RELATIVE CURRENTS AND LOXODROMIC ELEMENTS IN THE RELATIVE FREE FACTOR COMPLEX

by

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ABSTRACT

A mapping class group element can be understood by an inductive process - by passing to its action on the curve complexes of the subsurfaces in the complement of the curves it fixes. By the result of Masur and Minsky, the curve complex of any surface of finite type is hyperbolic. A fully irreducible outer automorphism ($\mathrm{Out}(\mathbb{F})$ analog of a pseudo-Anosov) acts with positive translation length on the free factor complex, which is also a hyperbolic space. But a reducible outer automorphism Φ fixes the invariant free factor A in the free factor complex and thus, the action is not very informative. In analogy to subsurfaces, we then look at the action of Φ on the free factor complex relative to A, which is a hyperbolic complex that captures the information in the complement of A. In this dissertation, we prove that a fully irreducible outer automorphism relative to a free factor system $\mathcal{A}$ acts with positive translation length on the free factor complex relative to $\mathcal{A}$. In order to prove this, we prove the following key results:

- Define relative currents and prove that Φ acts with uniform north-south dynamics on a certain subspace of the space of projectivized relative currents.
- Φ acts with uniform north-south dynamics on the closure of relative outer space.
- Define an intersection form between the space of projective relative currents and the closure of relative outer space.

For my parents and brother.

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CHAPTER 1

INTRODUCTION

The study of $\text{Out}(\mathbb{F})$, the outer automorphism group of the free group $\mathbb{F}$ of finite rank, is highly influenced by the study of the mapping class group of a surface. Like the action of a pseudo-Anosov homeomorphism on the curve complex, a fully irreducible outer automorphism acts with positive translation length on the free factor complex. But a reducible outer automorphism fixes a point on this complex. In this dissertation, we take a step towards understanding reducible outer automorphisms that are fully irreducible relative to a free factor system $\mathcal{A}$ by studying their action on three different spaces - the free factor complex relative to $\mathcal{A}$ [HM14], the space of relative currents (Chapter 4) and relative outer space [GL07].

1.1 What is $\text{Out}(\mathbb{F})$?

A free group $\mathbb{F}$ of rank n is the fundamental group of a wedge of n circles. In order to understand $\mathbb{F}$, it is important to know how it transforms under automorphisms and hence, it is natural to study the group of automorphisms $\text{Aut}(\mathbb{F})$. An inner automorphism is given by conjugation by an element of $\mathbb{F}$ and so the group of inner automorphisms, $\text{Inn}(\mathbb{F})$, is isomorphic to $\mathbb{F}$. Thus one studies the *outer automorphism group*, defined as follows:

$$\text{Out}(\mathbb{F}) := \text{Aut}(\mathbb{F}) / \text{Inn}(\mathbb{F}).$$

The group $\text{Out}(\mathbb{F})$ can be thought of as the mapping class group of a wedge of circles or the group of homotopy equivalences of a wedge of circles which do not preserve a fixed point.

Early fundamental contributions to the study of $\text{Out}(\mathbb{F})$ were made by Whitehead and Nielsen. It acquired a strong geometric flavor by the influence of Gromov and Thurston and got a boost when Culler and Vogtmann defined a space called *Culler-Vogtmann's outer space*, which is an analog of the Teichmüller space, on which $\text{Out}(\mathbb{F})$ acts. Later, Bestvina and Handel developed a powerful geometric tool by adapting Thurston's train track technology to study outer automoprhisms. For a detailed history of $\mathrm{Out}(\mathbb{F})$, the reader is referred to [Vog02].

1.2 Mapping class group as a guiding example

Mapping class group of a surface Σ is the group of orientation preserving homeomorphism of Σ taken up to isotopy. The group $\mathrm{MCG}(\Sigma)$ acts on a simplicial complex called the *curve complex* $\mathcal{C}(\Sigma)$ which is defined as follows: vertices are given by homotopy class of essential, simple closed curves, and a k-simplex is given by a collection of $k+1$ vertices which can be realized mutually disjointly. In 1999, Masur and Minsky [MM99] showed that $\mathcal{C}(\Sigma)$ is hyperbolic and since then, it has played a crucial role in understanding $\mathrm{MCG}(\Sigma)$. Some remarkable applications include rigidity results for $\mathrm{MCG}(\Sigma)$, bounded cohomology for subgroups of $\mathrm{MCG}(\Sigma)$ and finite asymptotic dimension for $\mathrm{MCG}(\Sigma)$. Several analogues of the curve complex for $\mathrm{Out}(\mathbb{F})$ have been defined and proven to be hyperbolic, like the *free factor complex*, the *free splitting complex* and the *cyclic splitting complex*. But none of them have proven to be as useful as the curve complex.

For instance, when a mapping class group element acts on $\mathcal{C}(\Sigma)$ with a fixed point, that is, it fixes a curve α, then one can look at its action on the curve complex of the subsurface given by the complement of α. Thus mapping class group elements can be understood by an inductive process. On the other hand, consider an outer automorphism which fixes a free factor A in the free factor complex of $\mathbb{F}$. Since the complement of A in $\mathbb{F}$ is not well defined, one cannot pass to the free factor complex of a free group of lower rank. In [HM14], Handel and Mosher define *free factor complex relative to a free factor system* $\mathcal{FF}(\mathbb{F},\mathcal{A})$ which is an $\mathrm{Out}(\mathbb{F})$-analog of the curve complex for a subsurface. They also prove that these relative complexes are hyperbolic for *nonexceptional* free factor systems.

In order to draw parallels with the theory of subsurfaces used to understand $MCG(\Sigma)$, we take a step towards understanding the action of a certain subgroup of $\mathrm{Out}(\mathbb{F})$ that acts on the relative free factor complex. Our main theorem is a relative version of a result of [MM99] that a mapping class group element acts *loxodromically*, that is with positive translation length, on the curve complex if and only if it is a pseudo-Anosov homeomorphism.

Let $\mathrm{Out}(\mathbb{F},\mathcal{A})$ be the subgroup of $\mathrm{Out}(\mathbb{F})$ containing outer automorphisms that fix $\mathcal{A}$.

After passing to a finite index subgroup, assume that each free factor in $\mathcal{A}$ is invariant under the elements of $\mathrm{Out}(\mathbb{F}, \mathcal{A})$. An outer automorphism $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ is *fully irreducible relative to* $\mathcal{A}$ if no power of Φ fixes a nontrivial free factor system of $\mathbb{F}$ properly containing $\mathcal{A}$.

Theorem A. *Let $\mathcal{A}$ be a nonexceptional free factor system and let $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$. Then Φ acts loxodromically on $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ if and only if Φ is fully irreducible relative to $\mathcal{A}$.*

1.3 Pseudo-Anosovs are loxodromic for the curve complex

In order to motivate the different chapters of this dissertation and explain the proof strategy for Theorem A, we present a proof of the following theorem.

Theorem 1.3.1 ([MM99]). *Let Σ be an oriented surface of finite type and let $\Psi \in \mathrm{MCG}(\Sigma)$. Then Ψ acts loxodromically on $\mathcal{C}(\Sigma)$ if and only of Ψ is a pseudo-Anosov homeomorphism.*

The following proof is due to Bestvina and Fujiwara [BF02, Proposition 11].

Proof. Let Λ^+ and Λ^- be the attracting and repelling measured laminations associated to Ψ. Let $\mathcal{PML}(\Sigma)$ be the space of all projective measured laminations, which contains the curve complex as a subset. The following facts will be used later:

- The pseudo-Anosov Ψ acts on $\mathcal{PML}(\Sigma)$ with uniform north-south dynamics, that is, there are two fixed points Λ^+ and Λ^- and any compact set not containing $\Lambda^-(\Lambda^+)$ converges to $\Lambda^+(\Lambda^-)$ under $\Psi(\Psi^{-1})$-iterates.

- The intersection number $i(\cdot, \cdot)$ between two curves in the curve complex extends to a continuous, symmetric bilinear form $i : \mathcal{PML}(\Sigma) \times \mathcal{PML}(\Sigma) \to \mathbb{R}$.

- The fixed points Λ^+ and Λ^- are uniquely self-dual, that is, $i(\Lambda^\pm, \mu) = 0$ if and only if $\mu = \Lambda^\pm$.

If U is a neighborhood of Λ^+, then there exists a neighborhood of V of Λ^+, such that $V \subset U$ and if $a \in U^C$, $b \in V$, then $i(a, b) > 0$. Indeed, if this is not true, then find a sequence of neighborhoods $U \supset V_1 \supset V_2 \supset \ldots$ and curves $a_i \in U^C$ and $b_i \in V_i$ such that $\{b_i\}$ converges to Λ^+, $\{a_i\}$ converges to $a \neq \Lambda^+$ and $i(a_i, b_i) = 0$. But by continuity of the intersection number, $i(a_i, b_i)$ converges to $i(a, \Lambda^+)$ which is not zero. Such a pair is called a

UV-pair. Now consider a sequence of nested neighborhoods of Λ^+, $U_0 \supset U_1 \supset U_2 \supset U_3 \supset \ldots \supset U_{2N}$ for some $N > 0$, such that the following hold:

- (U_i, U_{i+1}) is a UV-pair for all $0 \leq i < 2N$.
- $\exists\, k > 0$ such that for all $0 \leq i < 2N$, $\Psi^k(U_i) \subset U_{i+1}$

Let a be a curve such that $a \in U_0$ and $a \notin U_1$. Given $\alpha \in U_i^C$ such that $i(\alpha, \beta) = 0$, then $\beta \in U_{i+1}$. Thus $d(a, \Psi^{2Nk}(a)) > N$ in the curve complex.

□

1.4 Dissertation aim

The proof due to Bestvina and Fujiwara can also be employed to prove that a fully irreducible outer automorphism acts loxodromically on the free factor complex (original proof in [BF10]). However, in this case, we need north-south dynamics on a certain space of measured currents ([Mar95], [Uya14]), north-south dynamics on the closure of outer space ([LL03]) and an intersection number between measured currents and $\mathbb{F}$-trees in the closure of outer space ([KL09]). The case of the fully irreducible automorphism will be referred to as the 'absolute case'.

Keeping in mind that we want to prove Theorem A using the Bestvina and Fujiwara strategy, we aim to do the following in this dissertation:

- Define relative currents. (Chapter 4)
- Show that a fully irreducible outer automoprhism relative to $\mathcal{A}$, denoted Φ, acts with uniform north-south dynamics on a certain subspace of the space of projective relative currents. (Chapter 4)
- Show that Φ acts with uniform north-south dynamics on the closure of relative outer space. (Chapter 5)
- Define an intersection form between relative currents and trees in relative outer space. (Chapter 6)
- Classify loxodromic elements for the free factor complex relative to a free factor system. (Chapter 7)

CHAPTER 2

BACKGROUND ON OUT($\mathbb{F}$)

In this chapter, we will review some basics about Out($\mathbb{F}$) and define objects that will be used throughout.

2.1 Outer space

In [CV86], Culler and Vogtmann defined outer space *(unprojectivized outer space)*, $CV_\mathfrak{n}$ ($cv_\mathfrak{n}$), as the space of $\mathbb{F}$-equivariant homothety *(isometry)* classes of minimal, free and simplicial action of $\mathbb{F}$ by isometries on metric simplicial trees with no vertices of valence two.

An $\mathbb{F}$-tree is an $\mathbb{R}$-tree with an isometric action of $\mathbb{F}$. An $\mathbb{F}$-tree is called *very small* if the action is minimal, arc stabilizers are either trivial or maximal cyclic and tripod stabilizers are trivial. Outer space can be embedded into $\mathbb{R}^\mathbb{F}$ via translation lengths of elements of $\mathbb{F}$ in a tree in $cv_\mathfrak{n}$ [CM87]. The closure of $CV_\mathfrak{n}$ under the embedding into $\mathbb{P}\mathbb{R}^\mathbb{F}$ was identified in [BF94] and [CL95] with the space of all very small $\mathbb{F}$-trees. Denote by $\overline{CV}_\mathfrak{n}$ the closure of outer space and by $\partial CV_\mathfrak{n}$ its boundary.

2.2 Marked graph and topological representative

We recall some basic definitions from [BH92]. Identify $\mathbb{F}$ with $\pi_1(\mathcal{R},*)$ where $\mathcal{R}$ is a rose with $\mathfrak{n}$ petals and $\mathfrak{n}$ is the rank of $\mathbb{F}$. A *marked graph* G is a graph of rank $\mathfrak{n}$, all of whose vertices have valence at least two, equipped with a homotopy equivalence $m : \mathcal{R} \to G$ called a *marking*. The marking determines an identification of $\mathbb{F}$ with $\pi_1(G, m(*))$.

A homotopy equivalence $\phi : G \to G$ induces an outer automorphism of $\pi_1(G)$ and hence an element Φ of Out($\mathbb{F}$). If ϕ sends vertices to vertices and the restriction of ϕ to edges is an immersion, then we say that ϕ is a *topological representative* of Φ.

A *filtration* for a topological representative $\phi : G \to G$ is an increasing sequence of (not necessarily connected) ϕ-invariant subgraphs $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_K = G$. The closure of $G_r \setminus G_{r-1}$, denoted H_r, is a subgraph called the r^{th}-*stratum*. Let γ be a reduced path in G. Then $\phi(\gamma)$ is the image of γ under the map ϕ. Denote the tightened image of $\phi(\gamma)$ by $[\phi(\gamma)]$.

A path σ is a *periodic Nielsen path* if σ is nontrivial and $\phi^k(\sigma)$ is homotopic relative to end points to σ for some $k \geq 1$. The minimal such k is the period of σ and if the period is one, then σ is a Nielsen path. A (periodic) Nielsen path is *indivisible*, denoted INP, if it does not decompose as a concatenation of nontrivial (periodic) Nielsen subpaths. A path σ is a *pre-Nielsen* path if $\phi^k(\sigma)$ is a Nielsen path.

2.3 Train track map

We recall some more definitions from [BH92]. A *turn* in a marked graph G is a pair of oriented edges of G originating at a common vertex. A turn is nondegenerate if the edges are distinct, and it is degenerate otherwise. A *turn* $(\mathbf{e_1}, \mathbf{e_2})$ *is contained in a filtration element* G_r if both e_1 and e_2 are contained in G_r. If γ is an edge path given by $e_1 \cdot e_2 \cdots e_{m-1} \cdot e_m$, then we say that γ *contains the turn* $(\overline{e_{i-1}}, e_i)$ where $\overline{e_i}$ denotes opposite orientation.

For $\phi : G \to G$, a topological representative and an edge e, set $T\phi(e)$ equal to the first oriented edge of the edge path $\phi(e)$. Given a turn (e_1, e_2), we define $T\phi(e_1, e_2) = (T\phi(e_1), T\phi(e_2))$. We say a turn is *illegal* if under some iterate of $T\phi$, the turn maps to a degenerate turn, it is *legal* otherwise. A path γ is called *r-legal* if all of its illegal turns are contained in G_{r-1}.

We associate a matrix called *transition matrix*, denoted M_r, to each stratum H_r. The ij^{th} entry of M_r is the number of occurrences of the i^{th} edge of H_r in either direction in the image of the j^{th} edge under ϕ. A nonnegative matrix M is called *irreducible* if for every i, j, there exists $k(i, j) > 0$ such that the ij^{th} entry of M^k is positive. A matrix is called *primitive* or *aperiodic* if there exists $k > 0$ such that M^k is positive. A stratum is called *zero stratum* if the transition matrix is the zero matrix. If M_r is irreducible, then its Perron-Frobenius eigenvalue λ_r is greater than or equal to 1. A stratum with an irreducible transition matrix is *exponentially growing (EG)* if $\lambda_r > 1$, it is called *nonexponentially growing (NEG)* otherwise.

Definition 2.3.1 (Relative train track map)**.** A topological representative $\phi : G \to G$ of a free group outer automorphism Φ is a *relative train track map* with respect to a filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_K = G$ if G has no valence one vertices, if each nonzero stratum has an irreducible matrix and if each exponentially growing stratum satisfies the following conditions:

- If E is an edge in H_r, then the first and the last edges in $[\phi(E)]$ are also in H_r.
- If $\gamma \in G_{r-1}$ is a nontrivial path with endpoints in $H_r \cap G_{r-1}$, then $[\phi(\gamma)]$ is a nontrivial path with endpoints in $H_r \cap G_{r-1}$.
- For each r-legal path $\beta \subset H_r$, $[\phi(\beta)]$ is r-legal.

A reduced path $\sigma \subset G$ has *height* r if the highest stratum it crosses is G_r.

2.4 Completely split train track map (CT)

In [FH11], Feighn and Handel defined completely split train track maps for outer automorphisms, which are better versions of relative train track maps. Instead of giving a complete definition, we list some facts which are used in Chapter 3 and then describe a complete splitting. Let $\phi : G \to G$ be a completely split train track map. The following facts proved in different papers can be found in [HM13, Section 1.5.2].

Facts 2.4.1. 1. Every periodic Nielsen path has period one.

2. If H_r is an EG stratum, then there is at most one indivisible Nielsen path (INP) in G_r that intersects H_r nontrivially.

3. If H_r is an EG stratum and if ρ_r is an INP of height r, then ρ_r crosses each edge of H_r at least once, the initial oriented edges of ρ_r and $\overline{\rho}_r$ are distinct oriented edges of H_r, and:

   (a) ρ_r is not closed iff it crosses some edge of H_r exactly once and in this case:

      i. at least one end point of ρ is not in G_{r-1}.

      ii. There does not exist a height r fixed conjugacy class.

   (b) ρ_r is closed iff it crosses each edge of H_r exactly twice, and in this case:

      i. the endpoint of ρ_r is not in G_{r-1}.

      ii. the only height r fixed conjugacy classes are those represented by ρ_r, its inverse and their iterates.

If H_r is an EG stratum, then a nontrivial path in G_{r-1} with end points in $H_r \cap G_{r-1}$ is called a *connecting path*. If an NEG stratum H_i is a single edge e_i such that $\phi(e_i) = e_i u_i$ for a nontrivial closed Nielsen path u_i, then e_i is called a *linear edge*. Let $u_i = w_i^{d_i}$ for some $d_i \neq 0$ where w_i is root-free. If e_i and e_j are distinct linear edges such that $\phi(e_i) = e_i w^{d_i}$ and $\phi(e_j) = e_j w^{d_j}$ where $d_i \neq d_j$ and $d_i, d_j > 0$, then a path of the form $e_i w^p \overline{e_j}$ where $p \in \mathbb{Z}$ is called an *exceptional path*.

A decomposition of a path or a circuit σ into subpaths is a called a *splitting* if one can tighten the image of σ under ϕ by tightening the image of each subpath. In other words, there is no cancellation between images of two adjacent subpaths in the decomposition of σ.

Let e be an edge in an irreducible stratum H_r and let $k > 0$. A maximal subpath σ of $[\phi^k(e)]$ in a zero stratum H_i is said to be *r-taken*. A nontrivial path or circuit in G is said to be *completely split* if it has a splitting into subpaths, each of which is either a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path or a connecting path in a zero stratum H_i that is taken and is maximal in H_i.

A relative train track map is *completely split* if for every edge e in each irreducible stratum $\phi(e)$ is completely split and if σ is a taken connecting path in a zero stratum, then $[\phi(\sigma)]$ is completely split.

2.5 BFH laminations

In [BFH00], Bestvina, Feighn and Handel defined a dynamic invariant called the attracting lamination associated to an EG stratum of a relative train track map $\phi : G \to G$. The elements of the lamination are called *leaves*.

Let $\mathcal{B}$ be the space of lines defined as the quotient of $\partial^2\mathbb{F} := (\partial\mathbb{F} \times \partial\mathbb{F} - \Delta)/\mathbb{Z}_2$ by the action of $\mathbb{F}$, where Δ denotes the diagonal. We say $\beta' \in \mathcal{B}$ is weakly attracted to $\beta \in \mathcal{B}$ under the action of Φ if $[\Phi^k(\beta')]$ converges to β. A subset $U \subset \mathcal{B}$ is an *attracting neighborhood* of β for the action of Φ if $[\Phi(U)]$ is a subset of U and if $\{[\Phi^k(U)] : k \geq 0\}$ is a neighborhood basis for β in $\mathcal{B}$. A bi-infinite path σ in a marked graph is *birecurrent* if every finite subpath of σ occurs infinitely often as an unoriented subpath of each end of σ. An element of $\mathcal{B}$ is birecurrent if some realization in a marked graph is birecurrent.

A closed subset Λ^+ of $\mathcal{B}$ is called an *attracting lamination* for a free group outer automorphism Φ if it is the closure of a line β that is bireccurent, has an attracting neighborhood for the action of some iterate of Φ and is not carried by a Φ-periodic free factor of rank one. The line β is said to be a *generic leaf* of Λ^+. In this paper, we will look at the lift of the attracting lamination to $\partial^2\mathbb{F}$ and denote it also by Λ^+.

Lemma 2.5.1 ([BFH00, Lemma 3.1.9]). *Suppose that $\phi : G \to G$ is a relative train track map with respect to a filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_K = G$ representing Φ and H_r is an aperiodic EG stratum. Then there is an attracting lamination Λ_r^+ with generic leaf β so that H_r is the highest stratum crossed by a realization of β in G.*

In Chapter 6, we will give a more general definiton of lamination associated to $\mathbb{F}$ due to Coulbois, Hilion and Lustig.

2.6 Free factor system

A free factor system of $\mathbb{F}$ is a finite collection of proper free factors of $\mathbb{F}$ of the form $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$, $k \geq 0$ such that there exists a free factorization $\mathbb{F} = A_1 * \cdots * A_k * F_N$, where $[\cdot]$ denotes the conjugacy class of a subgroup. We refer to the free factor F_N as the *cofactor* of $\mathcal{A}$ keeping in mind that it is not unique, even up to conjugacy. There is a partial ordering $\sqsubset$ on the set of free factor systems given as follows: $\mathcal{A} \sqsubset \mathcal{A}'$ if for every $[A_i] \in \mathcal{A}$ there exists $[A'_j] \in \mathcal{A}'$ such that $A_i \subset A'_j$ up to conjugation. The free factor systems $\emptyset$ and $\{[\mathbb{F}]\}$ are called *trivial free factor systems*. Define *rank*$(\mathcal{A})$ to be the sum of the ranks of the free factors in $\mathcal{A}$ and let $\zeta(\mathcal{A}) = k + N$.

Example 2.6.1. The main geometric example of a free factor system is as follows: suppose G is a marked graph and K is a subgraph whose noncontractible connected components are denoted $C_1, \ldots, C_k$. Let $[A_i]$ be the conjugacy class of a free factor of $\mathbb{F}$ determined by $\pi_1(C_i)$. Then $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$ is a free factor system. We say $\mathcal{A}$ is *realized by* K and denote it by $\mathcal{F}(K)$.

2.7 Relative free factor complex

Let $\mathcal{A}$ be a nontrivial free factor system of $\mathbb{F}$. In [HM14], the complex of free factor systems of $\mathbb{F}$ relative to $\mathcal{A}$, denoted $\mathcal{FF}(\mathbb{F};\mathcal{A})$, is defined to be the geometric realization of the partial ordering $\sqsubset$ restricted to the set of nontrivial free factor systems $\mathcal{B}$ of $\mathbb{F}$ such that $\mathcal{A} \sqsubset \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$. The *exceptional* free factor systems are certain ones for which $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is either empty or zero-dimensional. They can be enumerated as follows:

- $\mathcal{A} = \{[A_1], [A_2]\}$ with $\mathbb{F} = A_1 * A_2$. In this case, $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is empty.
- $\mathcal{A} = \{[A]\}$ with $\mathbb{F} = A * \mathbb{Z}$. In this case, $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is 0-dimensional.
- $\mathcal{A} = \{[A_1], [A_2], [A_3]\}$ with $\mathbb{F} = A_1 * A_2 * A_3$. In this case, $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is also 0-dimensional.

Theorem 2.7.1 ([HM14])**.** *For any nonexceptional free factor system $\mathcal{A}$ of $\mathbb{F}$, the complex of free factor systems of $\mathbb{F}$ relative to $\mathcal{A}$ is positive dimensional, connected and hyperbolic.*

Definition 2.7.2 ($\mathrm{Out}(\mathbb{F}, \mathcal{A})$)**.** The group $\mathrm{Out}(\mathbb{F}, \mathcal{A})$ is the subgroup of $\mathrm{Out}(\mathbb{F})$ containing outer automorphisms that fix $\mathcal{A}$. After passing to a finite index subgroup, assume that each free factor in $\mathcal{A}$ is invariant under the elements of $\mathrm{Out}(\mathbb{F}, \mathcal{A})$.

$\mathrm{Out}(\mathbb{F}, \mathcal{A})$ acts on $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ as follows: for $\Psi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ and $\mathcal{D} \in \mathcal{FF}(\mathbb{F}, \mathcal{A})$, $\Psi \cdot \mathcal{D} = \Psi(\mathcal{D})$.

2.8 Fully irreducible relative to a free factor system

Let $\mathcal{A}$ be a nontrivial free factor system. An outer automorphism $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ is called *irreducible relative to* $\mathcal{A}$ if there is no nontrivial Φ-invariant free factor system that properly contains $\mathcal{A}$. If every power of Φ is irreducible relative to $\mathcal{A}$, then we say that Φ is *fully irreducible relative to* $\mathcal{A}$ (or relative fully irreducible).

Let $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$. Then by [BFH00, Lemma 2.6.7], there exists a relative train track map for Φ, denoted $\phi : G \to G$, and filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_r = G$ such that $\mathcal{A} = \mathcal{F}(G_s)$ for some filtration element G_s. If Φ is fully irreducible relative to $\mathcal{A}$, then $\mathcal{A} = \mathcal{F}(G_{r-1})$ and the top stratum H_r is an EG stratum with Perron-Frobenius eigenvalue $\lambda_\Phi > 1$.

Example 2.8.1. Here is an example of a relative fully irreducible outer automorphism when rank of cofactor of $\mathcal{A}$ is zero. Let $\mathbb{F} = \langle a, b, c\rangle$ and let $\mathcal{A} = \{[\langle a\rangle], [\langle b\rangle], [\langle c\rangle]\}$. Let Φ be an outer automorphism given by

$$\Phi(a) = a, \Phi(b) = aCbcA, \Phi(c) = CbcBc.$$

Let $\phi : G \to G$ be a relative train track representative of Φ with G as in **Figure 2.1**. The marking is given by

$$a \to e_1, b \to e_1e_4e_2E_4E_1, c \to e_5e_3E_5$$

and the map ϕ is given as follows

$$\begin{array}{lll} \phi(e_1) = e_1 & \phi(e_2) = e_2 & \phi(e_3) = e_3 \\ \phi(e_4) = e_5E_3E_5e_1e_4 & \phi(e_5) = e_5E_3E_5e_1e_4e_2E_4E_1e_5 & \end{array}$$

2.9 North-south dynamics

Let X be a topological space. Let $f : X \to X$ be a homeomorphism. The map f is said to have *pointwise north-south dynamics* if there are two points x^+ and x^- in X which are fixed by f and any $x \neq x^-(x^+)$ converges to $x^+(x^-)$ under $f(f^{-1})$-iterates.

The map f as above is said to have *uniform north-south dynamics* if the following hold: there are two fixed points x^+ and x^- and for any compact set K in $X \setminus x^-(x^+)$ and neighborhood $U^+(U^-)$ of $x^+(x^-)$, there exists $M^+(M^-)$ such that for all $m \geq M^+(M^-)$,

$$f^m(K) \subseteq U^+(f^{-m}(K) \subseteq U^-).$$

If the space X is compact, then by [HK53], point-wise north-south dynamics is equivalent to uniform north-south dynamics.

2.10 Loxodromic element

Let X be a metric space and let $f : X \to X$ be a homeomorphism. Then f is a loxodromic element if for some (any) $x \in X$,

$$\lim_{N\to\infty} \frac{d(x, f^N(x))}{N} > 0.$$

For example, a hyperbolic isometry of the hyperbolic plane is a loxodromic element, a pseudo-Anosov is loxodromic for the action on the curve complex [MM99] and a fully irreducible outer automorphism is loxodromic for the action on the free factor complex [BF94].

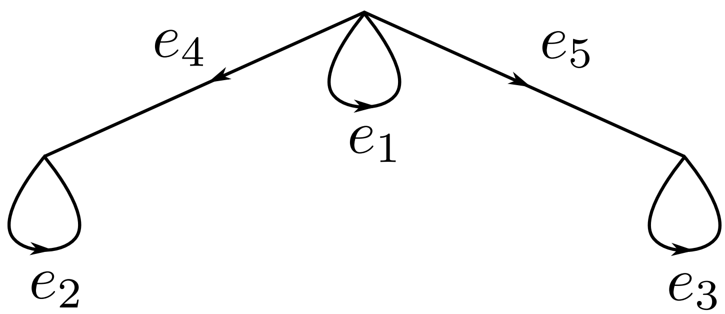

Figure 2.1. The graph G for Example 2.8.1

CHAPTER 3

CT TRAIN TRACK MAP AS A SUBSTITUTION

A train track representative $\psi : \Gamma \to \Gamma$ of a fully irreducible outer automorphism Ψ can be viewed as a substitution since the image of an edge in Γ is legal. If a is an edge of Γ such that $\psi(a)$ starts with a, then we get a ray ρ_a which is invariant under ψ. The results of [Que87] about primitive substitutions can be used to calculate the frequency of occurrence of subpaths in ρ_a, which turn out to be independent of the edge a. These frequencies of subpaths in turn give rise to a 'measured current' intrinsically associated to Ψ. Detailed discussion of currents follows in Chapter 4.

In the next chapter, we want to associate similarly defined currents to a fully irreducible outer automorphism relative to a free factor system. Since the transition matrix of a relative train track representative of such an automorphism is not primitive, the results of [Que87] cannot be applied directly. In this chapter, we

- generalize the results on substitution dynamics for primitive substitutions to more general substitutions,
- discuss how to view a completely split train track map as a substitution for the purpose of calculating frequency of subpaths in a fixed ray.

3.1 Preliminaries

Let $\mathbb{A}$ be a finite set with cardinality greater than or equal to two. Let ζ be a substitution on $\mathbb{A}$, that is, a map from $\mathbb{A}$ to the set of nonempty words on $\mathbb{A}$ which associates to a letter $e \in \mathbb{A}$ the word $\zeta(e)$ with length $|\zeta(e)|$. The substitution ζ induces a map on the set of all words on $\mathbb{A}$ by concatenation, that is,

$$\zeta(x_1 x_2 \ldots x_m) = \zeta(x_1)\zeta(x_2)\ldots\zeta(x_m)$$

where $x_1 x_2 \ldots x_m$ is a word on $\mathbb{A}$. Thus we define iterates ζ^n for all $n \geq 1$. To the substitution ζ, we associate its transition matrix, denoted M, where for $a, b \in \mathbb{A}$, $M(a, b)$

is the number of occurrence of a in $\zeta(b)$. The transition matrix for ζ^n is given by M^n. Likewise, define a map from $\mathbb{A}^{\mathbb{N}}$ to $\mathbb{A}^{\mathbb{N}}$, the set of all infinite words on $\mathbb{A}$, also denoted ζ, by the formula $\zeta(x_1x_2\ldots) = \zeta(x_1)\zeta(x_2)\ldots$. We are interested in possible fixed points or periodic points of ζ.

