$$

Notice that

$$p_K = -(1-n)\rho_K. \quad (37)$$

This model therefore has a constant negative $w_K = p_K/\rho_K = -(1-n)$. Then

$$p_K + \rho_K = n \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n} \rho_M. \quad (38)$$

The speed of sound in this model

$$c_s^2 = - \frac{n(1-n)}{n + \left(\frac{\rho_M}{\rho_{\text{Card}}} \right)^{1-n}} \quad (39)$$

is not guaranteed to be positive. So this model should be considered as an effective description at scales where $c_s^2 > 0$.

1. Basic equations in the Newtonian limit for the power law Cardassian model

For the ρ^n Cardassian model, the basic Newtonian equations become

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{\vec{\nabla} p}{\rho + p} + \vec{g}, \quad (40)$$

$$\vec{v} \cdot \vec{\nabla} p_K + \left[p_K + (1-n) \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n} \rho_M \right] \vec{\nabla} \cdot \vec{v} = 0, \quad (41)$$

$$\vec{\nabla} \times \vec{g} = 0, \quad (42)$$

$$\vec{\nabla} \cdot \vec{g} = -4\pi G \left[\rho_M - (2-3n) \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n} \rho_M \right], \quad (43)$$

$$\frac{\partial \rho_M}{\partial t} + \vec{\nabla} \cdot (\rho_M \vec{v}) = 0. \quad (44)$$

2. Problem on galactic scales

The fluid approach to the ρ^n model cannot be used on galactic scales. From Eqs. (40) and (38), we see that

$$\frac{d\vec{v}}{dt} = - \frac{\vec{\nabla} p_M + \vec{\nabla} p_K}{\rho_M [1 + n(\rho_{\text{Card}}/\rho_M)^{1-n}]} + \vec{g} \quad (45)$$

where we have dropped $p_M \ll \rho_M$ and $\rho_{\text{internal}} \ll \rho_M$ in the denominator. In standard cosmology, one would have

$$\frac{d\vec{v}}{dt} = - \frac{\vec{\nabla} p_M}{\rho_M} + \vec{g}, \quad (46)$$

where p_M is quite small and gravity dominates. Here, the fluid version of the ρ^n Cardassian cosmology has an additional term in both numerator and denominator. Since $\rho_{\text{Card}} \ll \rho_M$ throughout most of the interiors of galaxies, we can ignore the second term in the denominator. However, the

second term in the numerator can have drastic effects. Let us consider, e.g., the position of the Sun in the Milky Way. The local energy density of matter is roughly $\rho_M \approx 10^4 \rho_{\text{Card}}$. Hence we have $\rho_M \gg \rho_K$, so that one might expect to be able to ignore any effects of Cardassian terms. However, these expectations are not true, because of the Cardassian pressure, which in this case dominates: $|p_K| \gg p_M$. Hence there is a new force that acts due to the second term in the numerator of Eq. (45),

$$\left. \frac{d\vec{v}}{dt} \right|_{\text{new}} = - \frac{\vec{\nabla} p_K}{\rho_M} = +n(1-n) \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{1-n} \frac{\vec{\nabla} \rho_M}{\rho_M}. \quad (47)$$

This force destroys flat rotation curves (velocities tend to increase as one goes out to large radii in an unacceptable way). The argument in the preceding paragraph has been presented using nonrelativistic equations, and as such is incomplete. However, we have found that the result that a problematic new force arises on galactic scales remains true in a fully relativistic treatment based on the Oppenheimer-Volkov equation.

The fluid ρ^n Cardassian case must therefore be thought of as an effective model, which applies only on cosmological scales. The examples in the following two sections, on the contrary, can be treated as fluid models on all scales, including cosmological scales as well as galactic scales.