Lemma 3.1.1 (Lemma 5.1, [Que87]). *Let ζ be a substitution on an alphabet $\mathbb{A}$ such that*

$$\lim_{n\to\infty} |\zeta^n(a)| = +\infty$$

for every $a \in \mathbb{A}$. Then ζ admits periodic points, that is, there exists $\rho \in \mathbb{A}^{\mathbb{N}}$, $k \geq 1$ such that $\zeta^k(\rho) = \rho$.

Suppose ζ admits a fixed point, denoted $\rho \in \mathbb{A}^{\mathbb{N}}$, such that $\zeta^k(\rho) = \rho$ for all $k \geq 1$. From now on, only keep in the alphabet $\mathbb{A}$ the letters that actually appear in ρ.

For every $l > 0$, let $\mathbb{A}_l$ denote the set of all words on $\mathbb{A}$ of length l that appear in ρ. Define a substitution ζ_l on $\mathbb{A}_l$ as follows: let $w = x_1x_2\ldots x_l \in \mathbb{A}_l$. Define $\zeta_l(w) := w_1w_2\ldots w_{|\zeta(x_1)|}$ where $w_i \in \mathbb{A}_l$ and w_i is the length l subword of $\zeta(w)$ starting at the i^{th} position of $\zeta(x_1)$. In other words, $\zeta_l(w)$ consists of the ordered list of the first $|\zeta(x_1)|$ subwords of length l of the word $\zeta(w)$. The substitution ζ_l extends to a map on the set of all words on $\mathbb{A}_l$. Denote by $|\cdot|_l$ the length of words on $\mathbb{A}_l$. We have $|\zeta_l(w)|_l = |\zeta(x_1)|$. Let M_l denote the transition matrix for ζ_l. It is clear from definitions that $(\zeta^n)_l = (\zeta_l)^n$.

Lemma 3.1.2 (Lemma 5.2, [Que87]). *If $\rho = x_1x_2\ldots$ is a fixed point for ζ, then $\rho_l \in \mathbb{A}_l^{\mathbb{N}}$ is a fixed point of ζ_l where $\rho_l = (x_1x_2\ldots x_l)(x_2\ldots x_{l+1})\ldots$.*

3.2 Primitive substitution

A substitution is called *irreducible* if for every pair $a, b \in \mathbb{A}$, there exists $k := k(a, b)$ such that a occurs in $\zeta^k(b)$. A substitution is called *primitive* if there exists k such that for every pair $a, b \in \mathbb{A}$, a occurs in $\zeta^k(b)$.

Theorem 3.2.1 (Lemma 5.3, 5.4 [Que87]). *If the substitution ζ is primitive, then for every $l \geq 2$, ζ_l is also primitive with the same Perron-Frobenius eigenvalue as ζ.*

For u, w two words on $\mathbb{A}$, let $L_u(w)$ denote the number of times u occurs in w. The following two lemmas tell us about the frequency of occurrence of subwords of ρ in ρ.

Proposition 3.2.2 (Proposition 5.8, 5.9 [Que87]). *Let ζ be a primitive substitution. Let $a \in \mathbb{A}$. Then*

(a) for every $b \in \mathbb{A}$
$$\lim_{n\to\infty} \frac{L_b(\zeta^n(a))}{|\zeta^n(a)|} = d_b$$
where d_b is positive, independent of a and $\sum_{b\in\mathbb{A}} d_b = 1$.

(b) for every subword w of ρ,
$$\lim_{n\to\infty} \frac{L_w(\zeta^n(a))}{|\zeta^n(a)|} = d_w$$
where d_w is independent of a and is positive.

We want to generalize the above results to substitutions which are not necessarily primitive but are primitive on a subset of the alphabet.

3.3 Nonprimitive substitution

Consider an alphabet $\mathbb{A} = \bigsqcup_{i=0}^{k} B_i$. Define a *partial order on the alphabet* as follows. First define a partial order on subsets of $\mathbb{A}$ given by $B_i > B_j$ for $i < j$. For example, $B_0 > B_1$ and so on. Thus we get a partial ordering on the letters of $\mathbb{A}$ where $a > b$ if $a \in B_i$ and $b \in B_j$ where $i < j$. The alphabet $\mathbb{A}_l$ can now be given a partial lexicographic order as well. From now on, we will consider a substitution ζ on $\mathbb{A}$ with the following properties:

- For $a \in B_i$, $\zeta(a)$ contains letters only from B_j for $j \geq i$. This implies that the transition matrix M for ζ is lower triangular block diagonal with respect to the partial order on the set $\{B_i\}_{i=0}^{k}$. Denote the diagonal blocks of M also by B_i for $0 \leq i \leq k$ where B_0 is the top left block, followed by B_1 and so on.

- If B_i is a primitive block, then $\zeta(a)$ for $a \in B_i$ ends and begins in a letter in B_i.

- B_0 is primitive.

Lemma 3.3.1. *Let B_i be a primitive block of M. After possibly passing to a power of ζ, there exists $a \in B_i$ such that $\zeta(a)$ begins in a. Also $\rho_a := \lim_{n\to\infty} \zeta^n(a)$ is fixed by ζ, that is, $\zeta(\rho_a) = \rho_a$. If $b \in B_i$ is another letter which begins in b and ρ_b is fixed by ζ, then the set of subwords of ρ_a and ρ_b are the same.*

Proof. Consider a function $f : B_i \to B_i$ where for $a \in B_i$, $f(a)$ is the first letter of $\zeta(a)$. Since B_i is a finite set, some power of f has a fixed point. After possibly passing to a power, let $a \in B_i$ be a fixed point of f. Since $\zeta(a)$ begins with a, we have that $\zeta^n(a)$ begins with $\zeta^{n-1}(a)$ for every $n > 0$. Thus ρ_a is fixed by ζ. Since B_i is a primitive block, $\zeta^m(a)$ contains b for some $n > 0$. Thus subwords that appear in ρ_b also appear in ρ_a and vice versa. □

Example 3.3.2. Let $\mathbb{A} = \{a, b, c, d\}$. Let ζ be given as $\zeta(a) = abbab, \zeta(b) = bababbab, \zeta(c) = cad, \zeta(d) = dcad$. The transition matrices for ζ and ζ_2 are given by

$$M = \begin{array}{c} \begin{array}{cccc} c & d & a & b \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{bmatrix} \end{array}, \quad M_2 = \begin{array}{c} \begin{array}{cccccccc} ca & da & dc & ad & bd & ab & ba & bb \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 \end{bmatrix} \end{array}.$$

3.4 Eigenvalues for M and M_l

We now want to understand the spectrum of M_l and relate it to the spectrum of M. The main result from this section is Proposition 3.4.1.

Proposition 3.4.1. *For every $l \geq 2$, the eigenvalues of M_l are those of M with possibly some additional eigenvalues of absolute value less than equal or to one.*

The three lemmas that follow will be used to prove Proposition 3.4.1.

Notation 3.4.2. Since $(\zeta^n)_l = (\zeta_l)^n$, we have $(M^n)_l = (M_l)^n$, which is denoted by M_l^n unless the order needs to be specified. Denote the rows and columns of M by R_x and C_x for $x \in \mathbb{A}$, those of M_l by R_w and C_w and those of M_l^n by $R_{n,w}$ and $C_{n,w}$ for $w \in \mathbb{A}_l$.

Lemma 3.4.3. *Let $n \geq 2$. Let M, M_l, M_l^n be transition matrices for $\zeta, \zeta_l, \zeta_l^n$, respectively. Then*

(a) M_l is a lower triangular block diagonal matrix with respect to the partial order on $\mathbb{A}_l$.

(b) Let $w \in \mathbb{A}_l$ start with $x \in \mathbb{A}$. Then the sum of the entries of C_w is the same as the sum of the entries of C_x which is equal to $|\zeta(x)|$.

(c) *Let $w_1, w_2 \in \mathbb{A}_l$ be such that both words begin with $x \in \mathbb{A}$. Then the entries of C_{w_1} and C_{w_2} differ at most by $(l-1)$. The entries of C_{n,w_1} and C_{n,w_2} also differ at most by $(l-1)$.*

Proof. (a) Clear from definitions of M and M_l.

(b) Let w, x be as in the statement of the lemma. Then $|\zeta_l(w)|_l = |\zeta(x)|$, which implies that column sum of C_w is the same as that of C_x.

(c) Let w_1, w_2, x be as in the statement of the lemma. Then $\zeta_l(w_1)$ and $\zeta_l(w_2)$ differ only when the length l words starting at some position in $\zeta(x)$ are not subwords of $\zeta(x)$. If $|\zeta(x)| \geq l$, then the first time such a word occurs is when it starts at position $(l-1)$ from the end of $\zeta(x)$. If $|\zeta(x)| < l$, then $\zeta_l(w_1)$ and $\zeta_l(w_2)$ can differ in at most $|\zeta(x)| < l$ length l words. Thus there are at most $(l-1)$ such words. Replace ζ, ζ_l by $\zeta^n, (\zeta^n)_l$ above to conclude that entries of C_{n,w_1} and C_{n,w_2} also differ at most by $(l-1)$. □

Lemma 3.4.4. *If Q is a $s \times s$ matrix such that absolute values of all its entries are bounded above by $\delta > 0$, then the absolute values of the eigenvalues of Q are bounded above by $s\delta$.*

Proof. Let $\lambda \neq 0$ be an eigenvalue of Q and let $v = (v_1, \ldots, v_s)$ be a corresponding eigenvector. Let r_i denote rows of Q. Then $|r_i \cdot v| = |\lambda v_i|$, which gives $|\lambda v_i| \leq \delta \sum_{j=1}^{s} |v_j|$ for every $1 \leq i \leq s$. Adding all the inequalities together, we get $|\lambda| \leq s\delta$. □

Notation 3.4.5. We say a word w on $\mathbb{A}$ *crosses* B_i if w contains a letter in B_i. For every $B_i \subset \mathbb{A}$, let $\widetilde{B_i} \subset \mathbb{A}_l$ be the set of all words w that start with a letter in B_i and such that w does not cross B_j for any $j < i$. For every $B_i \subset \mathbb{A}$, let $\overline{B_i} \subset \mathbb{A}_l$ be the set of all words w that start with a letter in B_i and there exists a $j < i$ such that w crosses B_j (note that $\overline{B_0}$ is empty). Then $\widetilde{B_i} \cup \overline{B_i}$ is the union of all words of length l that start with a letter in B_i. The partial order on $\mathbb{A}_l$ defined earlier gives that $\widetilde{B_0} > \overline{B_1} > \widetilde{B_1} > \ldots > \overline{B_k} > \widetilde{B_k}$. The matrix M_l is lower triangular block diagonal with respect to this partial order on $\mathbb{A}_l$. For a subset $S \subset \mathbb{A}_l$, denote by S the transition matrix of ζ_l restricted to S.

Lemma 3.4.6. *(a) For every $0 \leq i \leq k$, the characteristic polynomial of B_i divides the characteristic polynomial of $\widetilde{B_i}$.*

(b) The eigenvalues of $\widetilde{B_i}$ are those of B_i with possibly some additional eigenvalues of absolute value less than or equal to one.

(c) *The eigenvalues of $\overline{B_i}$ have absolute value less than or equal to one.*

Proof. (a) Consider the matrix $P_i = \widetilde{B}_i - \lambda I$. We will do certain row and column operations on this matrix to reduce it to a lower triangular block diagonal matrix with $B_i - \lambda I$ as a diagonal block, which would imply that the characteristic polynomial of B_i divides the characteristic polynomial of $\widetilde{B}_i$. For later use, denote the other diagonal block of P_i by Q.

First perform the following row operations: for every $x \in B_i$, choose a word $w \in \widetilde{B}_i$ such that w starts with x. For every such w, replace the row R_w of $\widetilde{B}_i$ by the sum of rows R_u for all $u \in \widetilde{B}_i$ that start with x. Rearrange the rows and columns such that the top left block is indexed by the chosen words w. Denote the rearranged matrix by P_i'. The top left block of P_i' is exactly $B_i - \lambda I$. Indeed, suppose $w, u \in \widetilde{B}_i$ in the top left block of P_i' start with $x, y \in B_i$, respectively. Then $P_i'(w, v)$ is exactly the number of occurrences of x in $\zeta(y)$.

Now for any two columns C_{w_1} and C_{w_2} of P_i', where w_1, w_2 start with the same letter in B_i, the first few entries (as many as the number of rows in the top left block of P_i') are equal. Now perform column operations as follows: for every $x \in B_i$ and w the chosen word in the top left block, subtract C_w from C_u for every $u \neq w$ that start with x. Thus we have a lower triangular block diagonal matrix, again denoted P_i', with diagonal blocks $B_i - \lambda I$ and Q.

(b) Consider the lower block diagonal matrix P_i' from above. Eigenvalues of P_i' not coming from the block $B_i - \lambda I$ come from the lower block, denoted Q. By Lemma 3.4.3(c), the entries of Q are bounded in absolute value by $(l-1)$. We claim that the eigenvalues of Q are bounded in absolute value by one.

Let λ_0 be an eigenvalue of Q and hence of $\widetilde{B}_i$. Then for $n \geq 1$, λ_0^n is an eigenvalue of $(\widetilde{B}_i)^n$ which is a diagonal block of $(M_l)^n = (M^n)_l$. Thus λ_0^n is an eigenvalue of $(\widetilde{B}_i)^n$ that does not come from eigenvalue of B_i^n, the corresponding diagonal block of M^n. Applying part (a) to ζ^n, $(\widetilde{B}_i)^n$ can also be put in a lower triangular block diagonal form with diagonal blocks $B_i^n - \lambda I$ and Q' whose entries are bounded by $(l-1)$ and hence every eigenvalue bounded in absolute value by size of Q' times $(l-1)$ by Lemma 3.4.4.

Thus $|\lambda_0^n|$ is uniformly bounded, which can happen only when $|\lambda_0| \leq 1$.

Thus all eigenvalues of $\widetilde{B_i}$ are eigenvalues of B_i with the exception of some eigenvalues whose absolute value is less than or equal to one.

(c) Let λ be an eigenvalue of $\overline{B_i}$. Then λ^n is an eigenvalue of $(\overline{B_i})^n$, the diagonal block of $(M^n)_l$ corresponding to words that start with a letter in B_i and there exists a $j < i$ such that they cross B_j. For every n, the entries of $(\overline{B_i})^n$ are bounded by $(l-1)$. Indeed, if w is a length l word that starts with x, then only the words that start at some position less than l away from the last letter of $\zeta^n(x)$ belong to $(\overline{B_i})^n$. This implies that eigenvalues of $(\overline{B_i})^n$ are uniformly bounded. That is, $|\lambda^n|$ is uniformly bounded, which can happen only when $|\lambda| \leq 1$. □

Proof of Proposition 3.4.1. Since eigenvalues of a lower triangular block diagonal matrix are obtained from eigenvalues of each block, the proposition follows from Lemma 3.4.6. □

3.5 Frequency of words

The main result in this subsection is Proposition 3.5.5, which tells how to calculate the frequency of occurrence of words which cross B_0 in ρ.

Notation 3.5.1. Let λ be the top eigenvalue of the block B_0 of M. Consider a subset $\mathcal{B}_l := \widetilde{\overline{B_0}} \cup (\bigcup_{i=1}^k \overline{B_i})$ of $\mathbb{A}_l$. Then the set of all length l words that cross B_0 is a subset of $\mathcal{B}_l$. The transition matrix of ζ_l restricted to $\mathcal{B}_l$ is also lower triangular block diagonal with respect to the order $\widetilde{\overline{B_0}} > \overline{B_1} > \ldots > \overline{B_k}$ of words in $\mathcal{B}_l$. Then by Lemma 3.4.6, $\lambda > 1$ is the top eigenvalue of $\mathcal{B}_l$ with multiplicity one. Since $\mathcal{B}_l$ is a diagonal block of M_l, we have $M_l^n(w, \alpha) = \mathcal{B}_l^n(w, \alpha)$ for all $w, \alpha \in \mathbb{A}_l$ that cross B_0.

For w, v words on $\mathbb{A}$ or $\mathbb{A}_l$, let (w, v) denote the number of occurrences of w in v.

Lemma 3.5.2. *Let $a \in B_0$ and let $\rho_a = \lim_{n\to\infty} \zeta^n(a)$ be such that $\zeta(\rho_a) = \rho_a$. Let $w \in \mathbb{A}_l$ be a word that crosses B_0. Then*

$$\textit{frequency of occurrence of } w \textit{ in } \rho_a = \lim_{n\to\infty} \frac{(w, \zeta^n(a))}{\lambda^n} =: d_{w,a}$$

exists and is nonnegative. Here λ is the top eigenvalue of B_0.

Proof. Let $\alpha \in \mathbb{A}_l$ start with a. For n large, the number of occurrences of w in $\zeta^n(a)$ is approximately the same as the number of occurrences of w in $\zeta_l^n(\alpha)$. Also $(w, \zeta_l^n(\alpha)) = M_l^n(w, \alpha)$. We have

$$\lim_{n\to\infty} \frac{(w, \zeta^n(a))}{\lambda^n} = \lim_{n\to\infty} \frac{(w, \zeta_l^n(\alpha))}{\lambda^n} = \lim_{n\to\infty} \frac{M_l^n(w, \alpha)}{\lambda^n} = \lim_{n\to\infty} \frac{\mathcal{B}_l^n(w, \alpha)}{\lambda^n} =: d_{w,a}.$$

Indeed, the limit exists because λ is the top eigenvalue of $\mathcal{B}_l$. The limit is nonnegative because it is a sequence of nonnegative numbers. The limit does not depend on the exact choice of α because by Lemma 3.4.3(c), any two columns of M_l^n starting with the same letter in $\mathbb{A}$ differ by a bounded amount and thus give the same limit. □

Lemma 3.5.3 (Kirchhoff's Law)**.** *Let $a \in B_0$. Let $w \in \mathbb{A}_l$ cross B_0. Let we and ew be length one extensions of w by $e \in \mathbb{A}$. Then*

$$d_{w,a} = \sum_{e\in\mathbb{A}} d_{we,a} = \sum_{e\in\mathbb{A}} d_{ew,a}.$$

Proof. We have $(w, \zeta^n(a))$ and $\sum_{e\in\mathbb{A}}(we, \zeta^n(a))$ differ only when $\zeta^n(a)$ ends in w so the difference is at most one. Thus

$$\left| \frac{(w, \zeta^n(a))}{\lambda^n} - \sum_{e\in\mathbb{A}} \frac{(we, \zeta^n(a))}{\lambda^n} \right| \to 0 \text{ as } n \to \infty$$

which implies that $d_{w,a} = \sum_{e\in\mathbb{A}} d_{we,a}$. Similarly, $d_{w,a} = \sum_{e\in\mathbb{A}} d_{ew,a}$. □

Lemma 3.5.4. *Let $a, b \in B_0$ be distinct. Then*

$$d_{w,b} = \kappa d_{w,a}$$

for every word w that crosses B_0 where $\kappa = \kappa(a, b, \zeta|_{B_0})$.

Proof. Let's first consider the case when length of w is one. The substitution ζ restricted to B_0 is primitive with top eigenvalue $\lambda > 1$. Then

$$d_{w,a} = \lim_{n\to\infty} \frac{M^n(w, a)}{\lambda^n} = \lim_{n\to\infty} \frac{B_0^n(w, a)}{\lambda^n}.$$

Since B_0 is primitive, the limit of B_0^n/λ^n is a matrix P that is spanned by a positive eigenvector corresponding to λ. Since left eigenvector of B_0 is also positive, all columns of P are positive multiples of each other. Thus $d_{w,b} = P(w, b)$ is a scalar multiple of $d_{w,a} = P(w, a)$ which does not depend on w. Let this constant be denoted κ_1.

Now consider the case when length of w is l. We will first show that the constant κ_l, where $d_{w,b} = \kappa_l d_{w,a}$, does not depend on w and then show that $\kappa_l = \kappa_1$ for all $l \geq 2$. Since λ is the unique top eigenvalue of $\mathcal{B}_l$, $\lim_{n\to\infty} \mathcal{B}_l^n/\lambda^n$ is a matrix P whose column span is an eigenvector corresponding to λ. Thus $d_{w,b} = P(w,b)$ is a scalar multiple of $d_{w,a} = P(w,a)$ which does not depend on w. Let this constant be denoted κ_l.

Now we will show that $\kappa_l = \kappa_1$. Let w be a word of length one. We have $d_{w,b} = \sum_{e\in\mathbb{A}} d_{we,b}$. Also $d_{w,b} = \kappa_1 d_{w,a}$ and $d_{we,b} = \kappa_2 d_{we,a}$. Thus $\kappa_1 d_{w,a} = \kappa_2 \sum_{e\in\mathbb{A}} d_{we,a} = \kappa_2 d_{w,a}$, which implies $\kappa_2 = \kappa_1$. Repeat the same argument to get $\kappa_l = \kappa_1$ for every $l \geq 2$. $\square$

To summarize the results about substitutions, we have the following proposition.

Proposition 3.5.5. *Let ζ be a substitution on an alphabet $\mathbb{A}$ such that the transition matrix is lower triangular block diagonal with top left block B_0 primitive, and for every $e \in B_0$, $\zeta(e)$ starts and ends with a letter in B_0. Then there is a fixed infinite word ρ obtained by iterating a letter in B_0 under ζ. Moreover, the frequency of a word w on $\mathbb{A}$ in ρ that crosses B_0 is well defined up to scale and satisfies Kirchhoff's law.*

3.6 CT train track as a substitution

Let Φ be a free group outer automorphism. Let $\phi : G \to G$ be a completely split train track representative of Φ with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_K = G$. The transition matrix for ϕ, denoted M_ϕ, is lower triangular block diagonal. Let a be an edge in an EG stratum H_r such that up to taking powers $\phi(a)$ starts with a. Let $\rho_a = \lim_{n\to\infty} \phi^n(a)$. We want to understand the frequency of occurrence of paths in G_r that cross H_r and appear in ρ_a. We may not be able to treat ϕ as a substitution directly since there could be cancellations and inverse of edges would have to be treated separately. The proof of the next proposition explains how to view a completely split train track map as a substitution for the purpose of calculating frequencies of certain paths.

We set up some notation about exceptional paths that will be used in the next proposition. Let $e_1, e_2 \in G$ be two linear edges such that $\phi(e_1) = e_1\sigma^{d_1}$ and $\phi(e_2) = e_2\sigma^{d_2}$ where σ is an INP and $d_1 \neq d_2$. If $d_1, d_2 > 0$, then $x_m = e_1\sigma^m\bar{e}_2$ where $m \in \mathbb{Z}$ is an exceptional path. We say x_m has *height* $|m|$. Let $\delta = d_1 - d_2$. Then $\phi(x_m)$ is the exceptional path $x_{m+\delta}$.

Proposition 3.6.1. *Let $\phi : G \to G$ be a completely split train track map. Let a be an edge in an EG stratum H_r such that $\phi(a)$ starts with a. Let γ be a path in G_r that crosses H_r. Then*

$$\lim_{n\to\infty} \frac{(\gamma, \phi^n(a))}{\lambda^n} =: d_{\gamma,a}$$

exists and is nonnegative. Here λ is the Perron-Frobenius eigenvalue of the aperiodic EG stratum H_r. If $b \in H_r$ is another edge, then for every γ as above,

$$d_{\gamma,b} = \kappa d_{\gamma,a}$$

where κ is a constant with $\kappa = \kappa(a, b, \phi|_{H_r})$.

Proof. Let $\rho_a := \lim_{n\to\infty} \phi^n(a)$. The ray ρ_a is completely split and the terms of the complete splitting, called splitting units, of ρ_a form an alphabet $\mathbb{A}_\infty$ for a substitution. But $\mathbb{A}_\infty$ can be an infinite set if there are exceptional paths. We will define a finite alphabet $\mathbb{A}_\gamma$, which depends on γ, by identifying some elements in $\mathbb{A}_\infty$ in order to calculate the frequency of occurrence of γ in ρ_a. We will also show that the frequency of γ in ρ_a does not depend on the choice of the alphabet $\mathbb{A}_\gamma$. Let $\mathcal{N}$ be the set of all INPs, r-taken connecting paths and exceptional paths that appear in ρ_a.

Before defining the alphabet $\mathbb{A}_\gamma$, define a relation from the set of all finite paths in ρ_a that cross H_r, denoted $\mathcal{P}_r(\rho_a)$, to the set of all finite words on $\mathbb{A}_\infty$, denoted $\mathcal{W}(\mathcal{A}_\infty)$,

$$r : \mathcal{P}_r(\rho_a) \to \mathcal{W}(\mathcal{A}_\infty).$$

For a finite path $\gamma \in \mathcal{P}_r(\rho_a)$, the set $r(\gamma)$ consists of the following words:

(a) If an occurrence of γ in ρ_a is a concatenation of splitting units, then $r(\gamma)$ contains the corresponding word on $\mathbb{A}_\infty$.

(b) If an occurrence of γ in ρ_a is a subword of an INP σ, then $r(\gamma)$ contains the element of $\mathbb{A}_\infty$ determined by σ, denoted w_σ. There are only finitely many INPs that appear in ρ_a, therefore the number of occurrences of a path γ in an INP is bounded. If σ contains n occurrences of γ, then let $r(\gamma)$ contain n copies of w_σ. Note that a path γ in $\mathcal{P}_r(\rho_a)$ is not contained in an exceptional path or an r-taken connected path.

(c) If an occurrence of γ has partial overlaps with some elements of $\mathcal{N}$, then consider a path γ' such that γ' is the smallest subpath of ρ_a that is a concatenation of splitting units and which contains γ. Then $r(\gamma)$ contains the word on $\mathbb{A}_\infty$ corresponding to γ'.

Thus every occurrence of γ in ρ_a corresponds to the occurrence of some word in $r(\gamma)$. Note that $r(\gamma)$ can be an infinite set, for instance, when γ has partial overlap with infinitely many exceptional paths in ρ_a. We will define the alphabet $\mathbb{A}_\gamma$ such that the set of words in $r(\gamma)$ viewed in $\mathbb{A}_\gamma$ will be a finite set. For simplicity, let's assume that γ intersects only one family of exceptional paths, say determined by linear edges $e_1, e_2 \in G$.

- Let $\mathcal{H} = \{H_r = H_{i_1}, \ldots, H_{i_k}\}$ be the collection of strata crossed by edges in H_r. For every H_{i_j}, let $\mathbb{A}(H_{i_j})$ be the alphabet which contains an edge and its inverse as distinct letters if they both appear in ρ_a otherwise the edge with the orientation that appears.

  An edge in G is called a *Type 1* edge if it always appears with positive or negative orientation but not both in ρ_a. An edge which appears with both orientations in ρ_a is said to be of *Type 2*. If H_{i_j} is an EG stratum, then either all edges in H_{i_j} are Type 1 or all are Type 2 (see [Uya14] for proof). Thus, if we consider a substitution on $\mathbb{A}(H_r)$ representing ϕ restricted to H_r, then the substitution will be primitive.

- Now consider splitting units which are INPs, r-taken connecting paths and exceptional paths. Let $\mathbb{A}(\mathcal{N}_\gamma)$ be an alphabet defined as follows:

  (a) All oriented INPs and r-taken connecting paths that appear in ρ_a are contained in $\mathbb{A}(\mathcal{N}_\gamma)$. There can be infinitely many INPs in G_r but only finitely many appear in ρ_a.

  (b) Suppose γ contains an exceptional path determined by e_1, e_2 or a subsegment of an exceptional path determined by e_1, e_2. Let N be the maximum length of such an exceptional path that appears in γ, in $\phi(e)$ for all edges e in H_r and in an r-taken connecting path. Then $\mathbb{A}(\mathcal{N}_\gamma)$ contains exceptional paths determined by e_1, e_2 of height less than or equal to $N+1$ as distinct elements. All other exceptional paths determined by e_1, e_2 of height greater than $N+1$ correspond to a single element of $\mathbb{A}(\mathcal{N}_\gamma)$.

  (c) Suppose γ does not intersect an exceptional path determined by e_1, e_2. Then all exceptional paths determined by e_1, e_2 correspond to a single element of $\mathbb{A}(\mathcal{N}_\gamma)$.

- Let $\mathbb{A}_\gamma$ be defined as the set $\mathbb{A}(H_{i_1}) \sqcup \cdots \sqcup \mathbb{A}(H_{i_k}) \sqcup \mathbb{A}(\mathcal{N}_\gamma)$ and let $\zeta_{\gamma,\phi}$ be a substitution on $\mathbb{A}_\gamma$ determined by ϕ. Let $\tilde{r}(\gamma)$ be the set of words in $r(\gamma)$ viewed in the alphabet $\mathbb{A}_\gamma$. Then $\tilde{r}(\gamma)$ is a finite set of words on $\mathbb{A}_\gamma$. The frequency of occurrence of a path $\gamma \in \mathcal{P}_r(\rho_a)$ in ρ_a is given by the sum of the frequencies of the words in $\tilde{r}(\gamma)$.

If we replace $N+1$ by $N+C$ for any $C \geq 1$ in the above construction to get a different alphabet $\mathbb{A}'_\gamma$, then the frequency of γ calculated from the two alphabets is the same. More precisely, let $\mathbb{A}_\gamma$ and $\mathbb{A}'_\gamma$ be two alphabets which differ only in the naming of exceptional paths determined by e_1, e_2 of length greater than $N+1$. Let ζ and ζ' be the corresponding substitutions, and let $\tilde{r}(\gamma)$ and $\tilde{r}'(\gamma)$ be the set of words in $r(\gamma)$ viewed in $\mathbb{A}_\gamma$ and $\mathbb{A}'_\gamma$, respectively. An exceptional path maps to another exceptional path under ϕ. Therefore, ζ and ζ' have the same growth rate when restricted to $\mathbb{A}(H_r)$. Since the number of occurrences of γ does not change, the two substitutions yield the same frequency for words in $\tilde{r}(\gamma)$ and $\tilde{r}'(\gamma)$ and hence the same frequency for γ.

Thus, we have obtained an alphabet $\mathbb{A}_\gamma$. The completely split train track map ϕ indcues a substitution ζ_γ on this alphabet. Now Proposition 3.5.5 can be applied to ζ_γ to compute the frequency of occurrence of γ in ρ_a. Different substitutions constructed here for different words γ differ only in exceptional paths. Since an exceptional path maps to another exceptional path these different substitutions have the same growth rate when restricted to $\mathbb{A}(H_r)$. Also Kirchhoff's law still holds for frequencies of paths in ρ_a because $(\gamma, \phi^n(a))$ and $\sum_{e \in G_r} (\gamma e, \phi^n(a))$ differ by a bounded amount. $\square$

We do some examples below to exhibit how to view a completely split train track map as a substitution.

Example 3.6.2. Let R_3 be the rose on three petals with labels a, b, c. Consider a homotopy equivalence $\phi : R_3 \to R_3$ given by

$$\phi(a) = a, \phi(b) = Bac, \phi(c) = CBac.$$

Here capital letters denote inverses. The transition matrix for ϕ is

$$\begin{array}{c} \begin{array}{ccc} b & c & a \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

There are two strata $H_1 = \{a\}$ and $H_2 = \{b, c\}$. Every edge in H_2 is of Type 2. Let $\rho_C = \lim_{n\to\infty} \phi^n(C)$. We have $\mathcal{H} = \{H_2, H_1\}, \mathbb{A}(H_2) = \{b, c, B, C\}$ and $\mathbb{A}(H_1) = \{a, A\}$. Since there are no exceptional paths, use one alphabet $\mathbb{A} = \{b, c, B, C, a, A\}$ and a substitution ζ_ϕ on $\mathbb{A}$ whose transition matrix is given by

$$\begin{array}{c} \begin{array}{cccccc} b & c & B & C & a & A \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Example 3.6.3. Consider a homotopy equivalence $\phi : R_5 \to R_5$ given by

$$\phi(a) = ab, \phi(b) = bab, \phi(c) = cae, \phi(d) = dc\sigma d, \phi(e) = dcae$$

where $\sigma = abAB$ is a Nielsen path. There are two strata $H_1 = \{a, b\}$ and $H_2 = \{c, d, e\}$. Let $\rho_c = \lim_{n\to\infty} \phi^n(c)$. We have $\mathcal{H} = \{H_2, H_1\}, \mathbb{A}(H_2) = \{c, d, e\}, \mathbb{A}(H_1) = \{a, b\}$ and $\mathbb{A}(\mathcal{N}) = \{\sigma\}$. Since there are no exceptional paths, use one alphabet $\mathbb{A} = \{c, d, e, a, b, \sigma\}$ and a substitution ζ_ϕ on $\mathbb{A}$ whose transition matrix is given by

$$\begin{array}{c} \begin{array}{cccccc} c & d & e & a & b & \sigma \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

In this example, the frequency of occurrence of the edge path ca in ρ_c comes from the occurrence of the words ca and $c\sigma$ in ρ_c. Thus the frequency of ca in ρ_c is equal to $d_{ca,c} + d_{c\sigma,c}$.

Example 3.6.4. This example illustrates the discussion of exceptional paths in Proposition 3.6.1. Consider a homotopy equivalence $\phi : R_6 \to R_6$ given by

$$\begin{aligned} \phi(a) &= ab, & \phi(b) &= bab, \\ \phi(c) &= c\sigma^2, & \phi(d) &= d\sigma, \\ \phi(e) &= eaf, & \phi(f) &= fc\sigma Deaf, \end{aligned}$$

where $\sigma = abAB$. Some exceptional paths are $x_i = c\sigma^i D$ for $i > 0$. To calculate the frequency of words like fx_4 or $fc\sigma^4$ in ρ_f, we consider the alphabet

$$\mathbb{A} = \{e, f, a, b, c, D, x_1, x_2, x_3, x_4, x_5, \sigma, \overline{\sigma}\}$$

and substitution ζ such that

$$\begin{array}{ll}
\zeta(a) = ab, & \zeta(b) = bab, \\
\zeta(c) = c\sigma^2, & \zeta(d) = d\sigma, \\
\zeta(f) = fx_1eaf, & \zeta(e) = eaf, \\
\zeta(\sigma) = \sigma, & \zeta(\overline{\sigma}) = \overline{\sigma}, \\
\zeta(x_i) = x_{i+1} & \text{for } 1 \leq i \leq 3, \\
\zeta(x_4) = \zeta(x_5) = x_5. &
\end{array}$$

The path $\gamma = fc\sigma^4$ does not occur as a concatenation of splitting units in ρ_f. The path $\gamma' = fx_4$ is the smallest subpath of ρ_f that is a concatenation of splitting units and contains γ. Thus the frequency of occurrence of γ is the same as the frequency of occurrence of γ'.

3.7 Summary

In this chapter, we saw how to study substitutions which are not primitive but their restriction to a smaller alphabet is primitive. In particular, we saw how to compute frequencies of words that cross a particular subset of the alphabet, in an infinite word that is fixed by the substitution.

A CT train track representative of Φ, a fully irreducible outer automorphism relative to a free factor system $\mathcal{A}$, satisfies Proposition 3.6.1, with top stratum exponentially growing. We will define 'relative currents' in the next chapter and associate a relative current η^+_Φ to Φ. The relative current η^+_Φ will assign to every word in $\mathbb{F}$ which is not entirely contained in $\mathcal{A}$ the frequency calculated in Proposition 3.6.1. Explicit examples of these calculations are given in the next chapter.

CHAPTER 4

RELATIVE CURRENTS

In [Bon88], Bonahon first defined a space of *geodesic currents* for surfaces such that it contains the set of closed curves as a dense set. He studied the embedding of Teichmüller space in the space of geodesic currents and recovered Thurston's compactification of Teichmüller space. Currents for free groups were first studied by Reiner Martin [Mar95] in his thesis. Analogous to geodesic currents, the space of currents for $\mathbb{F}$ contains the set of conjugacy classes of elements of $\mathbb{F}$ as a dense set. Currents for free groups have also been studied in [Kap05], [Kap06], [KL09].

Let $\mathcal{A}$ be a free factor system of $\mathbb{F}$. In this chapter, we define a space of *currents relative to* $\mathcal{A}$ (also called *relative currents*) such that it contains the conjugacy classes of elements of $\mathbb{F}$ that are not contained in $\mathcal{A}$ as a dense set.

The main result of this chapter is a generalization of a theorem in [Mar95] (see also [Uya14]) which says that a fully irreducible outer automorphism acts with uniform north-south dynamics on a subspace of the space of projectivized currents. Let $\mathcal{MRC}(\mathcal{A})$ (see Definition 4.2.6) be a subspace of the space of projectivized relative currents.

Theorem B. *Let $\mathcal{A}$ be a nontrivial free factor system of $\mathbb{F}$ with $\zeta(\mathcal{A}) \geq 3$. Let $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ be fully irreducible relative to $\mathcal{A}$. Then Φ acts with uniform north-south dynamics on $\mathcal{MRC}(\mathcal{A})$.*

4.1 Preliminaries

We give a short introduction to currents for free groups and define some basic terms.

4.1.1 Boundary of $\mathbb{F}$

Given $\mathbb{F}$ and a fixed basis $\mathfrak{B}$ of $\mathbb{F}$, let $\mathrm{Cay}(\mathbb{F}, \mathfrak{B})$ be the Cayley graph of $\mathbb{F}$ with respect to $\mathfrak{B}$. The space of ends of the Cayley graph is called the *boundary of* $\mathbb{F}$, denoted by $\partial\mathbb{F}$. It is homeomorphic to the Cantor set. A one-sided cylinder set determined by a finite path γ starting at the base point is the set of all rays starting at the base point that cross γ. Such cylinder sets form a basis for the topology on $\partial\mathbb{F}$ and are in fact both open and closed.

Let Δ denote the diagonal in $\partial\mathbb{F} \times \partial\mathbb{F}$. Let $\partial^2\mathbb{F} := (\partial\mathbb{F} \times \partial\mathbb{F} - \Delta)/\mathbb{Z}_2$ *be the space of flip-invariant bi-infinite geodesics* in $\mathrm{Cay}(\mathbb{F}, \mathfrak{B})$. This space is also called the *double boundary* of $\mathbb{F}$. Finite paths γ in $\mathrm{Cay}(\mathbb{F}, \mathfrak{B})$ determine two-sided cylinder sets, denoted $C(\gamma)$, which form a basis for the topology of $\partial^2\mathbb{F}$. Two-sided cylinder sets are open and compact and hence closed. Compact open sets are given by finite disjoint union of cylinder sets. Also $\partial^2\mathbb{F}$ is locally compact but not compact. The action of $\mathbb{F}$ on $\partial^2\mathbb{F}$ is cocompact.

4.1.2 Currents for $\mathbb{F}$

In [Mar95], a *measured current* is defined as an additive, nonnegative, $\mathbb{F}$-invariant and flip-invariant function on the set of compact open sets in $\partial^2\mathbb{F}$. It is uniquely determined by its values on the cylinder sets given by words in $\mathbb{F}$. For each conjugacy class $\alpha \in \mathbb{F}$, a measured current μ_α can be defined as follows: for a cylinder set C in $\partial^2\mathbb{F}$, $\mu_\alpha(C)$ is defined as the number of lifts of α that are in C.

In [Mar95], Martin shows that the set of conjugacy classes of elements in $\mathbb{F}$ is dense in the space of measured currents, denoted $\mathcal{MC}(\mathbb{F})$. He also shows that the space of projective measured currents is compact. In this chapter, we aim to generalize the following theorem:

Theorem 4.1.1 ([Mar95])**.** *A fully irreducible outer automorphism acts with uniform north-south dynamics on the closure of the set of primitive conjugacy classes in the space of projectivized measured currents.*

4.1.3 Bounded cancellation constant and critical length

Lemma 4.1.2 ([Coo87])**.** *Let G be a marked metric graph and let $\phi : G \to G$ be a homotopy equivalence. There exists a constant* $\mathrm{BCC}(\phi)$*, called the bounded cancellation constant, depending only on ϕ such that for any path ρ in G obtained by concatenating two paths α, β, we have*

$$L(\phi(\rho)) \geq L(\phi(\alpha)) + L(\phi(\beta)) - \mathrm{BCC}(\phi)$$

where L is the length function on G.

Let $\mathrm{BCC}(\phi)$ be the bounded cancellation constant for $\phi : G \to G$, a relative train track representative of a relative fully irreducible outer automorphism Φ with top EG stratum

H_r. The transition matrix of H_r has a unique positive eigenvector whose smallest entry is one. For an edge e_i in H_r, the eigenvector has an entry $v_i > 0$. Assign a metric to G such that each edge e_i in H_r is isometric to an interval of length v_i and all edges in G_{r-1} have length one. Then the r-length of edges in H_r gets stretched by the PF eigenvalue λ_Φ under ϕ. Let l_r denote the r-length. Let α, β, γ be r-legal paths in G. Let $\alpha.\beta.\gamma$ be the path obtained by concatenating these r-legal paths. The only r-illegal turns possibly occur at the ends of the segments of β. Thus if $\lambda_\Phi l_r(\beta) - 2\,\mathrm{BCC}(\phi) > l_r(\beta)$, then iterations and tightening of $\alpha.\beta.\gamma$ will produce paths with r-length of legal segments corresponding to β going to infinity. We call $\frac{2\,\mathrm{BCC}(\phi)}{\lambda_\Phi - 1}$ the *critical length* for ϕ.

4.1.4 A subspace of $\partial^2\mathbb{F}$

Let $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$, $k > 0$, be a free factor system such that $\zeta(\mathcal{A}) \geq 3$.

Definition 4.1.3 (Relative basis)**.** Let $\mathfrak{B}_\mathcal{A}$ be a basis of $\mathbb{F}$ such that a basis of $\mathcal{A}$ is a subset of $\mathfrak{B}_\mathcal{A}$. Specifically,

$$\mathfrak{B}_\mathcal{A} = \{a_{11}, \ldots a_{11_s}, \ldots, a_{i1}, \ldots, a_{ii_s}, \ldots, a_{k1}, \ldots, a_{kk_s}, b_1, \ldots, b_p\}$$

where $a_{ij} \in A_i$ and $b_i \notin A$ for any $[A] \in \mathcal{A}$. Let $\sum_{i=1}^{k} i_s =: s$. Define a set $B_\mathcal{A}$ to be the collection of all words $a_{ij}^{\pm} a_{kl}^{\pm}$ of length two such that $i \neq k$ and all b_i. Note that if $\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathbb{F})$, then the set of b_i is empty. We call $\mathfrak{B}_\mathcal{A}$ a *relative basis* of $\mathbb{F}$.

Definition 4.1.4. Given a free factor A, a *one-sided infinite geodesic starting at the base point in* $\mathrm{Cay}(\mathbb{F}, \mathfrak{B}_\mathcal{A})$ *is in* ∂A if eventually it crosses only edges labeled by words in A. Note that ∂A is an $\mathbb{F}$-invariant set. Define $\partial\mathcal{A} = \bigsqcup_{i=1}^{k} \partial A_i$.

Definition 4.1.5 (Double boundary of $\mathcal{A}$)**.** Given a free factor A, define $\partial^2 A$ to be the set of bi-infinite geodesics in $\partial^2\mathbb{F}$ which are lifts of conjugacy classes of elements in A. Then define the double boundary of $\mathcal{A}$ as $\partial^2\mathcal{A} := \bigsqcup_{i=1}^{k} \partial^2 A_i$.

Definition 4.1.6. Let $\mathbf{Y} = \partial^2\mathbb{F} \setminus \partial^2\mathcal{A}$. It inherits the subspace topology from $\partial^2\mathbb{F}$. It can also be given a topology where cylinder sets in $\mathbf{Y}$ determined by finite paths that cross at least one word in $B_\mathcal{A}$ form a basis for the topology. The two topologies are in fact equivalent.

Lemma 4.1.7. $\mathbf{Y}$ *is locally compact.*

Proof. A space is locally compact if every point has a compact neighborhood. Let x be an element of $\mathbf{Y}$. Take a finite subpath of x that cannot be written as a string of words contained in a single $[A] \in \mathcal{A}$ and consider the cylinder set determined by that path. Then this cylinder set is a compact open set in $\mathbf{Y}$ containing x. $\square$

Lemma 4.1.8. *The action of* $\mathbb{F}$ *on* $\mathbf{Y}$ *is cocompact.*

Proof. Consider a compact set $C \subset \mathrm{Cay}(\mathbb{F}, \mathfrak{B}_{\mathcal{A}})$ given by a finite union of cylinder sets determined by all paths with one end point at the origin such that the label of each path is a word in $B_{\mathcal{A}}$. For every bi-infinite geodesic γ in $\mathbf{Y}$, there is a $g \in \mathbb{F}$ such that $g \cdot \gamma$ crosses a path starting at the origin determined by a word in $B_{\mathcal{A}}$. $\square$

4.2 Relative currents

In this section, we define a relative current. We show that the space of projective relative currents, denoted $\mathbb{P}\mathcal{RC}(\mathcal{A})$, is compact and that conjugacy classes in $\mathbb{F} \setminus \mathcal{A}$ are dense in $\mathbb{P}\mathcal{RC}(\mathcal{A})$.

4.2.1 Definition of relative current

Definition 4.2.1. With respect to the basis $\mathfrak{B}_{\mathcal{A}}$, let $\mathbb{F} \setminus \mathcal{A}$ denote the set of all words in $\mathbb{F}$ that are not contained in any free factor A_i, for $1 \leq i \leq k$. Note that $\mathbb{F} \setminus \mathcal{A}$ contains conjugates of words in A_i, as long as the conjugating elements are not in A_i.

Definition 4.2.2. Let $[\mathbb{F} \setminus \mathcal{A}]$ be the set of all conjugacy classes of elements in $\mathbb{F}$ that are not contained in any conjugacy class of a free factor in $\mathcal{A}$. Note that an element of $\mathbb{F} \setminus \mathcal{A}$ can be contained in the free product of distinct free factors representing elements of $\mathcal{A}$.

Let $\mathcal{C}(\mathbf{Y})$ be the collection of compact open sets in $\mathbf{Y}$. A *relative current* is an additive, nonnegative, $\mathbb{F}$-invariant and flip-invariant function on $\mathcal{C}(\mathbf{Y})$. Let $\mathcal{RC}(\mathcal{A})$ denote the space of relative currents. A subbasis for the topology of $\mathcal{RC}(\mathcal{A})$ is given by the sets $\{\eta \in \mathcal{RC}(\mathcal{A}) : |\eta(C) - \eta_0(C)| \leq \epsilon\}$ where $\eta_0 \in \mathcal{RC}(\mathcal{A})$, $C \in \mathcal{C}(\mathbf{Y})$ and $\epsilon > 0$.

$\mathrm{Out}(\mathbb{F}, \mathcal{A})$ acts on $\mathcal{RC}(\mathcal{A})$ as follows: let $\eta \in \mathcal{RC}(\mathcal{A})$, $\Psi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ and let $C \in \mathcal{C}(\mathbf{Y})$. Then

$$\Psi.\eta(C) = \eta(\Psi^{-1}(C)).$$

A relative current can also be defined as an $\mathbb{F}$-invariant, locally finite, inner regular measure (called Radon measure) on the σ-algebra of Borel sets of $\mathbf{Y}$.

Lemma 4.2.3. *A nonnegative, additive function on $\mathcal{C}(\mathbf{Y})$ corresponds to a Radon measure on the Borel σ-algebra of $\mathbf{Y}$.*

Proof. Given a nonnegative, additive function η on $\mathcal{C}(\mathbf{Y})$, define an outer measure $\eta^* : 2^{\mathbf{Y}} \to [0, \infty]$ as follows: for $A \in 2^{\mathbf{Y}}$

$$\eta^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \eta(C_i) : A \subseteq \bigcup_{i=1}^{\infty} C_i \text{ where } C_i \in \mathcal{C}(\mathbf{Y}) \text{ is a cylinder set} \right\}.$$

We have $\eta^*(C) = \eta(C)$ for $C \in \mathcal{C}(\mathbf{Y})$ because every cover of a compact set has a finite subcover and then use additivity of η. A cylinder set C in $\mathcal{C}(\mathbf{Y})$ is outer measurable, that is, for every $A \in 2^{\mathbf{Y}}$, we have $\eta^*(A) = \eta^*(A \cap C^c) + \eta^*(A \cap C)$. An outer measure is a measure on the σ-algebra of outer measurable sets which in this case is the same as the σ-algebra of Borel sets. Therefore, the outer measure η^* is a measure on the Borel σ-algebra of $\mathbf{Y}$. The space $\mathbf{Y}$ is locally compact and Hausdorff and every open set in $\mathbf{Y}$ is σ-compact, that is, a countable union of compact sets. Also η^* is a nonnegative Borel measure on $\mathbf{Y}$ such that it is finite on compact sets. Therefore by [Rud87, Theorem 2.18], η^* is a regular measure. $\square$

Thus the space of relative currents can be given a weak-$*$ topology, that is, $\eta_n \to \eta$ in $\mathcal{RC}(\mathcal{A})$ iff $\int_{\mathbf{Y}} f \, d\eta_n \to \int_{\mathbf{Y}} f \, d\eta$ for all compactly supported functions f on $\mathbf{Y}$. Since $\mathbf{Y}$ is a locally compact space, by the result in [Bou65, Chapter III, Section 1], $\mathcal{RC}(\mathcal{A})$ is complete.

4.2.2 Coordinates for the space of relative currents

Fix a relative basis $\mathfrak{B}_{\mathcal{A}}$ of $\mathbb{F}$. Given $w \neq 1 \in \mathbb{F}$, consider the unique oriented path, denoted γ_w, determined by w starting at the base point and let $C(w) := C(\gamma_w)$. This cylinder set contains *unoriented* bi-infinite geodesics that cross γ_w. For $w \in \mathbb{F} \setminus \mathcal{A}$, we have $C(w) \subset \mathcal{C}(\mathbf{Y})$. Orbits of cylinder sets of the form $C(w)$ under deck transformations cover $\mathbf{Y}$. We denote η applied to $C(w)$ by $\eta(w)$.

- Let $v \in \mathbb{F}$. Then $v \cdot C(w)$ is the set of all bi-infinite geodesics that cross an edge path labeled by w starting at the vertex labeled v in the Cayley graph. By $\mathbb{F}$-invariance of a relative current, $\eta(C(w)) = \eta(v \cdot C(w))$. Thus we work just with the cylinder sets determined by finite paths starting at the base point. Since every compact open set is a finite disjoint union of cylinder sets, a relative current is uniquely determined by its values on $\mathbb{F} \setminus \mathcal{A}$.

- Since a relative current is uniquely determined by its values on $\mathbb{F} \setminus \mathcal{A}$, a sequence of relative currents η_n converges to η iff $\eta_n(w) \to \eta(w)$ for all $w \in \mathbb{F} \setminus \mathcal{A}$.

- For any finite path γ in $\mathrm{Cay}(\mathbb{F}, \mathfrak{B}_{\mathcal{A}})$, we have $C(\gamma) = C(\overline{\gamma})$, where $\overline{\gamma}$ denotes the opposite orientation on γ. If w and γ_w are as above, then $C(w) = C(\gamma_w) = C(\overline{\gamma_w}) = w \cdot C(\overline{w})$. Thus $\eta(w) = \eta(\overline{w})$.

- Let $w = e_0 e_1 \dots e_l \in \mathbb{F} \setminus \mathcal{A}$ where each $e_i \in \mathfrak{B}_{\mathcal{A}}$. Then $C(w) = \cup C(we)$ where the union is taken over all basis elements in $\mathfrak{B}_{\mathcal{A}}$ except $e = \overline{e_l}$. Here $\overline{e}$ denotes the inverse of e. Also $C(w) = \cup \overline{e} \cdot C(ew)$ where e is any basis element other than $\overline{e_0}$. Thus additivity of a relative current can be stated as

$$\eta(w) = \sum_{e \neq \overline{e_l}} \eta(we) \qquad \text{or} \qquad \eta(w) = \sum_{e \neq \overline{e_0}} \eta(ew).$$

For example, let $\mathbb{F} = \langle a, b \rangle$ and $\mathcal{A} = \{[\langle a \rangle]\}$, we have

$$\eta(b) = \eta(ba) + \eta(bb) + \eta(b\overline{a}),$$
$$\eta(b) = \eta(ab) + \eta(bb) + \eta(\overline{a}b)$$

- Let $v, w \in \mathbb{F} \setminus \mathcal{A}$ be such that v is a subword of w. Then $\eta(w) \leq \eta(v)$.

Example 4.2.4 (Relative current)**.** Consider a conjugacy class $\alpha \in [\mathbb{F} \setminus \mathcal{A}]$ such that α is not a power of any other conjugacy class in $\mathbb{F}$. Then $\eta_\alpha(w)$ is the number of occurrences of w in the cyclic words α and $\overline{\alpha}$. Equivalently, one can also count the number of lifts of α that cross the path γ_w in the Cayley graph. We call such currents and their multiples *rational relative currents*. For example, let $\mathbb{F} = \langle a, b \rangle, \mathcal{A} = \{[\langle a \rangle]\}$ and let $\alpha = aba\overline{b}ab$. Then $\eta_\alpha(b) = 3, \eta_\alpha(ba) = 2, \eta_\alpha(abab) = 1$ and $\eta_\alpha(\overline{b}ab) = 1$.

Definition 4.2.5 (Length k-extension)**.** Given $w \in \mathbb{F}$, a *length k extension* of w is a word $w' = wx_1 \dots x_k$ where $x_i \in \mathfrak{B}_{\mathcal{A}}$, $x_i \neq \overline{x_{i+1}}$ and x_1 is not the inverse of the last letter of w.

Lemma 4.2.6. *Any nonnegative function η on $\mathbb{F} \setminus \mathcal{A}$ invariant under inversion and the action of $\mathbb{F}$, and satisfying the condition*

$$\eta(w) = \sum_{\substack{\text{length one} \\ \text{extension of } w}} \eta(v)$$

for all $w \in \mathbb{F} \setminus \mathcal{A}$ determines a relative current.

Proof. A set $C \in \mathcal{C}(\mathbf{Y})$ can be written as a disjoint union of cylinder sets $C(w_1), \dots C(w_k)$ with $w_i \in \mathbb{F} \setminus \mathcal{A}$. Then define $\eta(C) := \sum_{i=1}^{k} \eta(w_i)$. The value $\eta(C)$ does not depend on the choice of w_i. Thus we have an additive and nonnegative function on $\mathcal{C}(\mathbf{Y})$ which is invariant under the action of $\mathbb{F}$. $\square$

4.2.3 Projectivized relative currents

Let $\mathbb{P}\mathcal{RC}(\mathcal{A})$ be the space of projectivized relative currents. It has quotient topology induced from $\mathcal{RC}(\mathcal{A})$. A sequence of projective currents $[\eta_i]$ converges to $[\eta]$ in $\mathbb{P}\mathcal{RC}(\mathcal{A})$ iff there exist scaling constants a_i such that the sequence of relative currents $a_i\eta_i$ converge to η in $\mathcal{RC}(\mathcal{A})$.

Example 4.2.7. Let $\mathbb{F} = \langle a, b\rangle$ and let $\mathcal{A} = \{[\langle a\rangle]\}$. Consider the sequence $\eta_{a^n b} \in \mathcal{RC}(\mathcal{A})$. This sequence converges to a relative current η_∞ which is given by $\eta_\infty(a^n b a^m) = 1$ for all $n, m \geq 0$ and $\eta_\infty(w) = 0$ for all other $w \in \mathbb{F} \setminus \mathcal{A}$. Whereas in the space of measured currents as defined in [Mar95], the sequence $\mu_{a^n b}/n$ converges to the current μ_a.

Lemma 4.2.8. $\mathbb{P}\mathcal{RC}(\mathcal{A})$ *is compact.*

Proof. Consider a sequence of projective relative currents $[\eta_n]$. We have to show that it has a convergent subsequence. Fix a relative basis $\mathfrak{B}_{\mathcal{A}}$ and the associated set $B_{\mathcal{A}} = \{u_1, \dots, u_r\}$ (see Definition 4.1.3). Let η_n be a representative of $[\eta_n]$ normalized such that $\eta_n(u_i) \leq 1$ for all $u_i \in B_{\mathcal{A}}$ and $\eta_n(u_j) = 1$ for some $u_j \in B_{\mathcal{A}}$. We have $\eta_n(w) \leq \eta_n(u_i)$ where $w \in \mathbb{F} \setminus \mathcal{A}$ and crosses a path labeled $u_i \in B_{\mathcal{A}}$ in $\mathrm{Cay}(\mathbb{F}, \mathfrak{B}_{\mathcal{A}})$. The bounded sequence $\{(\eta_n(u_1), \dots, \eta_n(u_r))\}_{n\in\mathbb{N}}$ has a subsequence that converges to a nonzero element of $\mathbb{R}^r$. For every $w \in \mathbb{F} \setminus \mathcal{A}$, $\{\eta_n(w)\}_{n\in\mathbb{N}}$ is a bounded sequence and hence has a convergent subsequence. Now by the diagonal argument, conclude that $\{(\eta_n(w))_{w\in\mathbb{F}\setminus\mathcal{A}}\}_{n\in\mathbb{N}}$ has a subsequence that converges to a nonzero element. Thus $\{[\eta_n]\}_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{P}\mathcal{RC}(\mathcal{A})$. $\square$

4.2.4 Density of rational relative currents

Proposition 4.2.9. *The set of projectivized relative currents induced by conjugacy classes $\alpha \in \mathbb{F} \setminus \mathcal{A}$ are dense in $\mathbb{P}\mathcal{RC}(\mathcal{A})$.*

Let $\mathfrak{B}_{\mathcal{A}}$ be a relative basis of $\mathbb{F}$ and let $|w|$ denote the word length of $w \in \mathbb{F}$ with respect to $\mathfrak{B}_{\mathcal{A}}$. In the absolute case, the following lemma is the main step to prove density of rational measured currents in the space of measured currents for $\mathbb{F}$. But it does not directly apply to the relative setting as explained below.

Lemma 4.2.10 ([Mar95, Lemma 15])**.** *Let μ be a measured current and let $k \geq 2$. Let $P = 2\mathfrak{n}(2\mathfrak{n}-1)^{2\mathfrak{n}(2\mathfrak{n}-1)^{k-2}}$ be a constant, where $\mathfrak{n} = \operatorname{rank}(\mathbb{F})$. If $\mu(w_0) \geq P$ for some $w_0 \in \mathbb{F}$ with $|w_0| = k$, then there exists a conjugacy class $\alpha \in \mathbb{F}$ and the corresponding measured current μ_α with $\mu(w) \geq \mu_\alpha(w)$ for all $w \in \mathbb{F}$ and $|w| \leq k$.*

The proof of the above lemma relies on finding cycles in a certain labeled directed graph associated to μ defined as follows: vertices are given by words of length $k-1$ and edges are given by words of length k. A directed edge w joins vertex u to vertex v if u is the prefix of w and v is the suffix of w. An edge w is labeled by $\mu(w)$. Since μ satisfies additivity laws for all words in $\mathbb{F}$, this graph satisfies Kirchhoff's law at each vertex which is crucial to find cycles (which correspond to α) in the graph. The same graph defined for a relative current η_0 does not satisfy Kirchhoff's law at vertices which correspond to words in $\mathcal{A}$ because η_0 is not defined for words in $\mathcal{A}$.

Definition 4.2.11 (Signed measured current)**.** A *signed measured current* on $\partial^2\mathbb{F}$ is an $\mathbb{F}$-invariant and additive function on the set of compact open sets of $\partial^2\mathbb{F}$.

We now restate the above lemma for a signed measured current which is nonnegative on words in $\mathbb{F}$ of bounded length.

Lemma 4.2.12. *Let $k \geq 2$ and let η be a signed measured current such that $\eta(w) \geq 0$ for all $w \in \mathbb{F}$ with $|w| \leq k$. Let $P = 2\mathfrak{n}(2\mathfrak{n}-1)^{2\mathfrak{n}(2\mathfrak{n}-1)^{k-2}}$ be a constant. If $\eta(w_0) \geq P$ for some $w_0 \in \mathbb{F}$ with $|w_0| = k$, then there exists a conjugacy class $\alpha \in \mathbb{F}$ and the corresponding measured current*

η_α with $\eta(w) \geq \eta_\alpha(w)$ for all $w \in \mathbb{F}$ and $|w| \leq k$.

Definition 4.2.13 (*k*-extension of a current)**.** For $\eta_0 \in \mathcal{RC}(\mathcal{A})$, let η be a signed measured current such that $\eta(w) = \eta_0(w)$ for $w \in \mathbb{F} \setminus \mathcal{A}$ and $\eta(w) \geq 0$ for all words $w \in \mathbb{F}$ with $|w| \leq k$. We call such an η a *k-extension* of η_0.

Lemma 4.2.14. *Let η_0 be a relative current and let $k \geq 1$. Then there exists a signed measured current η which is a k-extension of η_0.*

To prove the above lemma, start by defining η on length one words in $\mathcal{A}$ arbitrarily and then extend the current to length two words by satisfying the additivity property. It needs to be checked that the constraints obtained from the additive property are consistent. A detailed proof is given in Appendix A. Assuming the above lemma is true, we now prove Proposition 4.2.9.