B. Polytopic Cardassian model

Another class of models has

$$\rho = \rho_{\text{internal}} + \rho_{\text{Card}} \left[1 + \left(\frac{\rho_M}{\rho_{\text{Card}}} \right)^q \right]^{1/q} \quad (48)$$

with $q \neq 0$. This model can be used on all scales (see below), but it does not quite fit the criteria of the Cardassian model as defined in Eq. (1); at late times in the future of the Universe, when $\rho_M \ll \rho_{\text{Card}}$, this model becomes cosmological constant dominated with $\Lambda = \rho_{\text{Card}}$. Phenomenologically, this energy density is very similar to a model that was derived earlier [11] motivated by gravitational leakage into extra dimensions.

In this model, the pressure is

$$p = p_M - \rho_{\text{Card}} \left[1 + \left(\frac{\rho_M}{\rho_{\text{Card}}} \right)^q \right]^{(1/q)-1}. \quad (49)$$

When the ordinary internal energy density ρ_{internal} and the ordinary pressure p_M can be neglected, this model obeys a polytopic equation of state

$$p = -\rho_{\text{Card}} \left(\frac{\rho}{\rho_{\text{Card}}} \right)^{1-q}, \quad (50)$$

with negative pressure and negative polytopic index $N = -1/q$. For $q > 1$, the speed of sound in this model is positive,

$$c_s^2 = \frac{q-1}{1 + \left(\frac{\rho_M}{\rho_{\text{Card}}}\right)^q}. \quad (51)$$

Thus this fluid model can be used on all scales.

We must make sure that at the scales of galaxies and galaxy clusters the Cardassian pressure can be neglected compared to the ordinary pressure. At large matter densities, the Cardassian pressure is $|p_K| \approx \rho_M^{1-q} \rho_{\text{Card}}^q$. In a galaxy or cluster with velocity dispersion σ , the ordinary pressure is $p_M \approx \rho_M \sigma^2$. We want $|p_K|/p_M \approx (\rho_{\text{Card}}/\rho_M)^q / \sigma^2 \ll 1$. Taking $\sigma \approx 300$ km/s and assuming p_K is unimportant out to ≈ 100 kpc where $\rho_M \approx 10^2 \rho_{\text{Card}}$, this condition amounts to $q \gtrsim 3$. It is remarkable that this value of q is compatible with supernova data [18].

C. Modified polytropic Cardassian model

A Cardassian model that can be used on all scales is

$$\rho = \rho_{\text{internal}} + \rho_M \left[1 + \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{qv} \right]^{1/q}. \quad (52)$$

For $q=1$ this reduces to the original ρ^n Cardassian model with $n=1-\nu$. The pressure follows as

$$p = p_M - \nu \rho_M \left[1 + \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{qv} \right]^{(1/q)-1} \left(\frac{\rho_{\text{Card}}}{\rho_M} \right)^{qv}. \quad (53)$$

This model is interesting because the two parameters ν and q are important on different scales. The parameter ν sets the current value of $w \approx -\nu$, and so can be chosen to fit the supernova data, while the parameter q governs the suppression of the Cardassian pressure at high densities, and can therefore be chosen not to interfere with galactic rotation curves and cluster dynamics. Concrete comparisons with data will be presented in [18].

V. PERTURBATIONS

A. Newtonian theory

Here we calculate the behavior of small fluctuations, using the nonrelativistic equations derived above in Eqs. (24)–(28). As discussed below, there are gauge choices that render the Newtonian theory inadequate, yet we can learn from it nonetheless. We will write down the general perturbation equations for any generalized Cardassian model, and then as an example will solve them in the ρ^n form of the Cardassian equations (of paper I) to illustrate the type of results that occur.

For the zero-order solution [superscript (0)] we take the simple spatially uniform solution with

$$\rho_M^{(0)} = \rho_{M,0} \left[\frac{a_0}{a(t)} \right]^3 \quad (54)$$

$$\vec{v}^{(0)} = \frac{\dot{a}(t)}{a(t)} \vec{r} \quad (55)$$

$$\vec{g}^{(0)} = - \frac{4\pi G(\rho^{(0)} + 3p^{(0)})}{3} \vec{r}. \quad (56)$$

We now seek a perturbed solution [superscript (1)] by adding to the zero-order solution the small perturbations $\rho_M^{(1)}$, $\vec{v}^{(1)}$, $p_M^{(1)}$ and $\vec{g}^{(1)}$. Following [17], we find that the hydrodynamic equations (24)–(28) then give, to first order in these perturbations,

$$\dot{\vec{v}}^{(1)} + \frac{\dot{a}}{a} \vec{v}^{(1)} = - \frac{1}{\rho_M + \rho_K} (\vec{\nabla} p_M^{(1)} + \vec{\nabla} p_K^{(1)}) + \vec{g}^{(1)}, \quad (57)$$

$$\dot{\rho}_M^{(1)} + 3 \frac{\dot{a}}{a} \rho_M^{(1)} + \rho_M \vec{\nabla} \cdot \vec{v}^{(1)} = 0, \quad (58)$$

$$\vec{\nabla} \times \vec{g}^{(1)} = 0, \quad (59)$$

$$\vec{\nabla} \cdot \vec{g}^{(1)} = -4\pi G \left[1 + \frac{\partial}{\partial \rho_M} (\rho_K + 3p_K) \right] \rho_M^{(1)}. \quad (60)$$

We have dropped the superscript (0) from zero-order quantities. In deriving these equations we have neglected ρ_{internal} and p_M with respect to ρ_M .

We take the perturbations to be adiabatic, so that the ordinary pressure perturbation is given by

$$p_M^{(1)} = v_s^2 \rho_M^{(1)} \quad (61)$$

where v_s is the ordinary speed of sound. In the generalized Cardassian model with pressure and energy density related by Eq. (23), the perturbation equations require

$$p_K^{(1)} = \rho_M \frac{\partial^2 \rho_K}{\partial \rho_M^2} \rho_M^{(1)}. \quad (62)$$

Newtonian perturbation equations for the ρ^n Cardassian case

For the case of paper I in Eq. (2), Eqs. (62) and (59) become

$$\vec{\nabla} \cdot \vec{g}^{(1)} = -4\pi G [1 + n(3n-2)b\rho_M^{n-1}] \rho_M^{(1)} \quad (63)$$

and

$$p_M^{(1)} = bn(n-1)\rho_M^{n-1}\rho_M^{(1)}. \quad (64)$$

B. Newtonian theory: Non-expanding case

First, let us consider the non-expanding case with unperturbed velocity $\vec{v} = (\dot{a}/a)\vec{r} = 0$. A combination of the perturbation equations (57)–(60) yields

$$\frac{\partial^2 \delta_M}{\partial t^2} = v_{s,\text{new}}^2 \nabla^2 \delta_M + 4\pi G \rho_{M,\text{new}} \delta_M. \quad (65)$$

This is the usual perturbation equation for

$$\delta_M = \frac{\rho_M^{(1)}}{\rho_M}, \quad (66)$$

with v_s^2 replaced by

$$v_{s,\text{new}}^2 = \frac{\rho_M}{\rho_M + \rho_K} \left(v_s^2 + \rho_M \frac{\partial^2 \rho_K}{\partial \rho_M^2} \right) \quad (67)$$

and ρ_M replaced by

$$\rho_{M,\text{new}} = \rho_M \frac{\partial}{\partial \rho_M} (\rho + 3p) = \rho_M + \rho_M \frac{\partial}{\partial \rho_M} (\rho_K + 3p_K). \quad (68)$$

The equations are spatially homogeneous, so we expect to find plane-wave solutions. We take $\delta_M \propto \exp[i(\vec{k} \cdot \vec{x} - \omega t)]$ to find

$$\omega^2 = k^2 v_{s,\text{new}}^2 - 4\pi G \rho_{M,\text{new}}. \quad (69)$$

If $\omega^2 < 0$, then the perturbation is unstable to collapse. Hence we find the Jeans length

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\pi} \frac{v_{s,\text{new}}}{\sqrt{G \rho_{M,\text{new}}}}. \quad (70)$$

Perturbations on scales larger than the Jeans length can collapse.