Proof of Proposition 4.2.9. We follow the same method of proof as in [Mar95, Proposition 16]. Let η_0 be a relative current and let $k \geq 2$. Choose $R > 0$ such that $R\eta_0(w_0) \geq P$ for some $w_0 \in \mathbb{F} \setminus \mathcal{A}$ with $|w_0| = k$. Consider a signed measured current η which is a k-extension of η_0. Then by Lemma 4.2.12 applied to $R\eta$, there exists an $\alpha_1 \in \mathbb{F}$ such that $R\eta(w) \geq \eta_{\alpha_1}(w)$ for all $w \in \mathbb{F}$ with $|w| \leq k$. If $R\eta(w) \leq \eta_{\alpha_1}(w) + P$ for all $w \in \mathbb{F}$ with $|w| \leq k$, then stop, otherwise again apply Lemma 4.2.12 to $R\eta - \eta_{\alpha_1}$ to obtain $\alpha_2 \in \mathbb{F}$ such that $R\eta(w) - \eta_{\alpha_1}(w) \geq \eta_{\alpha_2}(w)$ for all $w \in \mathbb{F}$ with $|w| \leq k$. By induction, $\sum \eta_{\alpha_i}(w) \leq R\eta(w) \leq \sum \eta_{\alpha_i}(w) + P$ for all words of length less than or equal to k.

It is necessary that at least one of the $\alpha_i \in \mathbb{F} \setminus \mathcal{A}$. Indeed, if they were all in $\mathcal{A}$, then $\sum \eta_{\alpha_i}(w_0) = 0$ which would mean $R\eta(w_0) \leq P$ which is a contradiction. Now we have

$$\left| \eta(w) - \frac{\sum \eta_{\alpha_i}(w)}{R} \right| \leq \frac{P}{R}$$

for all $w \in \mathbb{F}$ with $|w| \leq k$. For $w \in \mathbb{F} \setminus \mathcal{A}$ in fact, we have

$$\left| \eta_0(w) - \frac{\sum_{\alpha_i \notin \mathcal{A}} \overline{\eta}_{\alpha_i}(w)}{R} \right| \leq \frac{P}{R}$$

where $\overline{\eta}_{\alpha_i}$ is the restriction of η_{α_i} to $\mathbf{Y}$.

Since R can be chosen arbitrarily large, relative currents can be approximated by sums of rational relative currents for all $w \in \mathbb{F} \setminus \mathcal{A}$ with $|w| \leq k$. Now we can approximate $\sum_{\alpha_i \notin \mathcal{A}} \eta_{\alpha_i}$ by $\frac{1}{m}\eta_{\beta^m}$ where $\beta^m = w_1^m w_2^m \cdots w_l^m$ and w_i is in the conjugacy class of α_i. $\square$

4.2.5 $\mathcal{A}$-Whitehead graph

Definition 4.2.15 ($\mathcal{A}$-separable conjugacy class)**.** A conjugacy class $\alpha \in [\mathbb{F} \setminus \mathcal{A}]$ is $\mathcal{A}$-*separable* if it is contained in a nontrivial free factor system containing $\mathcal{A}$. Topologically, α is $\mathcal{A}$-separable if there is an $\mathbb{F}$-tree T with the set of vertex stabilizers given by $\mathcal{A}$ such that an axis of α does not cross every orbit of edges.

To detect when a conjugacy class is $\mathcal{A}$-separable, use Whitehead's algorithm and a theorem of Stallings [Sta99]. As defined in [Sta99], a collection $\mathcal{C}$ of conjugacy classes in $\mathbb{F}$ is *separable* if there exist free factors F, F' such that $\mathbb{F} = F * F'$ and each conjugacy class in $\mathcal{C}$ is contained in either F or F'. Let $\alpha_i \in A_i$, $0 < i \leq k$, be a conjugacy class such that α_i is not contained in any proper free factor of A_i. We say α_i is *filling* in A_i.

Lemma 4.2.16. *A conjugacy class $\alpha \in [\mathbb{F} \setminus \mathcal{A}]$ is $\mathcal{A}$-separable if and only if the collection of conjugacy classes $\mathcal{C} = \{\alpha, \alpha_1, \ldots, \alpha_k\}$ is separable.*

Proof. If $\mathcal{C}$ is separable, then there exist a decomposition $\mathbb{F} = F * F'$ such that each conjugacy class in $\mathcal{C}$ is contained either in F or F'. Suppose $\alpha_i \in F$. Then we claim that A_i is contained in F up to conjugation. Suppose not. Then $F \cap A_i \neq \emptyset$ up to conjugation. Also the intersection of two free factors is a free factor. So α_i is contained in a nontrivial free factor of A_i, which is a contradiction. Thus $\{[F], [F']\}$ is a nontrivial free factor system containing $\mathcal{A}$ that contains the conjugacy class w. On the other hand, if α is contained in a proper free factor system $\mathcal{D}$ containing $\mathcal{A}$, then $\mathcal{C}$ is separable. $\square$

Definition 4.2.17 (Whitehead Graph [Whi36])**.** Given a basis $\mathfrak{B}$ of $\mathbb{F}$, the Whitehead graph of a collection $\mathcal{C}$ of conjugacy classes, denoted $Wh(\mathcal{C})$, is defined as follows: the vertices are given by basis elements and their inverses. There is an edge connecting vertices x and y if $\overline{x}y$ is a subword of a conjugacy class in $\mathcal{C}$.

Theorem 4.2.18 ([Sta99, Theorem 4.2])**.** *Let $\mathcal{C}$ be a collection of conjugacy classes in $\mathbb{F}$. If $Wh(\mathcal{C})$ is connected and $\mathcal{C}$ is separable, then there is a cut vertex in $Wh(\mathcal{C})$.*

Definition 4.2.19 ($\mathcal{A}$-Whitehead Graph)**.** For each $[A_i] \in \mathcal{A}$, fix filling conjugacy classes $\alpha_i \in A_i$. The $\mathcal{A}$-Whitehead graph of a conjugacy class $\alpha \in [\mathbb{F} \setminus \mathcal{A}]$, denoted $Wh(w, \mathcal{A})$, is defined as the Whitehead graph of the collection $\{\alpha, \alpha_1, \ldots, \alpha_k\}$.

Note that even though we fix some filling conjugacy classes to define the relative Whitehead graph, detecting $\mathcal{A}$-separability of α is independent of them by Lemma 4.2.16.

Example 4.2.20. Let $\mathbb{F} = \langle a, b, c, d\rangle$ and $\mathcal{A} = \{\langle a, b\rangle\}$. Let $\alpha = cadb$ and $\alpha_1 = ab\bar{a}\bar{b}$. In the $\mathcal{A}$-Whitehead graph of α, a is a cut vertex with disjoint sets $\{a, \bar{c}\}$ and $\{\bar{a}, b, \bar{b}, c, d, \bar{d}\}$. See **Figure 4.1**. Let ϕ be the Whitehead automorphism given as $\phi(a) = a, \phi(b) = \bar{a}ba, \phi(c) = \bar{a}c, \phi(d) = \bar{a}da$. Then $\phi(cadb) = cdb$. Now the $\mathcal{A}$-Whitehead graph for $\alpha' = cdb$ is disconnected, which implies that $\alpha = cadb$ is $\mathcal{A}$-separable. See **Figure 4.2**. Indeed, α is contained in the free factor system $\{\langle c, adb\rangle, \langle a, b\rangle\}$.

4.2.6 A closed subspace of $\mathbb{P}\mathcal{RC}(\mathcal{A})$

In the absolute case, when a fully irreducible outer automorphism Ψ is a pseudo-Anosov on a surface with one boundary component, the measured current corresponding to the boundary conjugacy class in the space of projectivized measured currents $\mathcal{MC}(\mathbb{F})$ is fixed under the action of Ψ. Thus in [Mar95], a closed subspace is considered which is the closure of all primitive conjugacy classes in $\mathcal{MC}(\mathbb{F})$. For the same reason, we pass to a smaller closed Out($\mathbb{F}, \mathcal{A}$)-invariant subspace of $\mathbb{P}\mathcal{RC}(\mathcal{A})$. Let

$$\mathcal{MRC}(\mathcal{A}) = \overline{\{[\eta_\alpha] \in \mathbb{P}\mathcal{RC}(\mathcal{A}) | \alpha \text{ is } \mathcal{A}\text{-separable}\}}$$

Lemma 4.2.21. *$[\eta_\alpha] \in \mathbb{P}\mathcal{RC}(\mathcal{A})$ is in $\mathcal{MRC}(\mathcal{A})$ if and only if α is $\mathcal{A}$-separable.*

Proof. Let's assume that α is not $\mathcal{A}$-separable. Then by Theorem 4.2.18, the $\mathcal{A}$-Whitehead graph of α with respect to any relative basis is connected without a cut vertex. Let $w_\alpha \in \mathbb{F} \setminus \mathcal{A}$ be a cyclically reduced representative of α. Consider a relative current η_v where $v \in [\mathbb{F} \setminus \mathcal{A}]$ such that $\eta_v(w_\alpha^2) > 0$. This means that any $\mathcal{A}$-Whitehead graph of v contains the Whitehead graph of α as a subgraph and hence is connected without cut vertices. By Theorem 4.2.18 and Lemma 4.2.16, this implies that v is not $\mathcal{A}$-separable. Thus $\eta_v(w_\alpha^2) = 0$ for all $\mathcal{A}$-separable conjugacy classes v in $[\mathbb{F} \setminus \mathcal{A}]$, which in turn implies that $\eta(w_\alpha) = 0$ for any $[\eta] \in \mathcal{MRC}(\mathcal{A})$. Since $\eta_\alpha(w_\alpha^2) > 0$, we have that $\eta_\alpha \notin \mathcal{MRC}(\mathcal{A})$. $\square$

4.3 Stable and unstable relative currents

In this section, we associate a pair of relative currents to Φ, a fully irreducible outer automorphism relative to $\mathcal{A}$. A completely split train track representative of Φ will be used for this purpose. Since such a topological representative is often defined on a marked graph which may not be a Cayley graph, we first show how to define coordinates for relative currents using a marked graph.

Definition 4.3.1 (Coordinates with respect to a marked graph)**.** Let G be a marked metric graph in Culler-Vogtmann's outer space, such that G has a subgraph Γ with $\mathcal{F}(\Gamma) = \mathcal{A}$. Let $g : \mathcal{R} \to G$ be the marking of G. Here $\mathcal{R}$ is the quotient of $\mathrm{Cay}(\mathbb{F}, \mathfrak{B}_{\mathcal{A}})$ under the action of $\mathbb{F}$. Let $\widetilde{G}$ be the universal cover of G. The map g lifts to an $\mathbb{F}$-equivariant map $\widetilde{g}\colon \mathrm{Cay}(\mathbb{F}, \mathfrak{B}_{\mathcal{A}}) \to \widetilde{G}$. The map $\widetilde{g}$ identifies $\partial^2\widetilde{G}$ with $\partial^2\mathbb{F}$ and $\partial^2\Gamma$ with $\partial^2\mathcal{A}$. Given an edge path v in $\widetilde{G}$, let

$$C(v) := \{(x,y) \in \partial^2\mathbb{F} \mid v \subset (\widetilde{g}(x), \widetilde{g}(x))\}$$

be a compact open set of $\partial^2\mathbb{F}$ determined by the path v of $\widetilde{G}$. For a relative current η and a path v of $\widetilde{G}$ that is not entirely contained in the lift of Γ, $\eta(v)$ is defined to be equal to $\eta(C(v))$. Since η is $\mathbb{F}$-equivariant, we may consider v to be a reduced edge path in G itself. The collection of compact open sets $C(v)$ for all paths v in G that are not entirely contained in Γ contains the cylinder sets determined by words in $\mathbb{F}$ that determine a basis for topology of $\partial^2\mathbb{F}$. Since a relative current is uniquely determined by its values on elements in $\mathbb{F}$, it is also uniquely determined by its values on compact open sets determined by reduced paths v in G that are not entirely contained in Γ.

Lemma 4.3.2. *Let $\phi : G \to G$ be a completely split train track representative of Φ, a fully irreducible outer automorphism relative to $\mathcal{A}$. Let a be an edge in the top EG stratum H_r such that ρ_a is fixed under ϕ. Let v be any reduced edge path in G that crosses H_r. Let $d_{v,a}$ be the frequency of occurrence of v in ρ_a. Then the set of values*

$$d_{v,a} + d_{\overline{v},a} =: \eta^a_\phi(v)$$

define a unique current η^a_ϕ relative to $\mathcal{A}$. That is,

(a) $\eta^a_\phi(v) \geq 0$,

(b) $\eta^a_\phi(v) = \eta^a_\phi(\overline{v})$,

(c) $\eta^a_\phi(v) = \sum_{e \in E} \eta^a_\phi(ve)$ *where E is the set of edges of G and e is not equal to the inverse of the terminal edge of v.*

For an edge $b \neq a$ in H_r we have that $\eta_\phi^b = \kappa \eta_\phi^a$ for some constant $\kappa(a, b, \phi|_{H_r})$. Thus for every fully irreducible outer automorphism relative to $\mathcal{A}$, get a unique projective relative current, denoted $[\eta_\Phi^+] = [\eta_\phi^a]$.

Proof. By Proposition 3.6.1, we know that the values $d_{v,a}$ exist and are non-negative for all reduced paths v in G that cross H_r. The equation (b) holds by definition of $\eta_\phi^a(v)$. Proposition 3.6.1 provides a substitution determined by ϕ. Applying Proposition 3.5.5 to this substitution, we see that $\eta_\phi^a(v)$ satisfies Kirchoff's laws, that is, (c) holds. Since a relative current is uniquely determined by its values on compact open sets in $\partial^2\mathbb{F}$ determined by reduced paths in G that cross H_r, we get a unique relative current η_ϕ^a. Again by Proposition 3.6.1, we have $\eta_\phi^a(v) = \kappa\eta_\phi^b(v)$ for all reduced paths v in G that cross H_r and for some constant κ. Thus the projective class $[\eta_\phi^a] =: [\eta_\Phi^+]$ of the relative current η_ϕ^a is also unique. □

The projective relative current $[\eta_\Phi^+]$ is called the *stable current* for Φ. The stable current for ϕ^{-1}, denoted $[\eta_\Phi^-]$, is called the *unstable current* for Φ.

4.4 Examples

Relative currents are uniquely determined by their values on words in $\mathbb{F} \setminus \mathcal{A}$. Using the substitution dynamics techniques developed in Chapter 3, we show some examples of how to calculate $\eta_\Phi^+(w)$ for $w \in \mathbb{F} \setminus \mathcal{A}$ for some relative outer automorphisms Φ. The three examples that follow illustrate the cases when the growth in the stratum corresponding to $\mathcal{A}$ is less than, greater than and equal to the growth in the top EG stratum.

Example 4.4.1. Let $F_3 = \langle a, b, c \rangle$. Let G be the rose on three petals labeled a, b and c. Consider an outer automorphism Φ given by a train track representative $\phi : G \to G$ where

$$\phi(a) = a, \phi(b) = bac, \phi(c) = cbac.$$

Let $\mathcal{A} = \{[\langle a \rangle]\}$. The transition matrix for ϕ is given by

$$M = \begin{array}{c} \begin{array}{ccc} b & c & a \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Note that Φ is not fully irreducible relative to $\mathcal{A}$ because $\{[\langle b, ac\rangle], [\langle a\rangle]\}$ is Φ-invariant. But it is still instructive to understand the limiting behavior in this simple case.

Let $\rho_b = \lim_{n\to\infty} \phi^n(b)$ be a ray that is fixed by ϕ. View ϕ as a substitution ζ on the alphabet $\mathbb{A} = \{a, b, c\}$. Let $\mathbb{A}_l$ be the set of words of length l on $\mathbb{A}$ that appear in ρ_b. For example, $\mathbb{A}_2 = \{ba, ca, cb, ac\}$. Note that the sets $\mathbb{A}_l$ are independent of the specific choice b. Define a substitution ζ_l on $\mathbb{A}_l$ as follows: let $w \in \mathbb{A}_l$ start with $x \in \mathbb{A}$. Then $\zeta_l(w)$ consists of the ordered list of the first $|\zeta(x)|$ subwords of length l of the word $\zeta(w)$. For example, $\zeta_2(ba) = ba \cdot ac \cdot ca$. Let M_l be the transition matrix of ζ_l and let $\mathcal{B}_l$ be the transition matrix for ζ_l restricted to words in $\mathbb{F} \setminus \mathcal{A}$. We want to calculate the frequency of occurrences of words which are not in $\mathcal{A}$, in ρ_b.

Let $w \in \mathbb{A}_l$ and let β be a word of length l that starts with b. Then

$$\lim_{n\to\infty} \frac{(w, \phi^n(b))}{\lambda^n} = \lim_{n\to\infty} \frac{M_l^n(w,\beta)}{\lambda^n} = \lim_{n\to\infty} \frac{\mathcal{B}_l^n(w,\beta)}{\lambda^n} =: d_{w,b}$$

Here λ is the PF-eigenvalue of the top EG stratum. See Section 3.5 for detailed explanation. For example, in length one and two, we have

$$\mathcal{B}_1 = \begin{array}{c} \begin{array}{cc} b & c \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array}, \quad \mathcal{B}_2 = \begin{array}{c} \begin{array}{cccc} ba & ca & cb & ac \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Let $\beta = b$ and $\beta = ba$ for length one and length two words, respectively. Then

$$d_{b,b} = \frac{(5-\sqrt{5})}{10}, \quad d_{c,b} = \frac{1}{\sqrt{5}},$$
$$d_{ac,b} = \frac{1}{\sqrt{5}}, \quad d_{ba,b} = \frac{(5-\sqrt{5})}{10},$$
$$d_{ca,b} = \frac{(-5+3\sqrt{5})}{10}, \quad d_{cb,b} = \frac{(5-\sqrt{5})}{10}.$$

We get $d_{b,b} = d_{ba,b}$ and $d_{c,b} = d_{ca,b} + d_{cb,b}$, which indicates that additivity holds for η^+_Φ (defined in Lemma 4.3.2).

One way to calculate the above numbers is to compute the Jordan decomposition of the matrix $\mathcal{B}_l$. Say $\mathcal{B}_l = SJS^{-1}$. Consider another matrix J' which has a 1 in the spot for λ and zeros everywhere else. Compute $SJ'S^{-1}$ and read off entries from the column corresponding to β. For example, we have

$$B_2 = SJS^{-1} = S \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(3-\sqrt{5})}{2} & 0 \\ 0 & 0 & 0 & \frac{(3+\sqrt{5})}{2} \end{bmatrix} S^{-1},$$

$$J' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad SJ'S^{-1} = \begin{bmatrix} \frac{(5-\sqrt{5})}{10} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{(-5+3\sqrt{5})}{10} & \frac{(5-\sqrt{5})}{10} & \frac{(5-\sqrt{5})}{10} & 0 \\ \frac{(5-\sqrt{5})}{10} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{(5+\sqrt{5})}{10} & \frac{(5+\sqrt{5})}{10} & 0 \end{bmatrix}.$$

Example 4.4.2. Let $F_4 = \langle a, b, c, d \rangle$. Let G be the rose on four petals labeled a, b, c, d. Consider an outer automorphism Φ given by a train track representative $\phi : G \to G$ by

$$\phi(a) = abbab, \phi(b) = bababbab, \phi(c) = cad, \phi(d) = dcad.$$

Let $\mathcal{A} = \{[\langle a, b \rangle]\}$. The transition matrix for ϕ is given by

$$M = \begin{array}{c} \begin{matrix} c & d & a & b \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{bmatrix} \end{array}$$

Let $\rho_c = \lim_{n\to\infty} \phi^n(c)$. Consider ϕ as a substitution on the alphabet $\mathbb{A} = \{a, b, c, d\}$. Let $\mathbb{A}_l$ be the set of words of length l on $\mathbb{A}$ that appear in ρ_c. We want to calculate the frequency of occurrences of words, which cross c and d, in ρ_c. Let $w \in \mathbb{A}_l$ and let γ be a word of length l that starts with c. Using the same notation as in the previous example,

$$\lim_{n\to\infty} \frac{(w, \phi^n(c))}{\lambda^n} = \lim_{n\to\infty} \frac{M_l^n(w,\gamma)}{\lambda^n} = \lim_{n\to\infty} \frac{\mathcal{B}_l^n(w,\gamma)}{\lambda^n} =: d_{w,c}$$

For example, $\mathbb{A}_2 = \{ab, ba, bb, ad, bd, ca, da, dc\}$ and $\mathcal{B}_2 = \{ad, bd, ca, da, dc\}$. We get the matrices

$$\mathcal{B}_1 = \begin{array}{c} \begin{matrix} b & c \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array}, \quad \mathcal{B}_2 = \begin{array}{c} \begin{matrix} ca & da & dc & ad & bd \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

and compute the frequencies as in the previous example.

Example 4.4.3. Let $F_4 = \langle a, b, c, d\rangle$. Let G be the rose on four petals labeled a, b, c, d. Consider an outer automorphism Φ given by a train track representative $\phi : G \to G$ by

$$\phi(a) = ab, \phi(b) = bab, \phi(c) = cad, \phi(d) = dcad.$$

Let $\mathcal{A} = \{[\langle a, b\rangle]\}$. The transition matrix for ϕ is given by

$$M = \begin{array}{c} \begin{array}{cccc} c & d & a & b \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

Let $\rho_c = \lim_{n\to\infty} \phi^n(c)$. Consider ϕ as a substitution on the alphabet $\mathbb{A} = \{a, b, c, d\}$. As before,

$$\lim_{n\to\infty} \frac{(w, \phi^n(c))}{\lambda^n} = \lim_{n\to\infty} \frac{M_l^n(w,\gamma)}{\lambda^n} = \lim_{n\to\infty} \frac{\mathcal{B}_l^n(w,\gamma)}{\lambda^n} =: d_{w,c}$$

where λ is the PF-eigenvalue of the top stratum. We have $\mathbb{A}_2 = \{ab, ba, bb, ad, bd, ca, da, dc\}$ and $\mathcal{B}_2 = \{ad, bd, ca, da, dc\}$. We get the matrices

$$\mathcal{B}_1 = \begin{array}{c} \begin{array}{cc} b & c \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array}, \quad \mathcal{B}_2 = \begin{array}{c} \begin{array}{ccccc} ca & da & dc & ad & bd \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

and compute the frequencies as above.

4.5 Goodness

In [BFH97], Bestvina, Feighn and Handel studied the legal structure of conjugacy classes under forward and backward iterates of a train track representative of a fully irreducible outer automorphism. In [Bri00], Brinkmann generalized some of those results to relative train track maps which will be used in this section.

Throughout this section, $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ will be a fully irreducible outer automorphism relative to $\mathcal{A}$ and $\phi : G \to G$ a completely split train track representative of Φ with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_r = G$ such that $\mathcal{F}(G_{r-1}) = \mathcal{A}$, and H_r is the top EG stratum with PF eigenvalue $\lambda_\Phi > 1$. In this section, we use Facts 2.4.1 about completely split train track maps.

In [Bri00], Brinkmann considers the following metric on G: edges in H_r get length according to the PF eigenvector such that the smallest length is one and hence edges in H_r get stretched by λ_Φ under the application of ϕ. Edges in G_{r-1} get length one.

Throughout, we use the same notation for a conjugacy class in $\mathbb{F}$ and its representative in G which is taken to be cyclically reduced. For a reduced path ρ in G by $[\phi(\rho)]$, we mean the tightened image of ρ. Define $i_r(\rho)$ to be the number of r-illegal turns in ρ, $l_r(\rho)$ the r-length of ρ and $L_r(\rho)$ the length of the longest r-legal segment in ρ. Recall from Section 4.1.3 that $L_r^c = \dfrac{2\,\mathrm{BCC}(\phi)}{\lambda_\Phi - 1}$ is the critical r-length where $\mathrm{BCC}(\phi)$ is the bounded cancellation constant.

Denote by ρ^{-k} a path in G with the property that the tightened image of $\phi^k(\rho^{-k})$ is ρ. For a subpath ρ of a path σ, let $[\phi^k(\rho)]_\sigma$ denote the maximal subpath of $[\phi^k(\rho)]$ contained in $[\phi^k(\sigma)]$.

The following proposition is a generalization of [BFH97, Lemma 2.9].

Proposition 4.5.1 ([Bri00, Lemma 6.2]). *Let $\phi : G \to G$ be a relative train track map and let H_r be an EG stratum. For every $L > 0$, $\exists\, M(L) > 0$ such that if ρ is a path in G_r that crosses H_r, then one of the following holds:*

(a) *$[\phi^M(\rho)]$ contains an r-legal segment of r-length $> L$.*

(b) *$[\phi^M(\rho)]$ has fewer r-illegal turns.*

(c) *ρ can be expressed as a concatenation $\tau_1\rho'\tau_2$, where $l_r(\tau_1) \leq 2L$, $l_r(\tau_2) \leq 2L$, $i_r(\tau_1) \leq 1, i_r(\tau_2) \leq 1$, and ρ' splits as a concatenation of pre-Nielsen paths (with one r-illegal turn each) and segments in G_{r-1}.*

Lemma 4.5.2 (Backward iterations). *Let $\phi : G \to G$ be a completely split train track representative of a fully irreducible outer automorphism relative to $\mathcal{A}$. Given some number $L_0 > 0$, there exists $M > 0$, depending only on L_0 and H_r, such that for any subpath ρ of an $\mathcal{A}$-separable conjugacy class α realized in G_r with $1 \leq L_r(\rho) \leq L_0$ and $i_r(\rho) \geq 5$, we have*

$$\left(\frac{10}{9}\right)^n i_r(\rho) \leq i_r(\rho^{-nM})$$

for all $n > 0$.

Proof. In [Bri00, Lemma 6.4], the same statement is proved for atoroidal outer automorphisms and for any path in G_r. The same proof follows by using Facts 2.4.1 about completely split train track representatives.

Given $L = L_0 + L_r^c$, choose M as in Proposition 4.5.1. Subdivide the path ρ into subpaths $\rho_1, \ldots, \rho_m, \tau$ such that $i_r(\rho_i) = 5$ and $i_r(\tau) < 5$. Let ρ_i^{-M} be the pre-image of ρ_i under ϕ^M. Then ρ^{-M} is the concatenation of ρ_i^{-M} and τ^{-M}. We claim that $i_r(\rho_i^{-M}) \geq 6$ for all i. Suppose for contradiction that $i_r(\rho_i^{-M}) = 5$ for some i. Then by Proposition 4.5.1, ρ_i^{-M} splits as a concatenation of at least three pre-Nielsen paths and paths in G_{r-1}. By Facts 2.4.1, every Nielsen path has period one and there is at most one INP σ of height r. If σ is not closed, then at least one end-point of σ is not contained in G_{r-1}. Therefore, we cannot have three Nielsen paths in ρ_i^{-M} separated by paths in G_{r-1}. If σ is closed, then its end point is not in G_{r-1}. Since α is $\mathcal{A}$-separable, it cannot have two consecutive occurrences of σ in it. Indeed, since σ (which is not contained in G_{r-1}) is fixed by ϕ, it is not $\mathcal{A}$-separable. Therefore, its relative Whitehead graph is connected without cut points. If α has two consecutive occurrences of σ, then its relative Whitehead graph will also be connected without cut points, but α is $\mathcal{A}$-separable. Therefore, ρ and ρ_i^{-M} cannot have two consecutive occurrences of σ.

We claim that $i_r(\rho^{-M}) \geq 6m + i_r(\tau) \geq (10/9)i_r(\rho)$ and the lemma follows by induction. Indeed, we have

$$\frac{i_r(\rho^{-M})}{i_r(\rho)} \geq \frac{6s + i_r(\tau)}{5s + i_r(\tau)} \geq \frac{10}{9}$$

when $i_r(\tau) = 4$ and $s = 1$. Here $i_r(\rho) = 5s + i_r(\tau)$ because the concatenation points are legal. □

Lemma 4.5.3 ([Bri00, Lemma 6.5])**.** *Suppose H_r is an EG stratum. Given some $L > 0$, there exists some constant $C > 0$ such that for all paths $\rho \subset G_r$ with $1 \leq L_r(\rho) \leq L$ and $i_r(\rho) > 0$, we have*

$$C^{-1} i_r(\rho) \leq l_r(\rho) \leq C i_r(\rho).$$

The notion of goodness was introduced in [Mar95] and formalized in [BFH97].

Definition 4.5.4 (Goodness)**.** Given a loop or a path α in G_r that crosses H_r, we say that the *good portion*, denoted g, of α is the set of r-legal segments that are r-distance L_r^c away from

r-illegal turns. The *bad portion*, denoted b, is the part of α which is r-distance less than or equal to L_r^c from an r-illegal turn. The r-length of α is equal to the r-length of g (denoted $g_r(\alpha)$) plus the r-length of b (denoted $b_r(\alpha)$). Define *goodness* of α as

$$\mathfrak{g}(\alpha) = \frac{g_r(\alpha)}{l_r(\alpha)}.$$

Lemma 4.5.5 (Monotonicity of goodness)**.** *Let $\delta > 0$ and $\epsilon > 0$ be given. Then there exists an integer $M = M(\delta, \epsilon)$ such that for any $\mathcal{A}$-separable conjugacy class α that crosses H_r with $\mathfrak{g}(\alpha) \geq \delta$, we have $\mathfrak{g}(\phi^m(\alpha)) \geq 1 - \epsilon$ for all $m \geq M$.*

The proof of the above lemma which is the same as in the absolute case can be found in [Uya14, Lemma 3.10].

Definition 4.5.6 (Desired growth [Bri00])**.** Let σ be a path in G that crosses an EG stratum H_r. We say σ has desired growth if there exist $N > 0, \lambda > 1, \epsilon > 0$ and a collection of subpaths S of σ such that the following hold:

(a) For every integer $n > 0$ and for every $\rho \in S$, we have

$$\lambda^n l_r(\rho) \leq \max\{l_r([\phi^{nN}(\rho)]_\sigma), l_r(\gamma)\},$$

where γ is a subpath of σ^{-nN} such that $[\phi^{nN}(\gamma)]_{\sigma^{-nN}} = \rho$.

(b) There is no overlap between distinct paths in S.

(c) The sum of the lengths of the paths in S is at least $\epsilon l_r(\sigma)$.

Lemma 4.5.7. *Let $\alpha \in \mathbb{F}$ be an $\mathcal{A}$-separable conjugacy class that crosses H_r. Then α has desired growth either under forward iteration or under backward iteration.*

Proof. Let $L_0 > L_r^c$ be a constant. There are several cases to consider.

1. $\dfrac{l_r(\alpha)}{i_r(\alpha)} \geq L_0$. The proof of [Bri00, Proposition 7.1 (2)(b)(i)] shows that in this case, α has desired growth in the forward direction.

2. $\dfrac{l_r(\alpha)}{i_r(\alpha)} < L_0$.

   (a) $i_r(\alpha) \geq 5$. By [Bri00, Proposition 7.1 (2)(b)(ii)] and using Lemma 4.5.2, 4.5.3 we get desired growth in the backward direction.

(b) $i_r(\alpha) < 5$. We have that α is $\mathcal{A}$-separable and crosses H_r nontrivially. Therefore, α is not fixed and does not have two consecutive occurrences of a closed INP. Since $l_r(\alpha)$ is bounded from above, there are only finitely many possibilities for $\alpha \cap H_r$. Suppose the r-length of no segment of $\alpha \cap H_r$ grows under ϕ. Since there are only finitely many segments of H_r of bounded length, after passing to a power, assume that a segment α_i of $\alpha \cap H_r$ is fixed under ϕ. Also the end points of α_i are in $H_r \cap G_{r-1}$. There has to be an illegal turn in α_i, otherwise it would grow and in fact, it has to be an INP because it persists. But at least one end-point of an INP in G is not in G_{r-1}, thus we get a contradiction. Therefore, we can pass to a uniform power M such that $\phi^M(\alpha)$ satisfies (1) and hence has desired growth in forward direction.