In particular for the ρ^n Cardassian case of paper I in Eq. (2), we have

$$v_{s,\text{new}}^2 = \frac{v_s^2 - bn(1-n)\rho_M^{n-1}}{1 + nb\rho_M^{n-1}} \quad (71)$$

and

$$\rho_{M,\text{new}} = \rho_M [1 - bn(2-3n)\rho_M^{n-1}]. \quad (72)$$

Then the Jeans length becomes

$$\lambda_J = \sqrt{\frac{\pi}{G \rho_M [1 + nb\rho_M^{n-1}][1 - bn(2-3n)\rho_M^{n-1}]}}. \quad (73)$$

The Jeans length for ordinary cold dark matter was tiny, and the modifications here do not change it substantially enough to make it interesting.

C. Newtonian theory: Expanding case

Here we consider the expanding case with $\vec{v} = (\dot{a}/a)\vec{r} \neq 0$. We take the plane wave form for the solutions,

$$\rho_M^{(1)}(\vec{r}, t) = \rho_M(t) \delta(\vec{r}, t), \quad (74)$$

$$\delta(\vec{r}, t) = \delta(t) \exp\left[\frac{i\vec{r} \cdot \vec{q}}{a(t)}\right]. \quad (75)$$

After some algebra, we find

$$\ddot{\delta} + \frac{2\dot{a}}{a}\dot{\delta} + \left(v_{s,\text{new}}^2 \frac{q^2}{a^2} - 4\pi G \rho_{M,\text{new}} \right) \delta = 0, \quad (76)$$

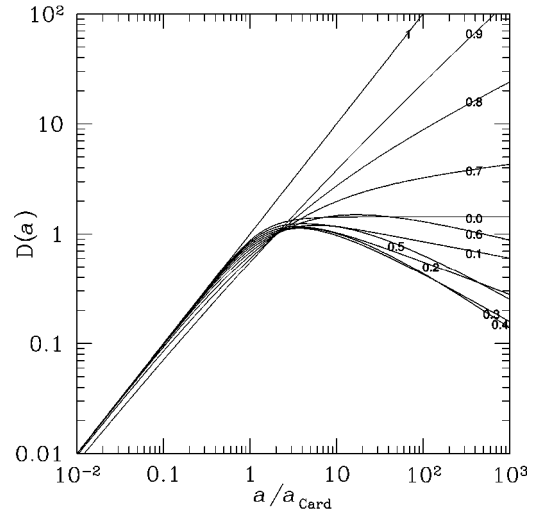


FIG. 1. Growth factor $D(a)$ for matter density perturbations in the original $\rho_K = b\rho_M^n$ Cardassian model as a function of the scale factor a . The normalization scale factor a_{Card} is defined by $\rho_M(a_{\text{Card}}) = \rho_K(a_{\text{Card}})$. Curves are labeled by the Cardassian index n . The curve with $n=1$ corresponds to the usual growth in a matter-dominated Universe, while $n=0$ corresponds to a cosmological constant.

where $v_{s,\text{new}}^2$ and $\rho_{M,\text{new}}$ are given in Eqs. (67) and (68).

For the power law ρ^n Cardassian model of Eq. (2), which is an effective model on large scales, we can drop the q^2 term. We have solved the resulting equation numerically after changing independent variable from t to $a(t)$. The solution for $\delta(a)$ (for modes with $\lambda \gg \lambda_J$) is given in Fig. 1 for varying values of the Cardassian index n . In the figure, a_{Card} is defined to be the scale factor at which the Cardassian term starts to dominate in the Friedmann equation, i.e. when $\rho_M(a_{\text{Card}}) = \rho_K(a_{\text{Card}})$. Today, a_0/a_{Card} is a factor of a few, but the figure extends to much higher values of a to show the behavior of the solutions. The vertical axis is normalized so that the solution becomes a/a_{Card} at early times; namely, using the solution in Ref. [19] for the growth of matter fluctuations in the radiation and matter-dominated eras of a cosmologically flat model,

$$D(a) = \frac{\delta(a)}{\delta_{\text{init}}} \frac{2a_{\text{eq}}}{3a_{\text{Card}}}, \quad (77)$$

where δ_{init} is the initial value of the perturbation and a_{eq} is the scale factor at matter-radiation equality. One can see in Fig. 1 that perturbation growth is suppressed once $a > a_{\text{Card}}$ for $0 \leq n < 1$. As a reminder, $n=1$ corresponds to ordinary matter whereas $n=0$ corresponds to a cosmological constant. To remind the reader, this figure corresponds to perturbation growth in an expanding Newtonian fluid in a box with the sides being pulled out.

D. General-relativistic theory

Since we do not neglect the pressure p with respect to the energy density ρ , the perturbations should in fact be studied in a general-relativistic theory. The problem arises of the

choice of gauge. In order to define the energy density perturbation throughout spacetime, one needs to choose a set of hypersurfaces; i.e., one needs to choose a gauge. Two choices are common:

(1) The comoving gauge corresponds to the set of comoving hypersurfaces, defined as those which are orthogonal to the comoving world lines, i.e., to the world lines which follow the flow of energy.

(2) The synchronous gauge corresponds to the hypersurfaces which are orthogonal to geodesics.¹

The value of density perturbations depends on the choice of gauge. However, typically, if one looks inside the horizon and well into the matter-dominated era, the value of the density perturbation becomes the same in all gauges.

In the case of Cardassian cosmology, we have the unusual circumstance that, even inside the horizon and well into the matter-dominated era, the value of the density perturbation depends on the choice of gauge.

The evolution of density perturbations in the Universe using a fluid flow approach was discussed in Ref. [16]. In the comoving gauge, with $w = p/\rho$ and the relation $\delta p/\delta\rho = c_s^2$ (true for barotropic Cardassian models), we find that the fractional perturbation $\delta = \delta\rho/\rho$ of momentum k obeys the equation [16]

$$H^{-2}\dot{\delta} + [2 - 3(2w - c_s^2)]H^{-1}\dot{\delta} - \frac{3}{2}(1 - 6c_s^2 + 8w - 3w^2)\delta = -\left(\frac{k}{aH}\right)^2 c_s^2 \delta. \quad (78)$$

The perturbation δ_s in the synchronous gauge is related to δ by [16]

$$\delta - \delta_s = 3H(1+w) \int_0^t \frac{\delta p}{\rho + p} dt, \quad (79)$$

if one drops the ‘‘gauge mode.’’ In ordinary cosmologies, where there is no Cardassian pressure term, the integral on the right-hand side has a fixed value after matter domination, obtained by setting $\delta p = 0$ in the matter-dominated era. A few Hubble times into the matter-dominated era it becomes negligible compared to the time scales of interest so that the comoving and synchronous gauges become identical. However, the values of the density perturbations in the two gauges do not become equal in Cardassian cosmology, when the Cardassian pressure δp_K is present. This term contributes to the integral in Eq. (79) all the way to the present time, and the comoving and synchronous gauges are not identical even today. This creates a problem of interpretation for fluctuations in the present Universe, which must be addressed in future studies.

¹There are, in fact, an infinity of synchronous gauges, but authors typically drop the so-called ‘‘gauge mode’’ solution so that the synchronous gauge becomes unique.

VI. ORIGIN OF INTERACTION ENERGY WITH NEGATIVE PRESSURE

Here we speculate on a possible origin for an interaction energy with a negative pressure. Dark matter particles may be subject to a new interparticle force which is long-range and confining,

$$F(r) \propto r^{\alpha-1} \quad (80)$$

with $\alpha > 0$. This force may be of gravitational origin or maybe a fifth force.