It can be seen in Brinkmann's proofs that the numbers N, λ, ϵ do not depend on a specific conjugacy class. $\square$

Let $\phi' : G' \to G'$ be a completely split train track representative of Φ^{-1}. Let $l_{r'}$, $i_{r'}$, $L^c_{r'}$ and C' be the corresponding notation related to ϕ'. There exists a constant B such that for any conjugacy class α, we have

$$\frac{l_{r'}(\alpha)}{B} \leq l_r(\alpha) \leq B l_{r'}(\alpha).$$

Let $\mathfrak{g}'$ denote the goodness with respect to the train track structure of ϕ'.

Lemma 4.5.8. *Given $\delta > 0$, there exists $M > 0$ such that for any $\mathcal{A}$-separable conjugacy class α that crosses H_r, either*

- $\mathfrak{g}(\phi^{nM}(\alpha)) \geq \delta$ *for all* $n \geq 1$ *or*
- $\mathfrak{g}'((\phi')^{nM}(\alpha)) \geq \delta$ *for all* $n \geq 1$.

Proof. Let $L_0 > L^c_r$ be the constant from Lemma 4.5.7. By the same lemma, there exist $N > 0, \lambda > 1$ and $\epsilon > 0$ such that any $\mathcal{A}$-separable conjugacy class that crosses H_r has desired growth. There are two cases:

(a) Let's first consider the case when α has desired growth in the forward direction. This happens when $l_r(\alpha) \geq L_0 i_r(\alpha)$. For case 2(b) in the proof of Lemma 4.5.7, pass to a uniform power of α which satisfies $l_r(\alpha) \geq L_0 i_r(\alpha)$. Let S be the collection of maximal r-legal subpaths of α of r-length at least $L_0 + 1$. Then by the choice of L_0 we have for $\rho \in S$,

$$l_r(\phi^{nN}(\rho)) \geq \lambda_\Phi^{nN} \frac{1}{L_0+1} l_r(\rho).$$

The paths in S account for a definite fraction $\epsilon > 0$ of α. Now

$$g_r(\phi^{nN}(\alpha)) \geq \sum_{\rho \in S} [l_r(\phi^{nN}(\rho))]_\alpha \geq \sum_{\rho \in S} \lambda_\Phi^{nN} \frac{1}{L_0+1} [l_r(\rho)]_\alpha \geq \lambda_\Phi^{nN} \frac{1}{L_0+1} \epsilon l_r(\alpha).$$

We also have $l_r(\phi^{nN}(\alpha)) \leq \lambda_\Phi^{nN} l_r(\alpha)$. Thus

$$\mathfrak{g}(\phi^{nN}(\alpha)) \geq \frac{\epsilon}{L_0+1}.$$

(b) If α has desired growth in the backward direction, then Lemma 4.5.2 and Lemma 4.5.3 imply

$$B l_{r'}((\phi')^{nN}(\alpha)) \geq l_r(\phi^{-nN}(\alpha)) \geq C^{-1} i_r(\phi^{-nN}(\alpha)) \geq \left(\frac{10}{9}\right)^n \frac{1}{C^2 B} l_{r'}(\alpha).$$

Now the number of r'-illegal turns in $(\phi')^{nN}(\alpha)$ is bounded above by those in α. We have

$$i_{r'}((\phi')^{nN}(\alpha)) \leq i_{r'}(\alpha) \leq C' l_{r'}(\alpha).$$

Also the bad portion of $(\phi')^{nN}(\alpha)$ is bounded from above by $2L_{r'}^c i_{r'}((\phi')^{nN}(\alpha))$. Thus

$$\mathfrak{g}'((\phi')^{nN}(\alpha)) \geq 1 - \frac{2L_{r'}^c C' B^2 C^2}{(10/9)^n} \geq 1 - \frac{2L_{r'}^c C' B^2 C^2}{(10/9)}.$$

Now by Lemma 4.5.5, find $M > 0$ such that either one of the goodness is greater than δ. □

4.6 North-south dynamics

We are now ready to prove a north-south dynamic result. Recall we have Φ a fully irreducible outer automorphism relative to $\mathcal{A}$ and a completely split train track representative $\phi : G \to G$. We also have a stable current $[\eta_\Phi^+]$ and an unstable current $[\eta_\Phi^-]$ in $\mathcal{MRC}(\mathcal{A})$.

Notation 4.6.1. Let G be a marked metric graph in CV_{n} and let $\tilde{G}$ be the universal cover of G. By identifying $\partial^2\mathbb{F}$ with $\partial^2\tilde{G}$, we can define relative currents on cylinder sets determined by paths in G.

Let α be the realization in G of a conjugacy class in $\mathbb{F}$ and let v be a reduced path in G. Let (v, α) be the number of occurrences of v in α. For a relative current η, let

$$\langle v, \eta \rangle := \eta(v), \quad \langle v, \alpha \rangle := (v, \alpha) + (\overline{v}, \alpha)$$

$$|\eta|_r := \sum_{e \in EH_r} \langle e, \eta \rangle$$

where EH_r is the set of edges in H_r.

Proposition 4.6.2. *Given a neighborhood U of $[\eta_\Phi^+]$ in $\mathcal{MRC}(\mathcal{A})$, there exists $0 < \delta < 1$ and $M(U) > 0$ such that for any $[\eta_\alpha] \in \mathcal{MRC}(\mathcal{A})$, with $\mathfrak{g}(\alpha) > \delta$, we have that $\phi^n([\eta_\alpha]) \in U$ for all $n \geq M$.*

Proof. The proof is similar to the proof of [Uya14, Lemma 3.11].

A relative current $[\eta]$ is in U if there exists $\epsilon > 0$, $R >> 0$ both depending on U such that for all reduced paths v with $0 < l_r(v) \leq R$, we have

$$\left| \frac{\langle v, \eta \rangle}{|\eta|_r} - \frac{\langle v, \eta_\Phi^+ \rangle}{|\eta_\Phi^+|_r} \right| \leq \epsilon.$$

We need to show there exists a $\delta > 0$ and $M(U) > 0$ such that for any conjugacy class α with $\mathfrak{g}(\alpha) \geq \delta$ and for any $n \geq M$, we have that

$$\left| \frac{\langle v, \eta_{\phi^n(\alpha)} \rangle}{|\eta_{\phi^n(\alpha)}|_r} - \frac{\langle v, \eta_\Phi^+ \rangle}{|\eta_\Phi^+|_r} \right| \leq \epsilon, \text{ which is the same as } \left| \frac{\langle v, \phi^n(\alpha) \rangle}{l_r(\phi^n(\alpha))} - \frac{\langle v, \eta_\Phi^+ \rangle}{|\eta_\Phi^+|_r} \right| \leq \epsilon.$$

Let α be written as a concatenation of some good edges denoted c_i in H_r, some bad edges denoted b_j in H_r and some subpaths in G_{r-1}. Since there are only finitely many edges in H_r and finitely many paths v up to intersection with H_r with $l_r(v) \leq R$, pick an integer $N_0(U) \geq 1$ such that

$$\left| \frac{\langle v, \phi^n(c_i) \rangle}{l_r(\phi^n(c_i))} - \frac{\langle v, \eta_\Phi^+ \rangle}{|\eta_\Phi^+|_r} \right| \leq \epsilon/4$$

for all edges c_i and all $n \geq N_0(U)$. The following is true by triangle inequality.

$$\left| \frac{\langle v, \phi^n(\alpha) \rangle}{l_r(\phi^n(\alpha))} - \frac{\langle v, \eta_\Phi^+ \rangle}{l_r(\eta_\Phi^+)} \right| \leq \left| \frac{\langle v, \phi^n(\alpha) \rangle}{l_r(\phi^n(\alpha))} - \frac{\sum \langle v, \phi^n(c_i) \rangle}{l_r(\phi^n(\alpha))} \right| + \left| \frac{\sum \langle v, \phi^n(c_i) \rangle}{l_r(\phi^n(\alpha))} - \frac{\sum \langle v, \phi^n(c_i) \rangle}{\sum l_r(\phi^n(c_i))} \right|$$
$$+ \left| \frac{\sum \langle v, \phi^n(c_i) \rangle}{\sum l_r(\phi^n(c_i))} - \frac{\langle v, \eta_\Phi^+ \rangle}{|\eta_\Phi^+|_r} \right|$$

We will look at individual terms in the inequality and find upper bounds.

- The following argument will show that one can choose $N_1(U) > 0$ and $\delta > 0$ such that if $n \geq N_1(U)$ and if $\mathfrak{g}(\alpha) > \delta$, then the contribution to occurrences of v in $\phi^n(\alpha)$ from mixed regions and from bad portions of α is arbitrarily small.

  The segment v can occur in $\phi^n(\alpha)$ either as a subsegment of some $\phi^n(c_i)$, or as a subsegment of $\phi^n(b_i)$, or v can cross those mixed regions in $\phi^n(\alpha)$ whose pre-image in α is also a mixed region. The number of such mixed regions is bounded by $l_r(\alpha)$. Thus
$$\left| \frac{\langle v, \phi^n(\alpha)\rangle}{l_r(\phi^n(\alpha))} - \frac{\sum\langle v, \phi^n(c_i)\rangle}{l_r(\phi^n(\alpha))} \right| \leq \frac{R l_r(\alpha)}{l_r(\phi^n(\alpha))} + \frac{\sum\langle v, \phi^n(b_i)\rangle}{l_r(\phi^n(\alpha))}.$$
  Using the definition of goodness, we have the following:
$$l_r(\phi^n(\alpha)) \geq \lambda_r^n \mathfrak{g}(\alpha) l_r(\alpha), \qquad b_r(\alpha) \leq l_r(\alpha_r)(1-\mathfrak{g}(\alpha)), \qquad \sum l_r(\phi^n(b_i)) \leq \lambda_r^n b_r(\alpha)$$
  where $b_r(\alpha)$ is the length of the bad portion of α. Thus
$$\frac{\sum l_r(\phi^n(b_i))}{l_r(\phi^n(\alpha))} \leq \frac{\lambda_r^n(1-\mathfrak{g}(\alpha))l_r(\alpha)}{\lambda_r^n \mathfrak{g}(\alpha) l_r(\alpha)} \leq \frac{(1-\mathfrak{g}(\alpha))}{\mathfrak{g}(\alpha)} \leq \epsilon/4.$$
  Choose $\delta > 0$ such that $\frac{1}{1+\epsilon/4} < \delta < 1$ and such that the above statement holds for $\mathfrak{g}(\alpha) > \delta$. We have
$$\frac{R l_r(\alpha)}{l_r(\phi^n(\alpha))} \leq \frac{R}{\lambda_r^n \mathfrak{g}(\alpha)}.$$
  Choose $N_1(R,\epsilon) = N_1(U) \geq 1$ such that $N_1 \geq \log_{\lambda_r}(R(1+4/\epsilon))$ and $\mathfrak{g}(\alpha) > \delta$ so that
$$\frac{R|\alpha|_r}{|\phi^n(\alpha)|_r} \leq \frac{R}{\lambda_r^n \mathfrak{g}(\alpha)} \leq \epsilon/4$$
  for all $n \geq N_1(U)$. Thus for all $n \geq N_1(U)$, we have
$$\left| \frac{\langle v, \phi^n(\alpha)\rangle}{|\phi^n(\alpha)|_r} - \frac{\sum\langle v, \phi^n(c_i)\rangle}{|\phi^n(\alpha)|_r} \right| \leq \epsilon/2$$

- $$\left| \frac{\sum\langle v, \phi^n(c_i)\rangle}{l_r(\phi^n(\alpha))} - \frac{\sum\langle v, \phi^n(c_i)\rangle}{\sum l_r(\phi^n(c_i))} \right| \leq \frac{\sum l_r(\phi^n(b_i))}{l_r(\phi^n(\alpha))} \frac{\sum\langle v, \phi^n(c_i)\rangle}{\sum l_r(\phi^n(c_i))} \leq (\epsilon/4)(1)$$
  for all $n \geq N_1(U)$.

- $$\left| \frac{\sum\langle v, \phi^n(c_i)\rangle}{\sum l_r(\phi^n(c_i))} - \frac{\langle v, \eta_\Phi^+\rangle}{|\eta_\Phi^+|_r} \right| \leq \left| \frac{\langle v, \phi^n(c_i)\rangle}{l_r(\phi^n(c_i))} - \frac{\langle v, \eta_\Phi^+\rangle}{|\eta_\Phi^+|_r} \right| \leq \epsilon/4$$
  for some edge c_i in H_r by using mediant inequality (for $a, b, c, d > 0$, we have that $\frac{c}{d} \leq \frac{a+c}{b+d} \leq \frac{a}{b}$) for all $n \geq N_0$.

Thus combining the three steps, we have that

$$\left| \frac{\langle v, \phi^n(\alpha) \rangle}{l_r(\phi^n(\alpha))} - \frac{\langle v, [\eta_\Phi^+] \rangle}{|\eta_\Phi^+|_r} \right| \leq \epsilon$$

for all $n \geq M(U)$ where $M(U) = max\{N_0, N_1\}$. □

Lemma 4.6.3. *Given neighborhoods U and V of $[\eta_\Phi^+]$ and $[\eta_\Phi^-]$ in $\mathcal{MRC}(\mathcal{A})$, respectively, there exists $M_1 > 0$ such that for any $\mathcal{A}$-separable conjugacy class α that crosses H_r, either $\phi^m([\eta_\alpha]) \in U$ or $(\phi')^m([\eta_\alpha]) \in V$ for all $m \geq M_1$.*

The proof follows from Lemma 4.5.8 and Lemma 4.6.2.

Proposition 4.6.4 ([LU15, Proposition 3.4])**.** *Let $\phi : X \to X$ be a homeomorphism of a compact space X and assume that X is sufficiently separable, for example metrizable. Let $Y \subset X$ be a dense set, and let $\mathcal{P}, \mathcal{Q}$ be two distinct ϕ-invariant points in X. Assume the following holds: for every neighborhood U of $\mathcal{P}$ and V of $\mathcal{Q}$, there exists an integer $M_2 \geq 1$ such that for all $m \geq M_2$ and all $y \in Y$, one has either $\phi^m(y) \in U$ or $\phi^{-m}(y) \in V$. Then ϕ^2 has uniform north-south dynamics from $\mathcal{P}$ to $\mathcal{Q}$.*

Proof. We recollect the proof from [LU15, Proposition 3.4] here. Let K be a compact set in $X \setminus Q$. The set K may or may not be disjoint from V. If not, then consider an open neighborhood W of K which is also disjoint from Q. Then $V_1 = V \setminus (V \cap \overline{W})$ is also a neighborhood of Q. Now consider $y \in Y \cap f^m(\overline{W})$. Then $f^{-m}(y)$ is not in V_1 because $\overline{W}$ is disjoint from V_1, therefore $f^m(y) \in U$. Since Y is dense in X, $Y \cap W$ is also dense in W. This is not true for a closed set or a compact set, we need an open set. Consider an open set $U_1 \subset U$ such that $\overline{U_1} \subset U$. We need this because we are working with a dense set and will need to take a closure. Therefore, $f^m(f^m(\overline{W}))$ is in U_1. Thus $f^{2m}(K)$ is in U. Similar argument works for f^{-1}. Thus f^2 has generalized north-south dynamics. □

Proposition 4.6.5 ([LU15, Proposition 3.5])**.** *Let $\phi : X \to X$ be as in Proposition 4.6.4 with distinct fixed points $\mathcal{P}$ and $\mathcal{Q}$ and assume that some power ϕ^s with $s \geq 1$ has uniform north-south dynamics from $\mathcal{P}$ to $\mathcal{Q}$. Then ϕ also has uniform north-south dynamics from $\mathcal{P}$ to $\mathcal{Q}$.*

Theorem B. *Let $\mathcal{A}$ be a nontrivial free factor system of $\mathbb{F}$ such that $\zeta(\mathcal{A}) \geq 3$. Let $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ be fully irreducible relative to $\mathcal{A}$. Then Φ acts with uniform north-south dynamics on $\mathcal{MRC}(\mathcal{A})$.*

Proof. The proof follows from Lemma 4.6.3, Proposition 4.6.4 and Proposition 4.6.5. $\square$

4.7 Summary

In this chapter, we defined relative currents and showed that a fully irreducible outer automorphism relative to $\mathcal{A}$ acts with uniform north-south dynamics on a certain subspace of the space of projectivised relative currents. In the next chapter, we will show that such an outer automorphism also acts with north-south dynamics on a relative version of Culler-Vogtmann's outer space. In Chapter 6, we will establish a duality between relative currents and trees in relative outer space.

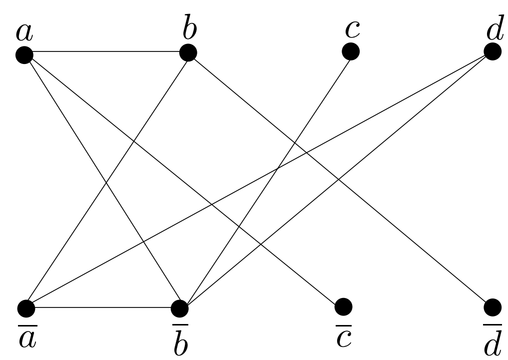

Figure 4.1. Whitehead graph for $\alpha = cadb$ and $\alpha_1 = ab\overline{a}\overline{b}$, Example 4.2.20

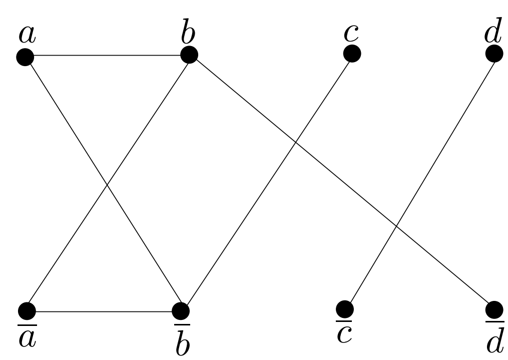

Figure 4.2. Whitehead graph for $\alpha = cdb$ and $\alpha_1 = ab\overline{a}\overline{b}$, Example 4.2.20

CHAPTER 5

NORTH-SOUTH DYNAMICS ON RELATIVE OUTER SPACE

In the surface theory, a pseudo-Anosov mapping class group element acts with uniform north-south dynamics on the compactified Teichmüller space. In [BFH97], Bestvina, Feighn and Handel showed that a fully irreducible outer automorphism acts with north-south dynamics in the interior of Culler-Vogtmann's outer space $CV_{\mathfrak{n}}$. Then in [LL03], Levitt and Lustig showed that in fact, north-south dynamics holds for all points in the closure of $CV_{\mathfrak{n}}$. The key technical tool they introduced was a map called $\mathcal{Q}$-map defined from the boundary of $\mathbb{F}$ to the completion of a tree in $CV_{\mathfrak{n}}$ union its boundary. In this chapter, we aim to generalize the north-south dynamics result to the action of a relative fully irreducible outer automorphism on the closure of relative outer space, $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$. The main result of this chapter is the following:

Theorem C. *Let $\mathcal{A}$ be a nontrivial free factor system of $\mathbb{F}$ such that $\zeta(\mathcal{A}) \geq 3$. Let $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ be fully irreducible relative to $\mathcal{A}$. Then Φ acts on $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ with uniform north-south dynamics.*

5.1 Relative outer space

In [GL07], Guirardel and Levitt define relative outer space for a countable group that splits as a free product

$$G = G_1 * \ldots * G_k * F_N$$

where $N + k \geq 2$. In [Hor14], Horbez shows that the closure of relative outer space is compact and characterizes the trees in the closure of relative outer space. In our setting, $G = \mathbb{F}$ and it splits as $\mathbb{F} = A_1 * \ldots * A_k * F_N$ for $k \geq 0$. Let $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$ be the associated free factor system of $\mathbb{F}$.

Subgroups of $\mathbb{F}$ that are conjugate into a free factor in $\mathcal{A}$ are called *peripheral* subgroups. An $(\mathbb{F}, \mathcal{A})$-tree is an $\mathbb{R}$-tree with an isometric action of $\mathbb{F}$, in which every peripheral subgroup fixes a unique point. A *Grushko* $(\mathbb{F},\mathcal{A})$-tree is a minimal, simplicial metric $(\mathbb{F},\mathcal{A})$-tree whose set of point stabilizers is exactly the free factor system $\mathcal{A}$ and edge stabilizers are trivial. Two $(\mathbb{F},\mathcal{A})$-trees are equivalent if there exists an $\mathbb{F}$-equivariant isometry between them. An $(\mathbb{F},\mathcal{A})$-tree T is *small* if arc stabilizers in T are either trivial, or cyclic and nonperipheral. A small $(\mathbb{F},\mathcal{A})$-tree T is *very small* if in addition, the nontrivial arc stabilizers in T are closed under taking roots and tripod stabilizers are trivial.

The *unprojectivized relative outer space* $\mathcal{O}(\mathbb{F},\mathcal{A})$ is the space of all equivalence classes of Grushko $(\mathbb{F},\mathcal{A})$-trees. *Relative outer space*, denoted $\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})$, is the space of homothety classes of trees in $\mathcal{O}(\mathbb{F},\mathcal{A})$.

Example 5.1.1. (a) Let $\mathbb{F} = A_1 * A_2$. In this case, relative outer space is just a point represented by a one edge splitting with vertex stabilizers A_1 and A_2 and trivial edge stabilizer.

(b) Let $\mathbb{F} = A_1 * \mathbb{Z}$. In this case, relative outer space is one-dimensional. A schematic is shown in part (i) of **Figure 5.1**. The central vertex v in (i) corresponds to the graph shown in (ii) and the end points of the one simplices in (i) correspond to graphs shown in (iii).

(c) Let $\mathbb{F} = A_1 * A_2 * A_3$. In this case, relative outer space is unbounded with respect to the simplicial metric.

The graph of groups decomposition of $\mathbb{F}$ represented in **Figure 5.2** is called a *relative rose*.

5.2 Preliminaries

Let Φ be a fully irreducible outer automorphism relative to $\mathcal{A}$.

Notation 5.2.1. Let $\phi'_0 : G' \to G'$ be a relative train track representative of Φ, where G' is a marked metric graph in CV_n, with filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_r = G'$ such that $\mathcal{A} = \mathcal{F}(G_{r-1})$ and the top stratum H_r is an EG stratum with Perron-Frobenius eigenvalue $\lambda_\Phi > 1$. Denote by Λ^+_Φ the attracting lamination associated to H_r and by $\Lambda^+_\Phi(G')$ its realization in G'. Let $T_{G'}$ be the universal cover of G' and let $\phi' : T_{G'} \to T_{G'}$ be a lift of the map $\phi'_0 : G' \to G'$ which satisfies $\Phi(g) \circ \phi' = \phi' \circ g$ for $g \in \mathbb{F}$.

Definition 5.2.2 ($\mathcal{A}$-train track map)**.** Let T_G be the tree in $\mathcal{O}(\mathbb{F}, \mathcal{A})$ obtained by equivariantly collapsing the maximal ϕ'-invariant proper forest of $T_{G'}$. Denote the collapse map by $\pi : T_{G'} \to T_G$. See **Figure 5.3**. The map $\phi' : T_{G'} \to T_{G'}$ descends to a map $\phi : T_G \to T_G$ representing Φ. Let $G = T_G/\mathbb{F}$ and $\phi_0 : G \to G$ be the corresponding map. We say ϕ_0 is an $\mathcal{A}$*-train track* representative of Φ.

5.3 Stable and unstable trees

$\mathrm{Out}(\mathbb{F}, \mathcal{A})$ acts on $\overline{\mathcal{O}(\mathbb{F}, \mathcal{A})}$ via

$$l_{T.\Psi}(\alpha) = l_T(\Psi(\alpha))$$

for $\Psi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ and for every conjugacy class $\alpha \in \mathbb{F}$, where $l_T(\alpha)$ is the translation length of α in T. A stable tree T^+_ϕ of Φ is defined as follows:

$$T^+_\phi = \lim_{p\to\infty} \frac{T_G \phi^p}{\lambda^p_\Phi}.$$

In other words,

$$l_{T^+_\phi}(\alpha) = \lim_{p\to\infty} \frac{l_{T_G}(\phi^p \alpha)}{\lambda^p_\Phi}.$$

The stable tree is well defined projectively and we denote the projective class by T^+_Φ. The unstable tree, denoted T^-_Φ, of Φ is defined to be the stable tree of Φ^{-1}. The fact that $T^\pm_\Phi$ do not depend on the choice of the train track map ϕ follows from the same arguments as in [BFH97, Lemma 3.4] whose relative version is stated below.

Proposition 5.3.1. *Let $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$. Suppose there exists a tree $T_0 \in \mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})$, an equivariant map $h : T_0 \to T$ and a bi-infinite geodesic $\gamma_0 \subset T_0$ representing a generic leaf γ of Λ^+_Φ such that $h(\gamma_0)$ has diameter greater than $2\,\mathrm{BCC}(h)$. Then*

(a) $h(\gamma_0)$ has infinite diameter in T.

(b) there exists a neighborhood V of T such that $\Phi^p(V)$ converges to T^+_Φ uniformly as $p \to \infty$.

The proof of Proposition 5.3.1 is essentially the same as in the absolute case in [BFH97, Lemma 3.4] and [LL03, Proposition 6.1]. After proving Proposition 5.3.1, our goal will be to prove that every tree $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ satisfies the assumptions of Proposition 5.3.1 if γ is allowed to be either in Λ^+_Φ or Λ^-_Φ.

Proposition 5.3.1 (a). Fix an equivariant map $\mu : T_G \to T_0$ with some bounded backtracking. Let γ_0 be the tightened image of γ, a generic leaf of Λ^+_Φ, under μ. Let $h : T_0 \to T$ be the $\mathbb{F}$-equivariant map as given in the proposition. If $AB \subset T_G$ is a segment, denote by $l_T(\nu(AB))$ the length of the tightened image of AB under $\nu = h \circ \mu$. Let $\mathrm{Lip}(\nu)$ be the Lipschitz constant of ν and let $\mathrm{BCC}(\nu)$ be the bounded backtracking constant. We have $\mathrm{BCC}(\nu) \leq \mathrm{Lip}(\mu)\,\mathrm{BCC}(\mu) + \mathrm{Lip}(h)\,\mathrm{BCC}(h)$.

By assumption, there is a segment A_0B_0 in γ_0 such that its image in T by h has length greater than $2\,\mathrm{BCC}(h)$. Let σ be the central subsegment of $h(A_0)h(B_0)$ whose length is $l_T(h(A_0)h(B_0)) - 2\,\mathrm{BCC}(h)$. We can find a segment $AB \subset \gamma$ such that its image by μ contains A_0B_0 and hence its tightened image by ν contains σ. Choose m_0 such that $\phi^{m_0}(e)$ contains a translate of AB for every edge e in T_G. If β is any leaf segment contained in Λ^+_Φ, then $l_T(\nu(\phi^{m_0}(\beta))) \geq l_T(\sigma)|\beta|$ where $|\beta|$ is the simplicial length of β in T_G.

We claim that $h(\gamma_0)$ has infinite diameter in T. Indeed, the attracting lamination is given by the closure of a generic leaf, say γ_0. A leaf $\gamma \in \overline{\gamma_0}$ if every subsegment of γ is contained in γ_0. Since $\overline{\gamma_0}$ is invariant under the action of ϕ, we have $\phi^{m_0}(\gamma_0) \in \overline{\gamma_0}$. This implies that $\phi^{m_0}(\beta)$ is a subsegment of γ_0. Thus $h(\gamma_0)$ has infinite diameter in T. □

We have that $h(\gamma_0)$ has infinite diameter in T. Consequently, for every edge $e \in T_G$, the length $l_T(\nu(\phi^p(e)))$ tends to infinity with p. Let β be an arbitrary edge path in T_G and let

$$d_+(\beta) = \lim_{p\to\infty} \frac{l_{T_G}(\phi^p(\beta))}{\lambda^p_\Phi}.$$

The following lemma is restating Lemma 7.1 and Lemma 7.2 in [LL03] in the relative setting and will be used to prove Proposition 5.3.1(b).

Lemma 5.3.2. *(a) There exists $c > 0$ such that for all $\beta \subset \Lambda^+_\Phi(T_G)$*

$$\lim_{p\to\infty} \frac{l_T(\nu(\phi^p(\beta)))}{\lambda^p_\Phi l_{T_G}(\beta)} = c.$$

(b) Let β be an arbitrary edge path in T_G. Then

$$\lim_{p\to\infty} \frac{l_T(\nu(\phi^p(\beta)))}{\lambda^p_\Phi d_+(\beta)} = c$$

and the convergence is uniform, that is, it is independent of β.

Proof. The proofs are essentially the same as for [LL03, Lemma 7.1, 7.2]. We provide the proofs here for completeness.

(a) For an edge e in T_G let N_e^p be the number of occurrences of e in $\phi^p(\beta)$. Since transition matrix of $\phi_0 : G \to G$ is primitive, $\lim_{p\to\infty} N_e^p/\lambda_\Phi^p$ has the form $c_e k_\beta$ where c_e depends only on e and k_β depends on β. Since $l_{T_G}(\phi^p(\beta)) = \lambda_\Phi^p l_{T_G}(\beta) = \sum N_e^p l_{T_G}(e)$, where the sum is taken over orbits of edges in T_G. Up to normalization, we have $k_\beta = l_{T_G}(\beta)$. We have that

$$\lim_{p\to\infty} \frac{N_e^p}{\lambda_\Phi^p l_{T_G}(\beta)} = c_e.$$

Given $\epsilon > 0$, fix p_0 such that $l_T(\nu(\phi^{p_0}(\beta))) > (1/\epsilon)\,\mathrm{BCC}(\nu)$. This is possible because by Proposition 5.3.1(a), a generic leaf of Λ_Φ^+ is unbounded in T. Consider $\phi^{p+p_0}(\beta)$ which is a union of translates of $\phi^{p_0}(e)$, with $\phi^{p_0}(e)$ appearing N_e^p times. We get the following:

$$\sum N_e^p(l_T(\nu(\phi^{p_0}(e))) - 2\,\mathrm{BCC}(\nu)) \leq l_T(\nu(\phi^{p+p_0}(\beta))) \leq \sum N_e^p l_T(\nu(\phi^{p_0}(e))).$$

Dividing throughout by $\lambda_\Phi^{p+p_0} l_{T_G}(\beta)$, we get

$$(1-2\epsilon)\sum \frac{c_e}{\lambda_\Phi^{p_0}} l_T(\nu(\phi^{p_0}(e))) \leq \frac{l_T(\nu(\phi^{p+p_0}(\beta)))}{\lambda_\Phi^{p+p_0} l_{T_G}(\beta)} \leq \sum \frac{c_e}{\lambda_\Phi^{p_0}} l_T(\nu(\phi^{p_0}(e))).$$

We claim that $l_T(\nu(\phi^{p_0}(e)))/\lambda_\Phi^{p_0}$ is bounded which implies that the limit in the statement of the lemma exists. Indeed, we have $l_{T_G}(\phi^{p_0}(e)) = \lambda_\Phi^{p_0} l_{T_G}(e)$. Under the map ν, we get that $l_T(\nu(\phi^{p_0}(e))) \leq \lambda_\Phi^{p_0} l_{T_G}(e)\,\mathrm{Lip}(\nu)$.

(b) Write β as a concatenation $\beta_1 \cdot \beta_2 \cdot \ldots \cdot \beta_k$ such that each β_i is a subsegment (or a translate of a subsegment) of $\Lambda_\Phi^+(T_G)$. The maximum amount of cancellation under the map ϕ^p is given by $\lambda_\Phi^p l_{T_G}(\beta) - l_{T_G}(\phi^p(\beta))$ which is less than or equal to $\lambda_\Phi^p(l_{T_G}(\beta) - d_+(\beta))$. Also if $\phi^p(\beta_i)$ and $\phi^p(\beta_{i+1})$ overlap in a segment of length D, then the cancellation between their tightened images under ν in T is bounded by $D\,\mathrm{Lip}(\nu) + 2\,\mathrm{BCC}(\nu)$. From this, we obtain,

$$\left| l_T(\nu(\phi^p(\beta))) - \sum_i l_T(\nu(\phi^p(\beta_i))) \right| \leq \lambda_\Phi^p(l_{T_G}(\beta) - d_+(\beta))\,\mathrm{Lip}(\nu) + k\,\mathrm{BCC}(\nu).$$

Dividing by $\lambda_\Phi^p l_{T_G}(\beta)$ and using part(a), we get

$$\lim_{p\to\infty} \left| \frac{l_T(\nu(\phi^p(\beta)))}{\lambda_\Phi^p l_{T_G}(\beta)} - \sum_i c \frac{l_{T_G}(\beta_i)}{l_{T_G}(\beta)} \right| \leq \left(1 - \frac{d_+(\beta)}{l_{T_G}(\beta)}\right) \mathrm{Lip}(\nu).$$

Replacing β by $\phi^{p_0}(\beta)$, we have

$$\lim_{p\to\infty}\left|\frac{l_T(\nu(\phi^{p+p_0}(\beta)))}{\lambda_\Phi^p l_{T_G}(\phi^{p+p_0}(\beta))}-c\right|\leq\left(1-\frac{d_+(\phi^{p_0}(\beta))}{l_{T_G}(\phi^{p_0}(\beta))}\right)\mathrm{Lip}(\nu).$$

We have $\lambda_\Phi^{p_0}d_+(\beta)=d_+(\phi^{p_0}(\beta))$. Thus for p_0 large, $\frac{d_+(\phi^{p_0}(\beta))}{l_{T_G}(\phi^{p_0}(\beta))}=\frac{\lambda_\Phi^{p_0}d_+(\beta)}{l_{T_G}(\phi^{p_0}(\beta))}$ is close to 1. Thus we get the desired limit. Also notice that the convergence only depends on the Lipschitz constant of ν. □

Proof of Proposition 5.3.1(b). Let $g\in\mathbb{F}$ be a nonperipheral conjugacy class. For $n\geq 1$, let β_n be a fundamental domain for the action of $g^n\in\mathbb{F}$ on T_G. Let $||g||_T$ be the translation length of g in T. Since $l_T(\nu(\phi^p(\beta_n)))-2\,\mathrm{BCC}(\nu)\leq||\Phi^p(g^n)||_T=||g^n||_{T\phi^p}\leq l_T(\nu(\phi^p(\beta_n)))$ and $d_+(\beta_n)=||g^n||_{T^+_\Phi}$, by Lemma 5.3.2, we get

$$\frac{||g^n||_{T\phi^p}}{c\lambda_\Phi^p}\to||g^n||_{T^+_\Phi}\text{ as }p\to\infty.$$

Since $||g||_T=\lim_{n\to\infty}||g^n||_T/n$, we get that T converges to T^+_Φ under forward iteration by Φ.

For T' close to T, there exists $h':T_0\to T'$, linear on edges such that images of edges have approximatey the same length in T' as in T. Thus $\mathrm{Lip}(h)$ is close to $\mathrm{Lip}(h')$ and thus $\mathrm{Lip}(\nu')$ is close to $\mathrm{Lip}(\nu)$. Since the convergence in Lemma 5.3.2(b) depends only on the lipschitz constant of ν, we can find a small neighborhood V of T where the convergence is uniform. □

Our goal now is to prove that every tree $T\in\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ satisfies the assumptions of Proposition 5.3.1 if γ is allowed to be either in Λ^+_Φ or Λ^-_Φ. We prepare ourselves for this task by proving some results about Whitehead graphs, transverse coverings and $\mathcal{Q}$ map in the next three sections which will then be put together in Section 5.7 to complete the proof of Theorem C.

5.4 Relative Whitehead graph

The main lemma in this section is Lemma 5.4.6 which is used in the proof of Lemma 5.7.1. We first recollect some observations in the absolute case about the Whitehead graph for a fully irreducible automorphism. We then define a relative Whitehead graph and make similar observations for a fully irreducible automorphism relative to $\mathcal{A}$.

Let $\psi : \Gamma \to \Gamma$ be a train track representative of a fully irreducible automorphism where $\Gamma \in CV_n$ and let Λ_ψ^+ be the attracting lamination.

Definition 5.4.1 (Whitehead graph [BFH97]). At a vertex v of Γ, the *Whitehead graph*, denoted $\mathrm{Wh}(v)$, is defined as follows: the vertices are given by the outgoing edges incident at v and two vertices are joined by an edge if the corresponding outgoing edges in Γ form a Λ_ψ^+*-legal turn*, that is, there is a ψ-iterate of an edge of Γ that crosses that turn.

If $\psi(v) = w$ where v, w are vertices in Γ, then ψ induces a simplicial map from $\mathrm{Wh}(v)$ to $\mathrm{Wh}(w)$.

Definition 5.4.2 ([BFH97]). A finitely generated subgroup H of $\mathbb{F}$ *carries* a lamination Λ if there exists a marked metric graph Γ_0, an isometric immersion $i : \Gamma_H \to \Gamma_0$ with $\pi_1(i(\Gamma_H)) = H$ and an isometric immersion $l : \mathbb{R} \to \Gamma_H$ such that $i \circ l$ is a generic leaf of $\Lambda(\Gamma_0)$.

Proposition 5.4.3 ([BFH97, Lemma 2.1, Proposition 2.4]). *(a) At every vertex of Γ, the Whitehead graph is connected.*

(b) Suppose $\pi : \Gamma' \to \Gamma$ is a finite sheeted covering space and $\psi' : \Gamma' \to \Gamma'$ is a lift of ψ. Then the transition matrix of ψ' is primitive and the Whitehead graph of ψ' at a vertex v of Γ' is the lift of the Whitehead graph of ψ at $\pi(v)$ and in particular is connected.

(c) If a finitely generated subgroup H of $\mathbb{F}$ carries Λ_ψ^+, then H is a finite index subgroup of $\mathbb{F}$.

We now look at an example of the Whitehead graph of a fully irreducible automorphism relative to $\mathcal{A}$ to see why a notion of a relative Whitehead graph is needed.

Example 5.4.4. Let $F_4 = \langle a, b, c, d \rangle$, $\mathcal{A} = \{[\langle a, b \rangle]\}$ and Φ a relative automorphism be given by

$$\Phi(a) = ab, \Phi(b) = b, \Phi(c) = cad, \Phi(d) = dcad.$$

Let $\phi_0' : G' \to G'$ be a relative train track representative of Φ where G' is the rose on four petals labeled a, b, c, d and vertex v. The Whitehead graph at v is shown in **Figure 5.4**.

The Whitehead graph at v is disconnected with two gates $\{c, \bar{c}, a, \bar{d}\}$ and $\{\bar{a}, b, \bar{b}, d\}$. If all the directions coming from the rose corresponding to $\langle a, b\rangle$ are identified, then we do get a connected graph.

We will now define a relative Whitehead graph. Let $\phi_0 : G \to G$ be the $\mathcal{A}$-train track representative of a relative fully irreducible automorphism Φ from Definition 5.2.2, with the attracting lamination Λ^+_Φ.

Definition 5.4.5 (Relative Whitehead graph)**.** Let v be a vertex of G of valence greater than one.

- If v has trivial stabilizer, then the relative Whitehead graph is defined as in Definition 5.4.1.
- If v has a nontrivial stabilizer, then do the following: attach a rose representing the vertex stabilizer at v, construct the Whitehead graph as in Definition 5.4.1 and then identify all the directions coming from the attached rose. Thus the vertices of the relative Whitehead graph are the outgoing edges incident to v and a vertex, denoted v_A, representing the nontrivial vertex stabilizer A.

In Example 5.4.4, after collapsing the maximal invariant subgraph of G', we get a graph G which is a rose with two petals and vertex stabilizer $A = \langle a, b\rangle$. The relative Whitehead graph at the vertex of G has vertices corresponding to $c, \bar{c}, d, \bar{d}, v_A$ and is shown in **Figure 5.5**.

Before stating the next lemma, let's look at two examples of covering spaces for the relative rose, one by a finite index subgroup and another by an infinite index subgroup. Let $F_6 = \langle a, b, c, d, e, f\rangle$ and $\mathcal{A} = \{[\langle a, b\rangle], [\langle c, d\rangle]\}$.

- Let $H = \langle a, b, ef\rangle$ be a subgroup of $\mathbb{F}$. The (infinite sheeted) cover of the relative rose corresponding to H is shown in **Figure 5.6**:
- A finite sheeted cover whose fundamental group contains $H = \langle a, b, ef\rangle$ is shown in **Figure 5.7**:

Lemma 5.4.6. *Let $\phi_0 : G \to G$ be an $\mathcal{A}$-train track representative of a fully irreducible automorphism relative to $\mathcal{A}$.*

(a) The relative Whitehead graph of ϕ_0 is connected at each vertex of G.

(b) Suppose $p : G'' \to G$ is a finite sheeted covering space such that for every vertex v of G'', $p_*(\mathrm{Stab}(v)) = \mathrm{Stab}(p(v))$, and $\phi'' : G'' \to G''$ is a lift of $\phi_0 : G \to G$. Then the relative Whitehead graph of ϕ'' at a vertex v of G'' is the lift of the relative Whitehead graph of ϕ at $p(v)$ and in particular is connected.

(c) Let H be a Φ-invariant, finitely generated subgroup of $\mathbb{F}$ such that for every $[A] \in \mathcal{A}$, $H \cap A$ equal to A, up to conjugation. If H carries Λ^+_Φ, then H has finite index in $\mathbb{F}$.

Proof. (a) The same proof as in the absolute case works by doing a blow-up construction ([BH92, Proposition 4.5]) at a vertex. We give a proof here for completeness. Suppose the relative Whitehead graph at a vertex of G is not connected. For simplicity, let's first assume G has only one vertex v with valence greater than one. Then this vertex is fixed under ϕ. Construct a new graph $\overline{G}$ by first deleting the vertex v and adding a new vertex v_i for each component of the relative Whitehead graph. Then connect all the new vertices to a common vertex $\overline{v}$ by edges E_i. Thus $\overline{G}$ is a blow-up of G at v. There is a homotopy equivalence $\overline{\phi} : \overline{G} \to \overline{G}$ such that no leaf of the lamination crosses the new edges E_i. The fundamental group of the complement of $\cup E_i$ gives a nontrivial Φ-invariant free factor system containing $\mathcal{A}$, which is a contradiction.

If G has more than one vertex of valence greater than one, then do the blow-up construction at all the vertices of valence greater than two and repeat the argument.

(b) The graph G'' gets a legal turn structure from the lift of G and it gets a legal turn structure from the map ϕ''. It needs to be shown that a turn in G'' whose image in G is Λ^+_Φ-legal is in fact crossed by a lift of a leaf of Λ^+_Φ to G''.

(i) Let a'', b'' be two edges incident at a vertex v'' of G'' where $p(a'') = a$ and $p(b'') = b$ are such that ab is a legal turn at $p(v'') = v$ in G. The same proof as [BFH97, Lemma 2.1] works in this case. We present a proof here for completeness. Since the transition matrix of $\phi_0 : G \to G$ is primitive, after passing to a power, assume that $\phi_0(a) = \ldots ab \ldots$. Thus a has a fixed point x. Since ϕ_0 is a homotopy equivalence, ϕ'' permutes the set $p^{-1}(x)$. After passing to a power, assume that ϕ'' also has a fixed point on a''. Thus a'' maps over $a''b''$ under ϕ''. Since the image under ϕ'' of an edge of G'' crosses the turn formed by a'' and b'', a leaf of the lamination associated to ϕ'' (which is the lift of Λ^+_Φ) crosses that turn.

(ii) Let v be a vertex of G'' with nontrivial vertex stabilizer. Let a'' be an edge at v'' such that $a = p(a'')$ and it forms a Λ^+_Φ-legal turn with the vertex stabilizer of $p(v'') = v$, that is, after passing to a power $\phi_0(e) = \ldots aw \ldots$ for some edge e and some path w in a blow-up of the vertex stabilizer of v. After passing to a further power, assume that $\phi_0(a) = \ldots aw \ldots$. Thus a has a fixed point. Now by the same argument as in the previous case, $\phi''(a'')$ maps over $a''w''$.

(c) Let Γ_H be the core of the covering space of G corresponding to a subgroup H as in the statement of the lemma. Here Γ_H is a finite graph. Let $i : \Gamma_H \to G$ be the isometric immersion. If H has infinite index in $\mathbb{F}$, then add more vertices and edges to Γ_H to complete it to a finite sheeted covering Γ'_H of G. Then pass to a further finite sheeted cover Γ''_H such that $\phi_0 : G \to G$ lifts to a map $\phi'' : \Gamma''_H \to \Gamma''_H$. By the previous part, the relative Whitehead graph is connected at every vertex of Γ''_H. Therefore, lifts of the leaves of $\Lambda^+_\Phi(G)$ cross every edge of Γ''_H. Under the projection $p : \Gamma''_H \to \Gamma'_H$, the edges added to Γ_H are crossed by leaves of Λ^+_Φ so H does not carry Λ^+_Φ. □

5.5 Transverse covering

Let $\phi_0 : G \to G$ be an $\mathcal{A}$-train track representative of a relative fully irreducible automorphism Φ. Let $\phi : T_G \to T_G$ be a lift to the universal cover T_G of G. In this section, we define a transverse covering for T_G which will be used in the proof of Lemma 5.6.12.

Define an equivalence relation on $\Lambda^+_\Phi(T_G)$ as follows: two leaves γ, γ' are equivalent if there is a sequence of leaves $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n = \gamma'$ such that γ_i and γ_{i+1} overlap in a nontrivial edge path in T_G. Let $\mathcal{Y}(\Lambda^+_\Phi) = \{Y_i\}_{i \in I}$ be the set of subtrees of T_G such that Y_i is the realization of leaves of $\Lambda^+_\Phi(T_G)$ in an equivalence class.

Definition 5.5.1 (Closed subtree [Gui04, Definition 2.4]). A subtree Y of a tree T is called closed if the intersection of Y with any segment of T is either empty or a segment of T.

Definition 5.5.2 (Transverse Covering [Gui04, Definition 4.6]). A transverse covering of an $\mathbb{R}$-tree T is a family $\mathcal{Y}$ of nondegenerate closed subtrees of T such that every arc in T is covered by finitely many subtrees in $\mathcal{Y}$ and any two distinct subtrees in $\mathcal{Y}$ intersect in at most one point.

Lemma 5.5.3. *The set $\mathcal{Y}(\Lambda_\Phi^+)$ forms a transverse covering of T_G.*

Proof. Since an element Y of $\mathcal{Y}(\Lambda_\Phi^+)$ contains a leaf of Λ_Φ^+, $\mathcal{Y}(\Lambda_\Phi^+)$ is a covering of T_G. We now need to check that every arc of T_G is covered by finitely many Y_i. Indeed, if an edge of T_G is covered by multiple Y_i, then by the definition of the equivalence relation, they are connected. Therefore, an edge of T_G is covered by one subtree Y_i and a finite arc is covered by finitely many subtrees in $\mathcal{Y}(\Lambda_\Phi^+)$. Also by definition, two distinct subtrees Y_i, Y_j intersect in at most one point. $\square$

Example 5.5.4. Recall the automorphism Φ from Example 5.4.4 given by $\Phi(a) = ab, \Phi(b) = b, \Phi(c) = cad, \Phi(d) = dcad$. Let $\phi' : T_{G'} \to T_{G'}$ be a relative train track representative of Φ. Say two leaves in $\Lambda_\Phi^+(T_{G'})$ are equivalent if they overlap in an edge in the top EG stratum. There are two different equivalence classes of leaves at a vertex in the universal cover $T_{G'}$. See **Figure 5.8**.

By collapsing the edges with labels a and b in G', we get a relative rose G with two petals and a nontrivial vertex stabilizer. The covering of $T_{G'}$ in **Figure 5.8** descends to a transverse covering of T_G. See **Figure 5.9**.

5.6 $\mathcal{Q}$ map

In [LL03], Levitt and Lustig define a map called the $\mathcal{Q}$ map from the boundary of $\mathbb{F}$ to a tree with dense orbits in $\overline{CV}_n$. This map is the key tool used to prove north-south dynamics for a fully irreducible automorphism on the closure of outer space. We will follow the same techniques to get a relative result. The main proposition in this section is Proposition 5.6.11.

Let T_0 be a metric simplicial $\mathbb{F}$-tree. Let $v(T_0)$ denote the volume of the quotient graph $T_0/\mathbb{F}$. Let T be a metric minimal very small $\mathbb{F}$-tree and let $\overline{T}$ be the metric completion of T. Let T be an $(\mathbb{F}, \mathcal{A})$-tree. The boundary of T, denoted ∂T, is defined as the set of infinite rays $\rho : [0, \infty) \to T$ up to an equivalence. Namely, two rays are equivalent if they intersect along a ray. If T_0 is a Grushko $(\mathbb{F}, \mathcal{A})$-tree, then there is a canonical identification between $\partial\mathbb{F} \setminus \partial\mathcal{A}$ (see Definition 4.1.4) and ∂T_0. Denote by ρ a ray in T_0 representing the point X

in ∂T_0. Given an equivariant map $h : T_0 \to T$, let $r = h(\rho)$. We say X *is T-bounded* if r is bounded in T (this does not depend on the choice of h as shown in [LL03, Proposition 3.1]). If r is unbounded, then we get a ray representing a point in ∂T.

Let $h : T_0 \to T$ be a continuous map between $\mathbb{R}$-trees. We say h has *bounded cancellation property* if there exists a constant $C \geq 0$ such that the h-image of any segment pq in T_0 is contained in the C neighborhood of the geodesic joining $h(p)$ and $h(q)$ in T. The smallest such C is called the bounded cancellation constant for h, denoted $\mathrm{BCC}(h)$. The following fact about BCC for very small trees is a generalization of Cooper's bounded cancellation lemma [Coo87], and can be found in [BFH97, Lemma 3.1] and [GJLL98].

Lemma 5.6.1. *Let T be an $\mathbb{R}$-tree with a minimal very small action of $\mathbb{F}$. Let T_0 be a free simplicial $\mathbb{F}$-tree, and $h : T_0 \to T$ an equivariant map. Then h has bounded cancellation, with $\mathrm{BCC}(h) \leq \mathrm{Lip}(h)v(T_0)$, where $\mathrm{Lip}(h)$ is the Lipschitz constant for h.*

Proposition 5.6.2 (Small BCC)**.** *Let $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ be a minimal $\mathbb{F}$-tree with dense orbits and trivial arc stabilizers. Given $\epsilon > 0$, there exists an $(\mathbb{F}, \mathcal{A})$-tree $T_0 \in \mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})$, $v(T_0) < \epsilon$, and an equivariant map $h : T_0 \to T$ whose restriction to each edge is isometric and $BCC(h) < \epsilon$.*

The proof of the above proposition when $T \in \overline{CV}_n$ and $T_0 \in CV_n$ in [LL03, Proposition 2.2] starts with an equivariant map $h : T_0 \to T$ which is isometric on edges. Then given an edge e of T_0, one replaces h by $h' : T_0' \to T$ with $v(T_0') \leq v(T_0) - 1/6|e|$. If $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$, then start with an equivariant map $h : T_0 \to T$ isometric on edges where $T_0 \in \mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})$ and do the same argument.

Proposition 5.6.3 ($\mathcal{Q}$ map)**.** *Let $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ be a minimal $(\mathbb{F}, \mathcal{A})$-tree with dense orbits and trivial arc stabilizers. Suppose $X \in \partial\mathbb{F} \setminus \partial\mathcal{A}$ is T-bounded. Then there is a unique point $\mathcal{Q}(X) \in \overline{T}$ such that for any equivariant map $h : T_0 \to T$ and any ray ρ representing X in $T_0 \in \mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})$, the point $\mathcal{Q}(X)$ belongs to the closure of $h(\rho)$ in $\overline{T}$. Also, every $h(\rho)$ is contained in a $2\,\mathrm{BCC}(h)$-ball centered at $\mathcal{Q}(X)$, except for an initial part.*

In [LL03, Proposition 3.1], the above lemma is proved for any tree with dense orbits in the closure of outer space hence it applies to our setting as well. Since the free factors in $\mathcal{A}$ are elliptic in T, take the tree T_0 in the original proof to be such that $T_0 \in \mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})$.

Remark 5.6.4 ([LL03, Remark 3.7]). If $\mathcal{Q}(X) = \mathcal{Q}(X')$ for a bi-infinite geodesic γ with end points X, X', then $h(\gamma)$ lies in a $2\,\mathrm{BCC}(h)$-neighborhood of $\mathcal{Q}(X)$.

Example 5.6.5. Let H be a vertex stabilizer in T and $X \in \partial H \subset \partial\mathbb{F}$. Then clearly, X is T-bounded and $\mathcal{Q}(X)$ is the point of T fixed by H. Another less trivial example is as follows: let L be an arational lamination on a surface with boundary. Let T be the dual tree to the lamination. Then in the universal cover of the surface for $\{X, X'\} \in L$, the point $\mathcal{Q}(X) = \mathcal{Q}$ is the point in T to which the leaf collapses as the dual tree is formed.

Definition 5.6.6 (Dual lamination of a tree [CHL08b])**.** Let T be a tree with dense orbits in ∂CV_{n}.

$$L_{\mathcal{Q}}(T) := \{\{X, X'\} \in \partial^2\mathbb{F} |\, \mathcal{Q}(X) = \mathcal{Q}(X')\}.$$

It is shown in [CHL08b] that $L_{\mathcal{Q}}(T)$ is the same as $L(T)$ (see Section 2.5 for definition).

For an algebraic lamination L, let *support* $s(L) \subset \partial\mathbb{F} \setminus \partial\mathcal{A}$ be the set of all $X \in \partial\mathbb{F}$ such that L contains some pair $\{X, X'\}$. The laminations $L_{\mathcal{Q}}(T^+_\Phi)$ and $L_{\mathcal{Q}}(T^-_\Phi)$ are $\mathbb{F}$-invariant and Φ-invariant.

Definition 5.6.7 (Eigenray)**.** Let $f_0 : \tau \to \tau$ be a relative train track map or an $\mathcal{A}$-train track map. Let $f : T_\tau \to T_\tau$ be a lift of f_0 to the universal cover T_τ of τ. Let v_0 be a fixed vertex in τ with a fixed direction e, where e is an edge in an EG stratum. Let v be a lift of v_0 to T_τ. Then a lift based at v of the ray $\lim_{n\to\infty} f_0^n(e)$ is called an *eigenray of f based at v*, denoted by $X_v \in \partial T_\tau$.

Recall from Definition 5.2.2 the $\mathcal{A}$-train track map $\phi_0 : G \to G$ representing Φ and a lift to the universal cover $\phi : T_G \to T_G$. Let $E\Lambda^+_\Phi$ be the set of all eigenrays of ϕ.

Remark 5.6.8. In the absolute case of a fully irreducible automorphism, any eigenray is in fact a half-leaf of Λ^+_Φ, that is, it is contained in a generic leaf of Λ^+_Φ. Thus it suffices to consider points in $s(\Lambda^+_\Phi)$ for the proof of [LL03, Lemma 5.2]. In the relative case, an eigenray based at a vertex with trivial stabilizer is a half-leaf of Λ^+_Φ but an eigenray based at a vertex with nontrivial vertex stabilizer might not be a half-leaf of Λ^+_Φ. It will be a half-leaf of a diagonal leaf of $L_{\mathcal{Q}}(T^-_\Phi)$ as explained below.

Lemma 5.6.9. *$s(L_{\mathcal{Q}}(T^-_\Phi))$ contains $s(\Lambda^+_\Phi)$ and $E\Lambda^+_\Phi$.*

Proof. The statement that $s(L_{\mathcal{Q}}(T^-_\Phi))$ contains $s(\Lambda^+_\Phi)$ follows from Lemma 6.6.1 where it is shown that $L_{\mathcal{Q}}(T^-_\Phi)$ contains Λ^+_Φ. Let $R_v : \mathbb{R}^+ \to T_G$ be a ray representing an eigenray X_v of ϕ based at a vertex v of T_G with nontrivial stabilizer. Let $R_v(\infty) = X_v \in \partial T_G$, which is identified with a point in $\partial\mathbb{F}$, also denoted by X_v. Let $\nu : T_G \to T^-_\Phi$ be an $\mathbb{F}$-equivariant map.

We first show that $\nu(R_v)$ is T^-_Φ-bounded. Suppose not. Then for every $C > 0$ and every $t_0 > 0$, there exist $t_2 > t_1 > t_0$ such that $d_{T^-_\Phi}(\nu(R_v(t_2)), \nu(R_v(t_1))) > C$. Now choose $C > 2BCC(\nu)$. Since R_v is an eigenray, a generic leaf l^+ of Λ^+_Φ crosses the segment $\sigma_v = [R_v(t_2), R_v(t_1)]$ of R_v. By Remark 5.6.4, the ν image of $l^+ = \{X, X'\}$ is in a $2\,\mathrm{BCC}(\nu)$ neighborhood of $\mathcal{Q}(X) = \mathcal{Q}(X')$. This implies that the diameter of σ_v under ν is less than $2\,\mathrm{BCC}(\nu)$, which is a contradiction.

Next we want to prove that $\mathcal{Q}(X_v) = \tilde{v}$ where $\tilde{v}$ is the point in T^-_Φ whose stabilizer contains the stabilizer of v. Given $\epsilon > 0$, let $h : T_0 \to T^-_\Phi$ be an $\mathbb{F}$-equivariant map with $\mathrm{BCC}(h) < \epsilon$ as given by Proposition 5.6.2. Let $\mu : T_G \to T_0$ be an $\mathbb{F}$-equivariant map and let $\nu = h \circ \mu$. Let $\overline{R}_v = \mu(R_v)$. Then by Proposition 5.6.3, $h(\overline{R}_v)$ is contained in a $2\,\mathrm{BCC}(h)$-neighborhood of $\mathcal{Q}(X_v)$ except an initial segment. Suppose $\mathcal{Q}(X_v) \neq \tilde{v}$. There exists a $g \in \mathbb{F} \setminus \mathcal{A}$ for which the following is true: let σ_g be the subsegment of R_v joining v and gv such that the length of $\overline{\sigma}_g := \mu(\sigma_g)$ is nonzero and $h(\overline{\sigma}_g)$ is not contained in a $2\,\mathrm{BCC}(h)$-neighborhood of $\mathcal{Q}(X_v)$. Since R_v is an eigenray, it contains translates of the segment σ_g. There exists some translate σ'_g of σ_g joining points u, gu on R_v such that $h(\overline{\sigma}'_g)$, where $\overline{\sigma}'_g := \mu(\sigma'_g)$, is in a $2\,\mathrm{BCC}(h)$-neighborhood of $\mathcal{Q}(X_v)$ because $h(\overline{R}_v)$ is T^-_Φ-bounded. But g acts by isometries on T^-_Φ so the diameters of $h(\overline{\sigma}_g)$ and $h(\overline{\sigma}'_g)$ cannot be different. Thus $\tilde{v}$ is in a $2\,\mathrm{BCC}(h)$-neighborhood of $\mathcal{Q}(X_v)$. Since ϵ, which bounds $\mathrm{BCC}(h)$, was arbitrary, we have that $\mathcal{Q}(X_v) = \tilde{v}$.

Now we show that for every vertex v of T_G with nontrivial stabilizer, there are at least two eigenrays X_v, X'_v based at v. This will imply that $\{X_v, X'_v\} \in L_{\mathcal{Q}}(T^-_\Phi)$ and hence $E\Lambda^+_\Phi \subset s(L_{\mathcal{Q}}(T^-_\Phi))$. If the image of v in $G = T_G/\mathbb{F}$ has at least two gates, then each gate will have a fixed direction which gives different eigenrays based at v. If there is only one gate at v, then in T_G the orbit of a given ray R_v under the stabilizer of v gives distinct eigenrays based at v. $\square$

Remark 5.6.10. From the above proposition, the following two types of leaves are contained in $L_{\mathcal{Q}}(T_{\Phi}^{-})$:

(a) leaves of the lamination Λ_{Φ}^{+}, which we call Λ_{Φ}^{+}-leaves, and,

(b) leaves obtained by concatenating two eigenrays, which are called *diagonal* leaves.

The next proposition, which is the relativization of [LL03, Proposition 5.1], is the main technical proposition of this section.

Proposition 5.6.11. *If $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ is a minimal $(\mathbb{F}, \mathcal{A})$-tree with dense orbits and trivial arc stabilizers, then at least one of the following is true:*

(a) there exists a generic leaf $\{X, X'\}$ of Λ_{Φ}^{+} or Λ_{Φ}^{-} such that $\mathcal{Q}(X) \neq \mathcal{Q}(X')$,

(b) there exists a diagonal leaf $\{X, X'\}$ of $L_{\mathcal{Q}}(T_{\Phi}^{-})$ or $L_{\mathcal{Q}}(T_{\Phi}^{+})$ such that $\mathcal{Q}(X) \neq \mathcal{Q}(X')$.

Since diagonal leaves are obtained by concatenating eigenrays, (b) implies (a) in the above proposition. Morally, the above proposition says that if $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ is a minimal $(\mathbb{F}, \mathcal{A})$-tree with dense orbits such that $L_{\mathcal{Q}}(T)$ contains both $L_{\mathcal{Q}}(T_{\Phi}^{+})$ and $L_{\mathcal{Q}}(T_{\Phi}^{-})$, then T is in fact a trivial tree. The proof of the proposition depends on Lemma 5.6.13 and Lemma 5.6.14. We need the following lemma for the proof of Lemma 5.6.13.