To be more quantitative, let us write the new interparticle potential as

$$U_{ij} = Ar_{ij}^\alpha, \quad (81)$$

where r_{ij} is the distance between particles and A is a normalization constant. The total new interaction energy of a system of N particles occupying a volume of radius R will be

$$U_{\text{new}} \simeq AN^2 R^\alpha, \quad (82)$$

to within a numerical factor of order 1 dependent on the geometry. The total gravitational potential energy of the same system is, also within a factor of order unity,

$$U_{\text{grav}} \simeq \frac{GM^2}{R}, \quad (83)$$

where M is the total mass of the system. To play a cosmological role at the present time, the new energy must be of the same order of the gravitational energy when $R \simeq R_H$, the current size of the horizon. Imposing that $U_{\text{new}} \simeq U_{\text{grav}}$ at $R \simeq R_H$ gives us the normalization

$$A = \frac{Gm^2}{R_H^{\alpha+1}}, \quad (84)$$

where $m = M/N$ is the mass of a single particle.

We can now find the magnitude of the new force on galactic scales. We have

$$U_{ij}(r) \simeq \frac{Gm^2}{R_H} \left(\frac{r}{R_H}\right)^\alpha. \quad (85)$$

Thus the new force per unit mass on a particle of mass m at distance R_g from a system of N particles is of order

$$\frac{F_{\text{new}}}{m} \simeq \frac{|\vec{\nabla} U_{\text{new}}|}{m} \simeq \alpha \frac{GM}{R_H^2} \left(\frac{R_g}{R_H}\right)^{\alpha-1}. \quad (86)$$

Compared with the gravitational force

$$\frac{F_{\text{grav}}}{m} \simeq \frac{GM}{R_g^2}, \quad (87)$$

this gives

$$\frac{F_{\text{new}}}{F_{\text{grav}}} \simeq \alpha \left(\frac{R_g}{R_H} \right)^{\alpha+1}. \quad (88)$$

For R_g of the order of galactic scales, i.e. $R_g \ll R_H$ the new force is negligible compared to the gravitational force. It is clear that for $\alpha > 0$ the new force is only important on very large scales. This Newtonian formulation must of course be modified at large distances because of the finite speed of light and issues of causality.

We want to comment on the equation of state of a system subject to long-range confining forces. If such a system reaches thermal equilibrium (that it does so in the presence of long-range confining forces is not at all clear), then simple statistical mechanics considerations based on the scaling of the partition function lead to the equation of state (see the Appendix for details)

$$p = -\frac{\alpha}{3}\rho. \quad (89)$$

This is the equation of state of the force mediators. For example, the Coulomb force (although not confining) has $\alpha = -1$ and equation of state $p = \rho/3$, which is that of photons. If $\alpha = 1$ (as in QCD) or $\alpha = 2$, the mediators are strings and two-dimensional objects, respectively, and their equations of state are $p = -(1/3)\rho$ and $p = -(2/3)\rho$, which are those of a network of strings and of domain walls, respectively. Finally, one obtains the vacuum equation of state $p = -\rho$ for $\alpha = 3$. Notice that the Cardassian index n in the power law ρ^n model is connected to the exponent α in the confining force law through $\alpha = 3(1 - n)$ [cf. e.g. Eqs. (37) and (89)]. That a confining force can give rise to an effective negative pressure is well known in particle physics, where the MIT bag model is just such an effective description of quark confinement. The cosmological negative pressure may be an indication that our observable Universe is in a big bag. This suggests that the dark energy may be the interaction energy associated to a long-range confining force.

VII. CONCLUSIONS

There is no unique four-dimensional or even higher-dimensional theory that gives Cardassian expansion. We have considered two different motivations for the modified Friedmann equations. First, we considered the possibility that these modifications arise from braneworld theories of gravity in higher dimensions [12,14]. In this paper we considered the alternative possibility that these modifications arise in a purely four-dimensional theory in which the right-hand side of the Friedmann equation has an extra contribution to the total energy density.

An interpretation of Cardassian expansion as an interacting dark matter fluid with negative pressure is developed. The Cardassian term on the right-hand side of the Friedmann equation (and of Einstein's equations) is interpreted as an interaction term. So the total energy density contains not only the matter density (mass times number density) but also interaction terms. These interaction terms give rise to an ef-

fective negative pressure which drives cosmological accelerating.