Lemma 5.6.12. *If e, e' are edges with a common initial vertex v in T_G, then there exists a sequence $e = e_0, e_1, \ldots, e_k = e'$ of distinct edges starting at v such that every edge path $\overline{e_i}e_{i+1}$ is crossed by either a Λ_{Φ}^{+}-leaf or a diagonal leaf of $L_{\mathcal{Q}}(T_{\Phi}^{-})$.*

Proof. If the vertex stabilizer of v is trivial, then by Lemma 5.4.6, the Whitehead graph of Λ_{Φ}^{+} is connected at the vertex v. Hence the lemma follows by using the Λ_{Φ}^{+}-leaves of $L_{\mathcal{Q}}(T_{\Phi}^{-})$. Now let's assume that the vertex stabilizer of v is nontrivial. Consider the transverse covering $\mathcal{Y}(\Lambda_{\Phi}^{+})$ of T_G from Section 5.5. Since an element Y of $\mathcal{Y}(\Lambda_{\Phi}^{+})$ contains a generic leaf of Λ_{Φ}^{+}, Y crosses the $\mathbb{F}$-orbit of every edge in T_G. Let Y_e and $Y_{e'}$ be the elements of $\mathcal{Y}(\Lambda_{\Phi}^{+})$ that contain e and e', respectively. Let E, E' be the set of edges with initial vertex v which are in Y_e and $Y_{e'}$, respectively.

If Y_e is equal to $Y_{e'}$, then the lemma follows by using Λ_{Φ}^{+}-leaves in $L_{\mathcal{Q}}(T_{\Phi}^{-})$. Suppose $Y_e \neq Y_{e'}$. Let $p : T_G \to G$ be the quotient map by the action of $\mathbb{F}$. Every gate at the vertex

$\pi(v)$ has a fixed direction. Thus we can find an eigenray X in T_G based at v with initial edge f in E (since Y_e crosses $\mathbb{F}$-orbit of every edge at v). Similarly, get an eigenray X' based at v and initial edge f' in E'. The diagonal leaf $\{X, X'\}$ of $L(T_\Phi^-)$ crosses $\overline{f}f'$. Now we have a sequence of edges $e = e_0, e_1, \ldots, e_l = f, e_{l+1} = f', e_{l+2}, \ldots, e_k = e'$ starting at v such that every edge path $\overline{e_i}e_{i+1}$ for $i \neq l$ is crossed by a Λ_Φ^+-leaf and $\overline{e_l}e_{l+1}$ is crossed by a diagonal leaf. □

Lemma 5.6.13. *Suppose $\mathcal{Q}(X) = \mathcal{Q}(X')$ for every generic leaf $\{X, X'\}$ of Λ_Φ^+ and for every diagonal leaf $\{X, X'\}$ of $L_\mathcal{Q}(T_\Phi^-)$. Let Z, Z' belong to $s(\Lambda_\Phi^+) \cup E\Lambda_\Phi^+$. Then the distance in $\overline{T}$ between $\mathcal{Q}(\Phi^p(Z))$ and $\mathcal{Q}(\Phi^p(Z'))$ tends to 0 as $p \to +\infty$.*

Proof. We follow the proof of Lemma 5.2 in [LL03]. If Z is in $s(\Lambda_\Phi^+)$, then there exists a ray ρ in T_G contained in $\Lambda_\Phi^+(T_G)$ with end point Z. If Z is in $E\Lambda_\Phi^+$, then there exists an eigenray ρ of ϕ with end point Z. Let's suppose $Z \in E\Lambda_\Phi^+$ and $Z' \in s(\Lambda_\Phi^+)$ with corresponding rays ρ and ρ' to exhibit the proof in both cases. Let e, e' be the initial edges of the two rays ρ and ρ'. By Lemma 5.6.12, we can find a sequence of edges $e = e_0, e_1, e_2, \ldots, e_k = e'$, in T_G connecting e to e' such that the finite subpaths $\gamma_i = e_i e_i'$ are subpaths of either Λ_Φ^+-leaves or diagonal leaves of $L_\mathcal{Q}(T_\Phi^-)$ where e_i' is the same as e_{i+1} but not necessarily with the same orientation. Note that the union of γ_i and γ_{i+1} is either a tripod or a segment of length 3. The rest of the proof follows exactly as in [LL03, Lemma 5.2]. □

The following lemma is the relativization of [LL03, Proposition 5.3]. Recall the $\mathcal{A}$-train track map $\phi_0 : G \to G$, and a lift to the universal cover $\phi : T_G \to T_G$ representing Φ where $T_G \in \mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})$.

Lemma 5.6.14. *Suppose $\mathcal{Q}(X) = \mathcal{Q}(X')$ for every generic leaf $\{X, X'\}$ of Λ_Φ^+ and for every diagonal leaf $\{X, X'\}$ of $L_\mathcal{Q}(T_\Phi^-)$. Then there exist maps $i_p : T_G \to \overline{T}$, $p \in \mathbb{N}$ such that $i_p \circ \phi^p$ is $\mathbb{F}$-equivariant and $\mathrm{BCC}(i_p) \to 0$ as $p \to \infty$.*

Proof. Assume that there are no vertices with trivial stabilizer in T_G. If there were some such vertices, then collapse a tree in $T_G/\mathbb{F}$ and factor through the quotient of T_G. For a representative v of an orbit of vertices in T_G, fix an eigenray X_v in $E\Lambda_\Phi^-$ such that $\mathcal{Q}(X_v) = \tilde{v}$, where $\tilde{v}$ is a point in T whose stabilizer contains the stabilizer of v. Then $\mathbb{F}$-equivariantly assign an eigenray to every vertex in the orbit of v. In this way, assign an eigenray to each vertex of T_G.

We will now define a map $i_p : T_G \to \overline{T}$ and show that $i_p(e) \to 0$ as $p \to \infty$ for every edge e of T_G. For a vertex $v \in T_G$, set $i_p(v) = \mathcal{Q}(\Phi^{-p}(X_v))$ and extend linearly on edges. Now for an edge e of T_G with end points v, u, we have, by applying Lemma 5.6.13 to Φ^{-1}, that distance between $i_p(v) = \mathcal{Q}(\Phi^{-p}(X_v))$ and $i_p(u) = \mathcal{Q}(\Phi^{-p}(X_u))$ goes to zero as $p \to \infty$. Thus $i_p(e) \to 0$ which implies that $\mathrm{BCC}(i_p) \to 0$. The map i_p satisfies a twisted equivariance relation $g \circ i_p = i_p \circ \Phi^p(g)$ for all $g \in \mathbb{F}$.

Also $i_p \circ \phi^p$ is $\mathbb{F}$-equivariant. Indeed,

$$\begin{aligned} g \circ i_p \circ \phi^p &:= g \circ j_p \circ \pi \circ \phi^p = j_p \circ \Phi^p(g) \circ \phi^p \circ \pi \\ &= j_p \circ \phi^p \circ g \circ \pi = j_p \circ \pi \circ \phi^p \circ g = i_p \circ \phi^p \circ g. \end{aligned}$$

Thus we have maps i_p as in the lemma. □

Proof of Proposition 5.6.11. Assume by contradiction that $\mathcal{Q}(X) = \mathcal{Q}(X')$ for every generic leaf $\{X, X'\}$ of Λ^+_Φ and Λ^-_Φ and every diagonal leaf of $L_{\mathcal{Q}}(T^-_\Phi)$ and $L_{\mathcal{Q}}(T^+_\Phi)$. Let e be an edge in T_G and let $\gamma \in \Lambda^+_\Phi$ be a leaf that crosses e. Then $\phi^p(\gamma)$ is also a leaf of the lamination. By assumption, the end points of γ map to the same point under the $\mathcal{Q}$ map. By Proposition 5.6.3 and Remark 5.6.4, $(i_p \circ \phi^p)(\gamma)$ is contained in a ball of radius $2\,\mathrm{BCC}(i_p \circ \phi^p)$ in $\overline{T}$. We have $\mathrm{BCC}(i_p \circ \phi^p) \le \mathrm{BCC}(i_p) + Lip(\phi^p)\,\mathrm{BCC}(\phi^p)$. Since γ is a leaf of Λ^+_Φ, ϕ^p restricted to γ has no cancellation thus $(i_p \circ \phi^p)(\gamma)$ is in fact contained in a ball of radius $2\,\mathrm{BCC}(i_p)$ in $\overline{T}$. Thus the diameter of $(i_p \circ \phi^p)(e)$ in $\overline{T}$ is bounded by $4\,\mathrm{BCC}(i_p)$.

Now let u be a conjugacy class, represented by a loop of edge-length k in $G = T_G/\mathbb{F}$. Since $i_p \circ \phi^p$ is $\mathbb{F}$-equivariant, the translation length of u in T is bounded by $4k\,\mathrm{BCC}(i_p)$. Since $\mathrm{BCC}(i_p) \to 0$ as $p \to \infty$, every u has zero translation length in T, which is a contradiction. □

5.7 Main theorem

We will now put together the results from Section 5.4 and Section 5.6 to prove the following lemma, which shows that the conditions mentioned in Proposition 5.3.1 are satisfied by all trees in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ if γ is allowed to be a leaf of Λ^+_Φ or Λ^-_Φ.

Lemma 5.7.1. *Let $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$. Then there exists a tree $T_0 \in \mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})$, an equivariant map $h : T_0 \to T$ and a bi-infinite geodesic $\gamma_0 \subset T_0$ representing a generic leaf γ of Λ^+_Φ or Λ^-_Φ such that $h(\gamma_0)$ has diameter greater than $2\,\mathrm{BCC}(h)$.*

Proof. There are three cases to consider for a tree T in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$.

- *T has dense orbits* (which implies that arc stabilizers are trivial by [LL03, Lemma 4.2]): Proposition 5.6.11 provides either a generic leaf $\{X, X'\}$ in Λ^+_Φ or Λ^-_Φ with $\mathcal{Q}(X) \neq \mathcal{Q}(X')$, or it provides an eigenray $X_v \in E\Lambda^+_\Phi$ or $E\Lambda^-_\Phi$ based at a vertex v of T_G such that $\mathcal{Q}(X_v) \neq \tilde{v}$, where $\tilde{v}$ is the vertex of T containing the stabilizer of v. Choose $h : T_0 \to T$ with $2\,\mathrm{BCC}(h) < d(\mathcal{Q}(X), \mathcal{Q}(X'))$ or $2\,\mathrm{BCC}(h) < d(\mathcal{Q}(X_v), \tilde{v})$ using Proposition 5.6.2. In the first case, let γ_0 be the geodesic joining end points corresponding to X, X' in T_0. In the second case, there exists a subsegment of an eigenray R_v corresponding to X_v whose diameter in T is at least $d_T(\mathcal{Q}(X_v), \tilde{v})$. Choose γ_0 to be any generic leaf (of either Λ^+_Φ or Λ^-_Φ) crossing that subsegment.

- *T does not have dense orbits and is also not simplicial:* then T contains simplicial parts and also subtrees T_v with the property that some subgroup $G_v \subset \mathbb{F}$ acts with dense orbits on T_v. Let $\pi : T \to T'$ be a collapse map such that T' has dense orbits. Choose γ_0 as in the previous case, using T'. Then by Proposition 5.3.1, γ_0 is unbounded in T' and hence it is T-unbounded. The map $h : T_0 \to T$ may be chosen arbitrarily.

- *T is simplicial:* we want to show that a generic leaf of Λ^+_Φ is unbounded in T. We need to show that a tail of a generic leaf of Λ^+_Φ or Λ^-_Φ does not live in ∂B for any vertex stabilizer B. By [GL95, Corollary III.4], vertex stabilizer in a tree in $\overline{CV}_n$ is finitely generated and has infinite index in $\mathbb{F}$. Also given T in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$, for every $[A] \in \mathcal{A}$, a vertex stabilizer in T either contains the full free factor A or intersects it trivially. By Lemma 5.4.6, a generic leaf of the attracting lamination cannot be carried by a vertex stabilizer of T, therefore it is unbounded in T. One can choose $h : T_0 \to T$ arbitrarily. □

Theorem C. *Let $\mathcal{A}$ be a nontrivial free factor system such that $\zeta(\mathcal{A}) \geq 3$. Let $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$ be fully irreducible relative to $\mathcal{A}$. Then Φ acts on $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ with uniform north-south dynamics:*

there are two fixed points T^+_Φ and T^-_Φ and any compact set that does not contain T^-_Φ (T^+_Φ) uniformly converges to $T^+_\Phi(T^-_\Phi)$ under $\Phi(\Phi^{-1})$-iterates .

Proof. By Lemma 5.7.1 and Proposition 5.3.1, every T in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ converges either to T^+_Φ under forward iterates or to T^-_Φ under backward iterates. We know that T^+_Φ is locally attracting and T^-_Φ is locally repelling. Thus given a tree $T \neq T^-_\Phi$, the set of its limit points under forward iterates cannot contain the repelling point T^-_Φ and hence T converges to T^+_Φ. Similarly, a tree $T \neq T^+_\Phi$ under backward iterates converges to T^-_Φ. Since $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ is a compact space, by [HK53], pointwise north-south dynamics implies uniform north-south dynamics. □

5.8 Summary

Now we know that a fully irreducible outer automorphism relative to $\mathcal{A}$ acts with uniform north-south dynamics on both the relative outer space $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ and a subspace $\mathcal{MRC}(\mathcal{A})$ of the space of projectivized relative currents. In the next chapter, we will see how relative currents and trees in relative outer space act as dual to each other. We will also see that the duality between the fixed points in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ and $\mathcal{MRC}(\mathcal{A})$ is especially nice.

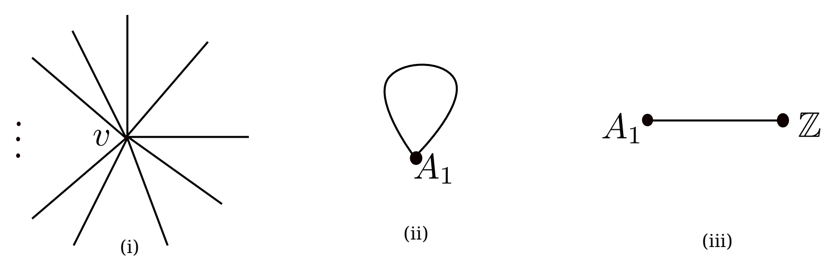

Figure 5.1. Relative outer space

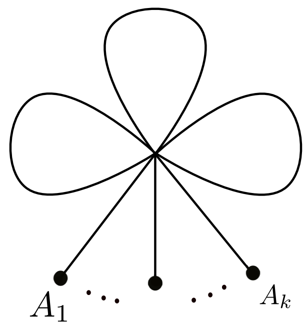

Figure 5.2. Relative rose

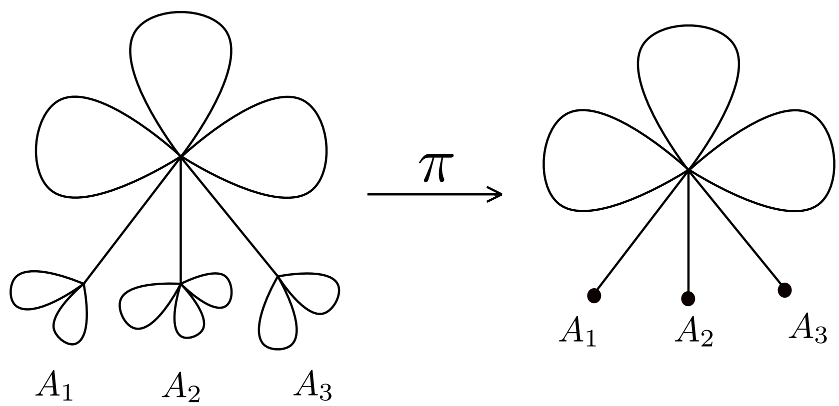

Figure 5.3. Collapse map π

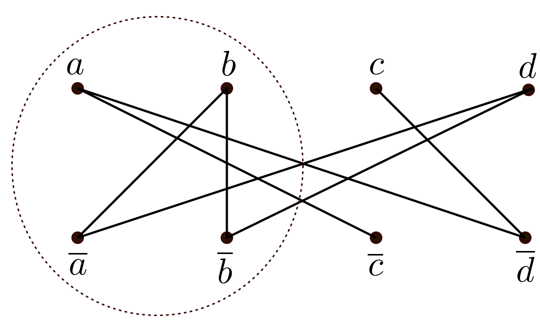

Figure 5.4. Whitehead graph for Example 5.4.4

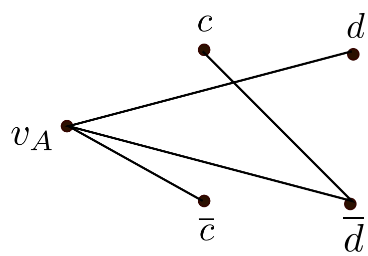

Figure 5.5. Relative Whitehead graph for Example 5.4.4

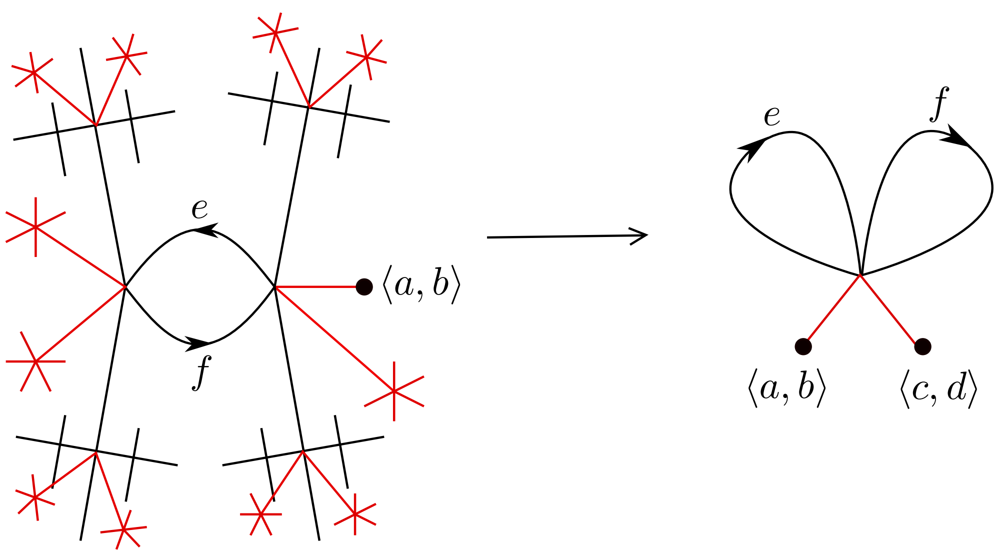

Figure 5.6. Infinite sheeted cover

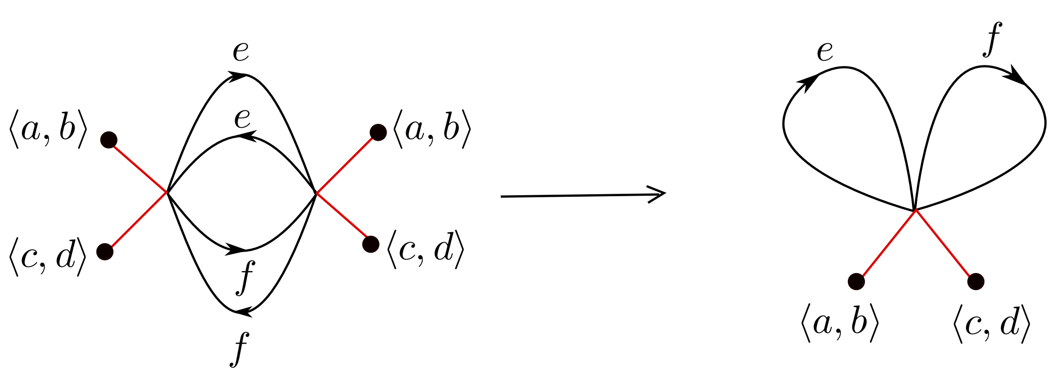

Figure 5.7. Finite sheeted cover

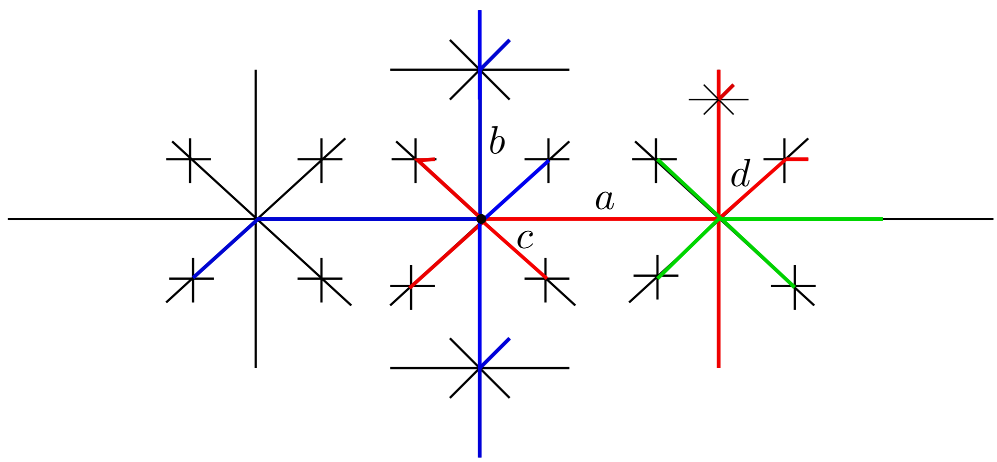

Figure 5.8. Three different equivalence classes in $T_{G'}$

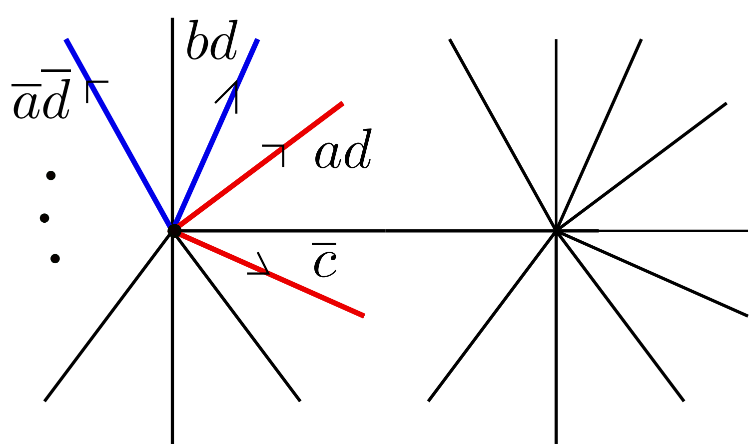

Figure 5.9. Different equivalence classes in T_G

CHAPTER 6

INTERSECTION FORM

Given two homotopy classes of simple closed curves on a surface, there is a well-defined notion of geometric intersection number of the two curves. Such curves are special examples of *measured geodesic laminations*. Thurston extended the notion of geometric intersection number between curves to a pair of measured geodesic laminations. Using this intersection number, the space of measured geodesic laminations can be viewed as its own dual space. In [KL09], Kapovich and Lustig showed the space of measured currents for $\mathbb{F}$ acts like a dual space to the closure of outer space. The goal of this chapter is to establish a similar duality between the space of relative currents and relative outer space (see Section 6.8).

6.1 Intersectin form for outer space and measured currents

In [KL09], Kapovich and Lustig established an intersection form between $\overline{cv}_{\mathrm{n}}$, the closure of unprojectivized outer space and $\mathcal{MC}(\mathbb{F})$, the space of measured currents. The precise statement is as follows:

Theorem 6.1.1 ([KL09, Theorem A])**.** *There is a unique* $\mathrm{Out}(\mathbb{F})$*-invariant, continuous length pairing that is* $\mathbb{R}_{\geq 0}$ *homogeneous in the first coordinate and* $\mathbb{R}_{\geq 0}$ *linear in the second coordinate.*

$$\langle \cdot, \cdot \rangle : \overline{cv}_{\mathrm{n}} \times \mathcal{MC}(\mathbb{F}) \to \mathbb{R}_{\geq 0}$$

Further, $\langle T, \eta_g \rangle = l_T(g)$ *for all* $T \in \overline{cv}_{\mathrm{n}}$ *and all rational currents* η_g *where* $g \in \mathbb{F} \setminus \{1\}$*.*

Kapovich and Lustig also give the following characterization of zero pairing:

Proposition 6.1.2 ([KL10, Theorem 1.1])**.** *Let* $T \in \overline{cv}_{\mathrm{n}}$*, and let* $\eta \in \mathcal{MC}(\mathbb{F})$*. Then* $\langle T, \eta \rangle = 0$ *if and only if* $\mathrm{supp}(\eta) \subseteq L(T)$*, where* $L(T)$ *is the dual lamination of* T *and* $\mathrm{supp}(\eta)$ *is the support of* η *in* $\partial^2\mathbb{F}$*.*

In this chapter, we define an intersection form for $\overline{\mathcal{O}(\mathbb{F},\mathcal{A})}$, the closure of relative outer space and $\mathcal{RC}(\mathcal{A})$, the space of relative currents.

6.2 First attempt

Generalizing the definition of intersection form due to Kapovich and Lustig, if $T \in \overline{\mathcal{O}(\mathbb{F},\mathcal{A})}$ and $\eta_\alpha \in \mathcal{RC}(\mathcal{A})$ is a rational relative current, then we can define $\langle T, \eta_\alpha\rangle := l_T(\alpha)$. But unfortunately, this pairing is not continuous. The following example was shown to us by Camille Horbez.

Example 6.2.1. Let $F_2 = \langle a, b\rangle$ with $\mathcal{A} = \{[\langle a\rangle]\}$. Let $T_k \in \mathcal{O}(\mathbb{F},\mathcal{A})$ be a simplicial tree such that $\Gamma_k = T_k/\mathbb{F}$ is a graph with two vertices joined by an edge and there is a loop at one of the vertices. Let $\langle a\rangle$ be the stabilizer of the vertex away from the loop. The graph Γ_k is marked such that the loop is labeled by $a^k b$. Let the loop and the edge have length 1. The limit of the sequence of trees T_k is the Bass-Serre tree of an HNN extension whose vertex stabilizer is given by $\langle a\rangle$ and it has a length 3 loop labeled b. Next consider a sequence of relative currents $\eta_k = \eta_{a^k b}$ converging to η_∞, which is given by $\eta_\infty(a^n bm^m) = 1$ for all $n, m \geq 0$ and $\eta_\infty(w) = 0$ for all other $w \in \mathbb{F} \setminus \mathcal{A}$. We have that $\langle T_k, \eta_k\rangle = 1$ and $\langle T_k, \eta_{k+1}\rangle = 3$ for all k. For continuity of the pairing, $\langle T_k, \eta_k\rangle$ and $\langle T_k, \eta_{k+1}\rangle$ should converge to a unique value, $\langle T, \eta_\infty\rangle$, but that does not happen in this example.

In Section 6.8, we will define a pairing for $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ and $\mathbb{P}\mathcal{RC}(\mathcal{A})$ along the lines of zero pairing criterion of Kapovich and Lustig.

6.3 CHL laminations

In [CHL08a], Coulbois, Hilion and Lustig defined three laminations associated to $\mathbb{F}$: algebraic laminations, symbolic laminations and laminary languages. They also established the equivalence of the three definitions. An *algebraic lamination* is a nonempty, closed and $\mathbb{F}$-invariant subset of $\partial^2\mathbb{F}$. Let $\Lambda^2(\mathbb{F})$ be the (compact, metric) space of algebraic lamination in $\mathbb{F}$.

Definition 6.3.1 (Convergence of laminations [CHL08a, Remark6.3])**.** A sequence of algebraic laminations L_n converges to a lamination L_∞ in $\Lambda^2(\mathbb{F})$ if the following holds: let L_n^s and L_∞^s be the symbolic laminations associated to L_n and L_∞, respectively, with respect to some (any) basis of $\mathbb{F}$. Given a symbolic lamination L^s, let $\mathcal{L}_m(L^s)$ be the set of words in L^s of length less than or equal to m. The sequence L_n converges to L_∞ if for every $m \geq 1$ there exists a $K(m) \geq 1$ such that for every $k \geq K(m)$, $\mathcal{L}_m(L^s_k) = \mathcal{L}_m(L^s_\infty)$.

6.4 Dual lamination for a tree

Definition 6.4.1 ($L(T)$)**.** For a tree $T \in \overline{CV}_{\mathfrak{n}}$, a *dual algebraic lamination* $L(T)$ is defined as follows in [CHL08b]: let

$$L_\epsilon(T) := \overline{\{(g^{-\infty}, g^{\infty}) \mid ||g||_T < \epsilon, g \in \mathbb{F}\}} \subset \partial^2\mathbb{F},$$

so $L_\epsilon(T)$ is an algebraic lamination and set $L(T) := \bigcap_{\epsilon>0} L_\epsilon(T)$.

For trees in $CV_{\mathfrak{n}}$, $L(T)$ is empty. If Λ^+_Ψ is the attracting lamination and T^-_Ψ is the unstable tree associated to Ψ, a fully irreducible outer automorphism, then $L(T^-_\Psi)$ is the diagonal closure of Λ^+_Ψ, that is, if $(X, X') \in \partial^2\mathbb{F}$ and $(X, X'') \in \partial^2\mathbb{F}$ are in Λ^+_Ψ which is a subset of $L(T^-_\Psi)$, and $X' \neq X''$, then (X', X'') is also in $L(T^-_\Psi)$.

For trees in $\partial CV_{\mathfrak{n}}$ with dense orbits, two more definitions are given in [CHL08b]:

Definition 6.4.2 ($L_\infty(T)$)**.** For a basis $\mathfrak{B}$ of $\mathbb{F}$, let $L^1_{\mathfrak{B}}(T) \subset \partial\mathbb{F}$ be the set of one-sided infinite words with respect to $\mathfrak{B}$ that are bounded in T. By [CHL08b, Proposition 5.2] this set is independent of the basis and henceforth will be denoted $L^1(T)$. The lamination $L_\infty(T)$ is the algebraic lamination defined by the recurrent laminary language in $\mathfrak{B}^\pm$ associated to $L^1(T)$. It is shown in the same paper that this definition is also independent of the basis.

Definition 6.4.3 ($L_{\mathcal{Q}}(T)$)**.** See Definition 5.6.6.

The equivalence of the three definitions of dual lamination of a tree in $\partial CV_{\mathfrak{n}}$ with dense orbits is established in [CHL08b]. Note that $L_\infty(T)$ can also be defined for trees which do not have dense orbits, but it might not be equal to $L(T)$.

6.5 Limits of trees and their dual laminations

In this section, we prove some results for trees in $\overline{CV}_{\mathfrak{n}}$. Since trees in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ are contained in $\overline{CV}_{\mathfrak{n}}$, the results of this section are applied to them later. The main proposition in this section is Proposition 6.5.5.

Consider a sequence of trees T_k in $\overline{CV}_n$ converging to a tree T. Then one may ask whether sequence of laminations $L(T_k)$ converges to $L(T)$ or not. An example in [CHL08b, Section 9] shows that $L_\infty = \lim_{k\to\infty} L(T_k)$ may not be equal to $L(T)$. Another example is recorded here.