These interactions may be due to interacting dark matter, e.g. with a long-range confining force or a fifth force between particles. Alternatively, such interactions may be an effective description of higher dimensional physics. We have said that matter alone can be responsible for accelerated behavior. However, if the Cardassian behavior results from integrating out extra dimensions, then one may ask what behavior of the radii of the extra dimensions is required. Similarly, if we follow a QCD bag or other description of self-interacting dark matter, one may wonder if an equivalent vacuum description can be constructed. Further work in search of a fundamental origin of Cardassian expansion must be studied to answer these questions in detail.

A fully relativistic fluid model of Cardassian expansion has been developed, in which energy, momentum, and particle number are conserved, the modified Poisson's equations have been derived, and a preliminary study of density fluctuations in the early Universe has been presented.

One of the goals of this study is to allow predictions of various observables that will serve as tests of the model. The Cardassian model will have unique predictions, particularly due to the modified Poisson's equations. For example, one can now calculate the effect on the integrated Sachs-Wolfe effect in the cosmic microwave background [20]. In addition, one can now calculate the effect on cluster abundances as a function of redshift. These predictions can then be tested against existing and upcoming measurements of these quantities. Comparison with existing and upcoming supernova data is being studied in another paper [18]. We reiterate that this fluid approach is only one of the ways that Cardassian expansion may result.

ACKNOWLEDGMENTS

We thank Ted Baltz, Richard Easther, Josh Frieman, and Yun Wang for useful conversations. K.F. acknowledges support from the Department of Energy via the University of Michigan. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949 at the Kavli Institute for Theoretical Physics, University of California, Santa Barbara. K.F. thanks the Aspen Center for Physics for hospitality during her stay. P.G. thanks the Michigan Center for Theoretical Physics for hospitality.

APPENDIX: EQUATION OF STATE FOR A SYSTEM SUBJECT TO CONFINING FORCES

We give here the details of the derivation of the equation of state $p = -(\alpha/3)\rho$ for the mediators of a confining interparticle potential $U = Ar^\alpha$. Our derivation assumes that the gas is nonrelativistic and in thermodynamical equilibrium. We stress that it is not at all clear that a system of particles subject to long-range confining forces may reach thermal equilibrium. For our application to dark energy, it is not even clear that it could reach equilibrium on a time scale short compared to the age of the Universe. Therefore, the content of this appendix may be of somewhat academic interest.

Nevertheless, we present it for completeness.

The partition function of a nonrelativistic gas of N particles subject to a confining interparticle potential $U=Ar^\alpha$ is

$$Z(V,T)=\prod_{i=1}^N \int_V d^3r_i \int \frac{d^3p_i}{(2\pi)^3} \exp\left[-\sum_j \frac{p_j^2}{2mT} - \sum_{j<k} \frac{Ar_{jk}^\alpha}{T}\right] \quad (\text{A1})$$

where T is the temperature and V is the volume occupied by the system.

If we rescale $V\rightarrow\lambda^3V$ and $T\rightarrow\lambda^\alpha T$, and then change integration variables $r\rightarrow\lambda r'$, $p\rightarrow\lambda^{\alpha/2}p'$, we can prove that the partition function scales as

$$Z(\lambda^3V,\lambda^\alpha T)=\lambda^{3N+(3/2)\alpha N}Z(V,T). \quad (\text{A2})$$

Now the free (ideal gas) partition function

$$Z_0(V,T)=V^N\left(\frac{mT}{2\pi}\right)^{(3/2)N} \quad (\text{A3})$$

scales in the same way,

$$Z_0(\lambda^3V,\lambda^\alpha T)=\lambda^{3N+(3/2)\alpha N}Z_0(V,T). \quad (\text{A4})$$

It follows that

$$Z(V,T)=Z_0(V,T)Z_1(T^3/V^\alpha), \quad (\text{A5})$$

where $Z_1(x)$ is a function of the ratio T^3/V^α , which is invariant under the rescaling $V\rightarrow\lambda^3V$, $T\rightarrow\lambda^\alpha T$.

Pressure, entropy, and energy density can then be computed from the free energy

$$F=-T\ln Z(V,T)=-T\ln Z_0(V,T)-T\ln Z_1(T^3/V^\alpha) \quad (\text{A6})$$

as

$$P=-\left(\frac{\partial F}{\partial V}\right)_T=\frac{NT}{V}-\frac{\alpha NT}{V}f(T^3/V^\alpha), \quad (\text{A7})$$

$$U=F-T\left(\frac{\partial F}{\partial T}\right)_V=\frac{3}{2}NT+3NTf(T^3/V^\alpha), \quad (\text{A8})$$

where

$$f(x)=\frac{1}{N}\frac{x}{Z_1}\frac{dZ_1}{dx}. \quad (\text{A9})$$

The first terms on the right-hand sides of Eqs. (A7) and (A8) correspond to the ideal gas. The second terms are the pressure p and the energy ρV due to the confining forces. They obey the relation

$$p=-\frac{\alpha}{3}\rho. \quad (\text{A10})$$

Although we have not quantized the interaction, we draw on the analogy with electromagnetism described in the main text and call Eq. (A10) the equation of state for the force mediators.

-
- [1] Supernova Cosmology Project Collaboration, S. Perlmutter *et al.*, *Astrophys. J.* **517**, 565 (1999).
[2] Supernova Search Team Collaboration, A.G. Riess *et al.*, *Astron. J.* **116**, 1009 (1998).
[3] C.B. Netterfield *et al.*, *Astrophys. J.* **571**, 604 (2002); R. Stomp *et al.*, *Astrophys. J. Lett.* **561**, L7 (2001); N.W. Halverson *et al.*, *Astrophys. J.* **568**, 38 (2002); C. Pryke *et al.*, *ibid.* **568**, 46 (2002).
[4] 2dF Survey, L. Verde *et al.*, *Mon. Not. R. Astron. Soc.* **335**, 432 (2002).
[5] K. Freese, F.C. Adams, J.A. Frieman, and E. Mottola, *Nucl. Phys.* **B287**, 797 (1987).
[6] P.J.E. Peebles and B. Ratra, *Astrophys. J., Lett. Ed.* **325**, L17 (1988); B. Ratra and P.J.E. Peebles, *Phys. Rev. D* **37**, 3406 (1988).
[7] J. Frieman, C. Hill, A. Stebbins, and I. Waga, *Phys. Rev. Lett.* **75**, 2077 (1995).
[8] L. Wang and P. Steinhardt, *Astrophys. J.* **508**, 483 (1998).
[9] R. Caldwell, R. Dave, and P. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998).
[10] G. Huey, L. Wang, R. Dave, R. Caldwell, and P. Steinhardt, *Phys. Rev. D* **59**, 063005 (1999).
[11] C. Deffayet, *Phys. Lett. B* **502**, 199 (2001); C. Deffayet, G. Dvali, and G. Gabadadze, *Phys. Rev. D* **65**, 044023 (2002).
[12] K. Freese and M. Lewis, *Phys. Lett. B* **540**, 1 (2002).
[13] K. Freese, *Nucl. Phys. B (Proc. Suppl.)* **124**, 50 (2003).
[14] D.J. Chung and K. Freese, *Phys. Rev. D* **61**, 023511 (2000).
[15] S. Hawking, *Astrophys. J.* **145**, 544 (1966).
[16] D. Lyth and E. Stewart, *Astrophys. J.* **361**, 343 (1990).
[17] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[18] Y. Wang, K. Freese, P. Gondolo, and M. Lewis, *Astrophys. J.* **594**, 25 (2003).
[19] E.J. Groth and P.J.E. Peebles, *Astron. Astrophys.* **41**, 143 (1975).
[20] P. Gondolo, K. Freese, and M. Lewis (in preparation).