Example 6.5.1. Let $\mathbb{F} = \langle a, b\rangle$ be the free group of rank two. Let T_k be a simplicial $\mathbb{F}$-tree given as follows: it is the universal cover of the one-edge free splitting with vertex stabilizers given by $\langle a^k b\rangle$ and $\langle a\rangle$. The sequence T_k converges to a tree T which is the Bass-Serre tree of the HNN extension with vertex group $\langle a\rangle$ and edge labeled b. The algebraic lamination $L(T_k)$ is the set of periodic lines determined by a and $a^k b$ which converges to the periodic lines determined by a, denoted $\ldots aaaa \ldots$, and the lines of the form $\ldots aaaa\, b\, aaaa \ldots$. On the other hand, $L(T)$ is given by the periodic lines determined by a. Thus $L_\infty = \lim_{k\to\infty} L(T_k)$ is not equal to $L(T)$. But the birecurrent line in L_∞ is contained in $L(T)$. This is in fact always true by a result of [CHL06] (see Proposition 6.5.5).

The following lemma is needed for the proof of Lemma 6.5.5.

Lemma 6.5.2. *Let T be a tree in $\overline{CV}_n$. Then the birecurrent leaves of $L_\infty(T)$, which is the algebraic lamination defined by the birecurrent laminary language associated to $L^1(T)$, are contained in $L(T)$.*

Proof. Consider different cases according to whether T is simplicial or has dense orbits.

T has dense orbits: by [CHL08b, Proposition 5.8], a stronger statement is true, which says that $L_\infty(T) = L(T)$.

T is simplicial with trivial edge stabilizers but is not free: let $\hat{T}$ be a free simplicial tree with a collapse map $c : \hat{T} \to T$ with $\mathrm{BCC}(c)$ equal to zero. The map c extends to $\partial\hat{T}$ and we denote its restriction to $\partial\hat{T}$ by $Q : \partial\hat{T} \to T \sqcup \partial T$. There is a canonical identification between $\partial^2\mathbb{F}$ and $\partial^2\hat{T}$. If $X \in \partial\hat{T}$ is carried by a vertex stabilizer of T, then $Q(X)$ is precisely (since c has no cancellation) the vertex in T with that stabilizer, otherwise $Q(X)$ is a point in ∂T. Let $l = \{X, X'\}$ be a birecurrent leaf in $L_\infty(T)$. Since X and X' are T-bounded, $Q(X)$ and $Q(X')$ are vertices in T. If $Q(X) \neq Q(X')$, then l crosses an edge e in $\hat{T}$ that maps to a nondegenerate edge in T. Since l is birecurrent, l crosses translates of e infinitely often, which implies that X or X' is not T-bounded. Thus $Q(X) = Q(X')$. Thus l is carried by a

vertex stabilizer of T and hence $l \in L(T)$.

T is simplicial with nontrivial edge stabilizers: by results of [Swa86] and [She55], for T there exists $\hat{T}$ a free simplicial tree with an $\mathbb{F}$-equivariant map $c : \hat{T} \to T$ which is a composition of a collapse map and a fold map. The edge paths in $\hat{T}$ that possibly backtrack under the map c are the ones that cross a minimal subtree (of $\hat{T}$) of an edge stabilizer of T. By [BFH97, Lemma 3.1], $\mathrm{BCC}(c) \leq \mathrm{Lip}(c)\,\mathrm{vol}(\hat{T})$. By scaling the metric on $\hat{T}$, we may assume that $\mathrm{Lip}(c)$ is less than or equal to 1. Since the volume of the free simplicial tree $\hat{T}$ is bounded, $\mathrm{BCC}(c)$ is finite.

As before, consider the map $Q : \partial\hat{T} \to T \sqcup \partial T$. Let $X \in \partial\hat{T}$ be represented by a one-sided infinite word x starting at the basepoint in $\hat{T}$. If the tail of x is carried by a vertex stabilizer of T, then except an initial segment, $c(x)$ crosses the corresponding vertex in T infinitely often with possibly some bounded backtracking. Thus set $Q(X)$ to be that vertex. If the tail of x is carried by an edge stabilizer H, then except an initial segment, $c(x)$ is a vertex of T whose stabilizer contains H and set $Q(X)$ to be that vertex. Even though there are finitely many vertices in T whose stabilizer contains H, there is only one minimal subtree for H in $\hat{T}$, which maps to a unique vertex in T. Thus in this case, $Q(X)$ only depends on the choice of $\hat{T}$. If the tail of x is neither carried by a vertex stabilizer nor an edge stabilizer, then $Q(X)$ is an element of ∂T.

Now for a birecurrent leaf $l = \{X, X'\}$ such that X and X' are T-bounded, we get that $Q(X) = Q(X')$. Thus the leaf l maps to a vertex of T under the map c with possiblly bounded backtracking from edges in $\hat{T}$ that fold under the map c. Hence l is in $L(T)$.

When T is neither simplicial nor does it have dense orbits: let T' be the simplicial tree which is the graph of actions (see [Gui04] for definition) of T corresponding to the Levitt decomposition [Lev94] of T. Let $l = \{X, X'\}$ be a birecurrent leaf in $L_\infty(T)$. Since $X, X' \in L^1(T)$, we get that X, X' are also T'-bounded. Since l is birecurrent, by the previous two cases, l is carried by a vertex stabilizer H of T'. Since vertices of T' correspond to subtrees with dense orbits in T, the leaf l is contained in some subtree T_d of T with dense orbits and stabilizer H. Since T_d is a subtree of T, X and X' are also T_d-bounded.

The subgroup H is finitely generated because point stabilizers in the very small tree T' have bounded rank [GL95]. Therefore, there exists a finite graph Γ_H and an immersion $i : \Gamma_H \to R_{\mathfrak{B}}$, where $R_{\mathfrak{B}}$ is a rose with petals labeled by elements of a basis $\mathfrak{B}$ of $\mathbb{F}$, such that $\pi_1(i(\Gamma_H)) = H$. Since H carries l, which can be viewed as a map $l : \mathbb{Z} \to R_{\mathfrak{B}}$, there exists a map $l_H : \mathbb{Z} \to \Gamma_H$ such that $i \circ l_H = l$. Since l is birecurrent, we claim that l_H is also birecurrent. Consider a word w in l_H such that $i(w)$ is a subword of l. Since l is birecurrent, $i(w)$ appears infinitely often in both ends of l. Let $w_1, w_2, \ldots, w_n$ be the pre-images of all occurrences of $i(w)$ in l_H. There are only finitely many such w_i because Γ_H is a finite graph. Thus at least one of the w_i appears infinitely often in both ends of l_H. But we need to show that every such w_i appears infinitely often in l_H. So consider a finite subword u of l_H that contains at least one appearance of each w_i. Such a word exists because there are only finitely many w_i. Now $i(u)$ appears infinitely often in both ends of l. Therefore, some pre-image u_1 of $i(u)$ in l_H appears infinitely often. Since every pre-image of $i(u)$ contains all the w_is, each w_i appears infinitely often in both ends of l_H. Thus l_H is birecurrent.

Let $l_H = \{X_H, X'_H\}$. Since i is an immersion and X, X' are T_d-bounded, X_H, X'_H are also T_d-bounded. Thus l_H is in $L_\infty(T_d)$, which is equal to $L(T_d)$ by the first case. Since T_d is a subtree of T and l is contained in T_d, we get that $l = i \circ l_H$ is in $L(T)$. $\square$

Example 6.5.3 (Proof of Lemma 6.5.2 - T is simplicial with nontrivial edge stabilizer)**.** Consider the one-edge cyclic splitting T with vertex stabilizers $\langle a, b\rangle$ and $\langle b, c\rangle$ and edge stabilizer $\langle b\rangle$. Let $\hat{T}$ be the blow-up of a one-edge free splitting with stabilizers $\langle a, b\rangle$ and $\langle c\rangle$.

Definition 6.5.4. A lamination L is called birecurrent if every leaf of L is birecurrent.

Proposition 6.5.5 ([CHL06])**.** *Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of trees in $\overline{CV}_{\mathfrak{n}}$ converging to a tree T. Also suppose that the sequence of laminations $L(T_k)$ converges to L_∞ in $\Lambda^2(\mathbb{F})$. Let L_r be a birecurrent sublamination of L_∞. Then $L_r \subseteq L(T)$.*

Proof. We will use notation from [CHL08b]. If the trees T_k are free simplicial, then their dual lamination is empty and the lemma is true vacuously. So let's assume that $L(T_k)$ is nonempty. Let $l = \{X, X'\}$ be a leaf of L_∞. Fix a basis $\mathfrak{B}$ of $\mathbb{F}$ and realize X in this basis as a one-sided infinite word. For $l \geq 1$, let $X_l \in \mathbb{F}$ be the prefix of length l of X. We first show that $X \in L^1(T)$, that is, for a point $p \in T$, the sequence $X_l p$ is bounded in T. Suppose not. Then for any $C > 0$, $p \in T$, $K_0 > 0$, there exists $q > r > K_0$ such that $d_T(X_q p, X_r p) > C$. Let $u = X_r^{-1} X_q$. Then $d_T(up, p) > C$. By Gromov-Hausdorff topology on $\overline{CV}_{\mathfrak{n}}$, given

$p, up \in T$, let $p_k, s_k \in T_k$ be approximations of p and up relative to some exhaustions (see [Hor16, Lemma 4.1] for details). Then $d_{T_k}(up_k, s_k)$ goes to zero and $d_{T_k}(s_k, p_k)$ goes to $d_T(up, p)$ as $k \to \infty$. Thus given $\delta > 0$, there exists a $K_1 > 0$ such that for all $k > K_1$, $d_T(up, p) - \delta \leq d_{T_k}(up_k, p_k)$, or in other words, $d_{T_k}(up_k, p_k) \geq C - \delta$.

Now by the convergence criterion (Definition 6.3.1), for any $m \geq 1$, there exists a $K_2(m) > 0$ such that for all $k \geq K_2$, $\mathcal{L}_m(L(T_k)) = \mathcal{L}_m(L_\infty)$. Let m be the word length of u with respect to the fixed basis. Since $u \in \mathcal{L}_m(L_\infty)$, we get that $u \in \mathcal{L}_m(L(T_k))$ for all $k > \max(K_0, K_1, K_2)$. By [CHL08b, Remark 4.2], this means that, for every $\epsilon > 0$, there exists a cyclically reduced w in $\mathbb{F}$ such that $||w||_{T_k} < \epsilon$ and u is a subword of w. Also by [CHL08b, Lemma 3.1(c)]

$$d_{T_k}(up_k, p_k) \leq 2\,\mathrm{BCC}(\mathfrak{B}, p_k) + ||w||_{T_k},$$

where $\mathrm{BCC}(\mathfrak{B}, p_k)$ is the bounded cancellation constant of the $\mathbb{F}$-equivariant map from $\mathrm{Cay}(\mathbb{F}, \mathfrak{B})$ to T_k such that the base point of $\mathrm{Cay}(\mathbb{F}, \mathfrak{B})$ is mapped to p_k. We claim that $\mathrm{BCC}_k := \mathrm{BCC}(\mathfrak{B}, p_k)$ is bounded above by a constant. Let $\mathrm{BCC}_T := \mathrm{BCC}(\mathfrak{B}, p)$. Since up is in the BCC_T neighborhood of an axis of w in T, then by [Hor16, Lemma 4.1 (c)], for sufficiently large k, s_k is in the $\mathrm{BCC}_T + 1$ neighborhood of axis of w in T_k. Given $\delta' > 0$, for sufficiently large k, $d_{T_k}(up_k, s_k) \leq \delta'$. Therefore, up_k is in a $\mathrm{BCC}_T + 1 + \delta'$ neighborhood of axis of w in T_k. Since this is true for any cyclically reduced word w and a subword u, we get that $BCC_k \leq \mathrm{BCC}_T + 1 + \delta'$.

By choosing C large enough, we get a contradiction since

$$C - \delta \leq d_{T_k}(up_k, p_k) \leq 2BCC_k + ||w||_{T_k} \leq 2(BCC_T + 1 + \delta') + \epsilon$$

for all k sufficiently large. Thus X and similarly X' are both in $L^1(T)$. Therefore, $l \in L_\infty(T)$.

If $l = \{X, X'\}$ is birecurrent and $l \in L_\infty(T)$, then by Lemma 6.5.2, $l \in L(T)$. Thus $L_r \subseteq L(T)$. $\square$

Lemma 6.5.6. *Let $\{T_k\}_{k \in \mathbb{N}}$ be a sequence of trees in $\overline{CV}_n$ converging to a tree T such that T has dense orbits. Also suppose that the sequence of laminations $L(T_k)$ converges to L_∞ in $\Lambda^2(\mathbb{F})$. Then $L_\infty \subseteq L(T)$.*

Proof. If the trees T_k are free simplicial, then $L_\infty = \emptyset$. Thus after passing to a subsequence, assume that $L(T_k) \neq \emptyset$. Since T has dense orbits, by [LL03, Proposition 2.2] (see Proposition 5.6.2), given $\epsilon > 0$, there exists a free simplicial $\mathbb{F}$-tree S and an $\mathbb{F}$-equivariant map $h : S \to T$ which is isometric on edges ($\mathrm{Lip}(h) = 1$) and $\mathrm{BCC}(h) < \mathrm{vol}(S) < \epsilon$. We will now construct $\mathbb{F}$-equivariant maps $h_k : S \to T_k$ for k sufficiently large such that $\mathrm{BCC}(h_k)$ is bounded above by a linear function of ϵ.

For trees $S \in CV_n$ and T in $\overline{CV}_n$, let $\mathrm{Lip}(S,T)$ be the infimum of the Lipschitz constant of all $\mathbb{F}$-equivariant maps $f : S \to T$. By [Alg12, Proposition 4.5], [Hor16, Theorem 0.2], $\mathrm{Lip}(S,T)$ is equal to

$$\Lambda(S,T) := \sup_{g \in \mathbb{F}\setminus\{1\}} \frac{||g||_T}{||g||_S}.$$

By [Alg12, Proposition 4.5], [Hor16, Proposition 6.15, 6.16], the supremum above can be taken over a set of candidates $\mathcal{C}(S) \subset \mathbb{F}$. Since S is free simplicial, the set $\mathcal{C}(S)$ is finite.

For every $\delta > 0$ and the finite set $\mathcal{C}(S)$ of elements of $\mathbb{F}$, there exists a $K > 0$ such that for all $k \geq K$ and for all $g \in \mathcal{C}(S)$,

$$||g||_{T_k} \leq ||g||_T + \delta.$$

Thus $\Lambda(S, T_k) \leq \Lambda(S,T) + \delta'$ where δ' is the maximum of $\delta/||g||_S$ over all $g \in \mathcal{C}(S)$. This implies that $\mathrm{Lip}(S, T_k) \leq \mathrm{Lip}(S,T) + \delta' \leq \mathrm{Lip}(h) + \delta' \leq 1 + \delta'$. By [Hor16, Theorem 0.4], $\mathrm{Lip}(S, T_k)$ is realized, that is, there exists an $\mathbb{F}$-equivariant map $h_k : S \to \overline{T_k}$, where $\overline{T_k}$ is the metric completion of T_k, such that $\mathrm{Lip}(h_k) = \mathrm{Lip}(S, T_k) \leq 1 + \delta'$ for all $k \geq K$. Also

$$\mathrm{BCC}(h_k) \leq \mathrm{Lip}(h_k)\,\mathrm{vol}(S) \leq (1+\delta')\epsilon.$$

Now consider a sequence of leaves $l_k \in L(T_k)$ converging to a leaf $l \in L_\infty$. Then by Proposition 5.6.3 ($\mathcal{Q}$ map), the diameter of $h_k(l_k)$ in $\overline{T_k}$ is bounded by $2\,\mathrm{BCC}(h_k)$ which is less than $2(1+\delta')\epsilon$. Thus, in the limit, the diameter of $h(l)$ in $\overline{T}$ is bounded above by $2(1+\delta')\epsilon$. Since ϵ and δ were arbitrary, we get that $l \in L(T)$. □

6.6 Stable and unstable trees

Lemma 6.6.1. $\Lambda^{\mp}_{\Phi} \subseteq L(T^{\pm}_{\Phi})$, $\Lambda^{\pm}_{\Phi} \nsubseteq L(T^{\pm}_{\Phi})$.

Proof. We have $T^+_\Phi = \lim_{n\to\infty} \dfrac{T_G \phi^n}{\lambda^n_\Phi}$. Let w be a nontrivial conjugacy class in $\mathbb{F} \setminus \mathcal{A}$. Assume $l_{T^+_\Phi}(w) = 1$. Let $g_m = \Phi^{-m}(w)$. Then $l_{T^+_\Phi}(g_m) = 1/\lambda^m_\Phi$ which implies $(g_m^{-\infty}, g_m^{\infty})$ is contained in $L_{1/\lambda^m_\Phi}(T^+_\Phi)$. Thus $l_- = \lim_{m\to\infty} g_m$ is contained in $L(T^+_\Phi) = \bigcap_{m\to\infty} L_{1/\lambda^m_\Phi}(T^+_\Phi)$. Since l_- is a generic leaf of Λ^-_Φ and $L(T^+_\Phi)$ is a closed subset of $\partial^2\mathbb{F}$, conclude that $\Lambda^-_\Phi \subseteq L(T^+_\Phi)$.

Let $g_m = \Phi^m(w)$ such that g_m converges to a generic leaf $l_+ \in \Lambda^+_\Phi$. We have $l_{T^+_\Phi}(g_m) = \lambda^m_\Phi l_{T^+_\Phi}(w)$ which grows as m goes to infinity. Thus $l_+ \notin L(T^+_\Phi)$. □

Lemma 6.6.2. *The stable and unstable trees $T^\pm_\Phi$ have dense orbits.*

Proof. By a result of [Hor14, Proposition 4.16], which is a relativization of Levitt's decomposition theorem for trees in CV_n [Lev94], we have the following: if T^+_Φ does not have dense orbits, then T^+_Φ splits uniquely as a graph of actions, all of whose vertex trees have dense orbits, such that the BassSerre tree $\mathcal{G}_{T^+_\Phi}$ of the underlying graph of groups is very small (Section 5.1), and all its edges have positive length. Up to taking powers, $\mathcal{G}_{T^+_\Phi}$ is Φ-invariant. If $\mathcal{G}_{T^+_\Phi}$ has an edge with trivial stabilizer, then by collapsing all other edges, we get a Φ-invariant free factor system, which is a contradiction. If the edge stabilizers are nontrivial, then they are nonperipheral. Then by theorems of Shenitzer [She55] and Swarup [Swa86], there is a smallest free factor system containing the edge stabilizer and $\mathcal{A}$, which will have to be Φ-invariant. This is a contradiction. □

6.7 Support of a relative current

Definition 6.7.1 (Support of a relative current)**.** Support of a relative current η is defined as the closure in $\mathbf{Y}$ (see Section 4.1.4 for definition) of the intersection of the complement of all open sets $U \subset \mathbf{Y}$ such that $\eta(U) = 0$. For $\eta \in \mathbb{P}\mathcal{RC}(\mathcal{A})$, $\mathrm{supp}(\eta)$ is a closed, nonempty and $\mathbb{F}$-invariant subset of $\mathbf{Y}$.

Since $\mathbf{Y}$ is not a closed subset of $\partial^2\mathbb{F}$, $\mathrm{supp}(\eta) \subset \mathbf{Y}$ may not be a closed subset of $\partial^2\mathbb{F}$. Let $\overline{\mathrm{supp}(\eta)}$ denote its closure in $\partial^2\mathbb{F}$. Then $\overline{\mathrm{supp}(\eta)} \setminus \mathrm{supp}(\eta)$ is contained in $\partial^2\mathcal{A}$ which is nonempty when lines in $\mathrm{supp}(\eta)$ accumulate on lines in $\partial^2\mathcal{A}$.

Example 6.7.2. Let $F_2 = \langle a, b\rangle$, $\mathcal{A} = \{[\langle a\rangle]\}$ and consider the sequence of relative currents $\eta_{a^k b}$ converging to η_∞ in $\mathbb{P}\mathcal{RC}(\mathcal{A})$ as in Example 6.2.1. Then $\mathrm{supp}(\eta_\infty)$ is given by bi-infinite geodesics determined by $\ldots aaa\, b\, aaa \ldots$. Thus the set $\overline{\mathrm{supp}(\eta_\infty)}$ also contains the bi-infinite lines given by $\ldots aaaa \ldots$. Geometrically, consider a lamination L on a torus with one puncture (with fundamental group identified with $F_2 = \langle a, b\rangle$) as follows: the lamination L contains the simple closed curve a and another leaf l which goes around b and spirals towards a from both sides. In the absolute case, the support of the current $\mu_{a^k b}$

is the curve a and the curve c_k obtained by Dehn twisting b around a, k times. The absolute currents $\mu_{a^k b}$ projectively converge to the absolute current μ_a whose support is just the curve a. But in the relative case, the support of the relative current $\eta_{a^k b}$ is the curve c_k and the relative currents $\eta_{a^k b}$ converge to η_∞ whose support is the leaf l. Thus the closure of l also contains the curve a.

We have that $\overline{\mathrm{supp}(\eta)}$ is a closed, nonempty, $\mathbb{F}$-invariant subset of $\partial^2\mathbb{F}$. Recall Notation 5.2.1 for a relative train track representative of Φ.

Lemma 6.7.3. *$\Lambda^+_\Phi \cap \mathbf{Y}$ is minimal in $\mathbf{Y}$, that is, $\Lambda^+_\Phi \cap \mathbf{Y}$ contains no proper closed (in $\mathbf{Y}$), nonempty $\mathbb{F}$-invariant subset.*

Proof. By [BFH00, Lemma 3.1.15], we have the following: suppose δ is a generic leaf in Λ^+_Φ that is not entirely contained in G_{r-1}. Then the closure of δ in $\partial^2\mathbb{F}$ is all of Λ^+_Φ. Suppose $\Lambda^+_\Phi \cap \mathbf{Y}$ contains a proper closed (in $\mathbf{Y}$), nonempty, $\mathbb{F}$-invariant subset S. A generic leaf δ in S is not entirely contained in G_{r-1} where $\mathcal{F}(G_{r-1}) = \mathcal{A}$. Since $\mathbf{Y}$ gets subspace topology from $\partial^2\mathbb{F}$, the closure of δ in $\mathbf{Y}$ is all of $\Lambda^+_\Phi \cap \mathbf{Y}$, which is a contradiction. □

Lemma 6.7.4. *We have $\mathrm{supp}(\eta^\pm_\Phi)$ as a subset of $\mathbf{Y}$ is equal to $\Lambda^\pm_\Phi \cap \mathbf{Y}$ and $\overline{\mathrm{supp}(\eta^\pm_\Phi)} \subseteq \Lambda^\pm_\Phi \cup \partial^2\mathcal{A}$.*

A proof of a similar fact in the case of a fully irreducible automorphism can be found in [CP12, Proposition 6.1].

Proof. Let a be a primitive conjugacy class in $\mathbb{F} \setminus \mathcal{A}$ realized as α in $G' = T'_G/\mathbb{F}$ (see Notation 5.2.1). Then α is a union of N r-legal paths for some $N > 0$. For every $m \geq 0$, $\alpha_m := (\phi')^m(\alpha)$ contains at most N segments of leaves of $\Lambda^+_\Phi \cap \mathbf{Y}$. Let the complement of $\Lambda^+_\Phi \cap \mathbf{Y}$ in $\mathbf{Y}$ be covered by cylinder sets $C(\gamma)$ where γ is a subpath of G' that crosses H_r and is not crossed by any leaf of Λ^+_Φ. For every $m \geq 0$, α_m contains at most N occurrences of γ (at concatenation points of the r-legal segments). Thus $\eta_{\alpha_m}(C(\gamma)) \leq N$. Since $\eta_{\alpha_m}/\lambda^m_\Phi \to \eta^+_\Phi$ as $m \to \infty$, we have that $\eta^+_\Phi(C(\gamma)) = 0$. Thus $\mathrm{supp}(\eta^+_\Phi) \subseteq \Lambda^+_\Phi \cap \mathbf{Y}$. By Lemma 6.7.3, $\Lambda^+_\Phi \cap \mathbf{Y}$ is minimal in $\mathbf{Y}$, therefore we have $\mathrm{supp}(\eta^+_\Phi) = \Lambda^+_\Phi \cap \mathbf{Y}$. Since Λ^+_Φ is a closed subset of $\partial^2\mathbb{F}$, we get that $\overline{\mathrm{supp}(\eta^+_\Phi)} \subseteq \Lambda^+_\Phi \cup \partial^2\mathcal{A}$. □

Lemma 6.7.5. *Let $\{\eta_k\}_{k\in\mathbb{N}}$ be a sequence of relative currents converging to a relative current η. Suppose the sequence $\operatorname{supp}(\eta_k)$ converges to $S_\infty \subset \mathbf{Y}$. Then $\operatorname{supp}(\eta) \subseteq S_\infty$.*

Proof. Consider a word $w \in \mathbb{F} \setminus \mathcal{A}$ such that $\eta(w) > 0$. Then given $\epsilon > 0$, there exists an $N_0 > 0$ such that for every $k > N_0$, $\eta_k(w) > \epsilon$. Thus $C(w) \cap \operatorname{supp}(\eta_k)$ is nonempty for every $k \geq N_0$ which implies that $C(w) \cap S_\infty$ is nonempty. Since this is true for any word $w \in \mathbb{F} \setminus \mathcal{A}$ with $\eta(w) > 0$, we get that $\operatorname{supp}(\eta) \subset S_\infty$. □

6.8 Intersection form

We are now ready to define an intersection form for closure of relative outer space and the space of projectivized relative currents.

Definition 6.8.1. Define a function $I : \overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})} \times \mathbb{P}\mathcal{RC}(\mathcal{A}) \to \{0,1\}$ as follows:

$$I(T,\eta) = 0 \quad \text{if } \overline{\operatorname{supp}(\eta)} \subseteq L(T),$$

$$I(T,\eta) = 1 \quad \text{if } \overline{\operatorname{supp}(\eta)} \nsubseteq L(T).$$

Lemma 6.8.2. *The function I satisfies the following properties:*

(a) $I(T\Psi,\eta) = I(T,\Psi\eta)$ for $\Psi \in \operatorname{Out}(\mathbb{F},\mathcal{A})$.

(b) Let $T_k \to T$ in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ and $\eta_k \to \eta$ in $\mathbb{P}\mathcal{RC}(\mathcal{A})$ such that $I(T_k,\eta_k) = 0$ for all k. If either T has dense orbits or $\overline{\operatorname{supp}(\eta)}$ is a birecurrent lamination, then $I(T,\eta) = 0$.

Remark 6.8.3. It is not true in general that if $I(T_k,\eta_k) = 0$ for all k, then $I(T,\eta) = 0$. Consider the sequence of trees T_k as in Example 6.5.1 and the sequence of currents η_k as in Example 6.7.2. Then $I(T_k,\eta_k) = 0$ but $I(T,\eta) \neq 0$.

Proof. (a) We have $\operatorname{supp}(\Psi\eta) = \Psi\operatorname{supp}(\eta)$ and $L(T\Psi) = \Psi^{-1}L(T)$ which gives the desired equality.

(b) Let $\mathcal{S}$ be the closure of $\lim_{n\to\infty}\operatorname{supp}(\eta_n)$ and let $L(T_n)$ converge to L_∞. Then $\mathcal{S} \subseteq L_\infty$ and by Lemma 6.7.5, $\overline{\operatorname{supp}(\eta)} \subseteq \mathcal{S}$. If T has dense orbits, then by Lemma 6.5.6, $L_\infty \subseteq L(T)$. Thus $\overline{\operatorname{supp}(\eta)} \subseteq L(T)$. If $\overline{\operatorname{supp}(\eta)}$ is a birecurrent lamination, then by Proposition 6.5.5, it is contained in $L(T)$. □

Lemma 6.8.4 (Uniqueness of dual)**.** *Let Φ be a fully irreducible outer automorphism relative to $\mathcal{A}$. Let $T \in \overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ and $\eta \in \mathbb{P}\mathcal{RC}(\mathcal{A})$. Then*

(a) $I(T^{\pm}_{\Phi}, \eta^{\mp}_{\Phi}) = 0$.

(b) If $I(T^{\pm}_{\Phi}, \eta) = 0$, *then* $\eta = \eta^{\mp}_{\Phi}$.

(c) If $I(T, \eta^{\mp}_{\Phi}) = 0$, *then* $T = T^{\pm}_{\Phi}$.

Proof. (a) By Lemma 6.6.1, $\Lambda^{\mp}_{\Phi} \subset L(T^{\pm}_{\Phi})$. Also $\partial^2\mathcal{A} \subset L(T^{\pm}_{\Phi})$ because $\mathcal{A}$ is elliptic in $T^{\pm}_{\Phi}$. Thus by Lemma 6.7.4, $\overline{\mathrm{supp}(\eta^{\mp}_{\Phi})} \subseteq L(T^{\pm}_{\Phi})$.

(b) By Lemma 6.6.1 and Lemma 6.7.4, $\mathrm{supp}(\eta^{+}_{\Phi}) \nsubseteq L(T^{+}_{\Phi})$, therefore $I(T^{+}_{\Phi}, \eta^{+}_{\Phi}) \neq 0$. Now suppose $I(T^{+}_{\Phi}, \eta) = 0$ for some $\eta \neq \eta^{-}_{\Phi}$. Then by definition, $\overline{\mathrm{supp}(\eta)} \subseteq L(T^{+}_{\Phi})$. By the $\mathrm{Out}(\mathbb{F},\mathcal{A})$ action, we also get that $\overline{\mathrm{supp}(\Phi^n(\eta))} \subseteq L(T^{+}_{\Phi})$. By Theorem B, $\Phi^n(\eta)$ converges to η^{+}_{Φ}, therefore in the limit $\overline{\mathrm{supp}(\eta^{+}_{\Phi})} \subseteq L(T^{+}_{\Phi})$, which is a contradiction.

(c) Similar argument as above using Theorem C. □

6.9 Summary

Even though we were not successful in defining an intersection number along the lines of Kapovich and Lustig, we were able to generalize the zero pairing criterion. Our definition of intersection form was sufficient to establish uniqueness of pairing for the stable and unstable trees and currents obtained from north-south dynamics on $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ and $\mathcal{MRC}(\mathcal{A})$, respectively. The intersection form defined here is not continuous in general (Remark 6.8.3), but Lemma 6.8.2 gives continuity at pairs containing the stable and unstable trees or currents. The fact that the intersection form behaves well for the four special points is enough to carry out the proof of Theorem A in the next section.

CHAPTER 7

LOXODROMIC ELEMENTS IN RELATIVE FREE FACTOR COMPLEX

In this chapter, we will prove Thereom A. The proof is based on [BF02, Proposition 11].

Lemma 7.1 (UV-pair). *Let Φ be fully irreducible relative to $\mathcal{A}$. For every neighborhood U of T_Φ^+ in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$, there exists a neighborhood V of η_Φ^- in $\mathbb{P}\mathcal{RC}(\mathcal{A})$ such that for every $T \in U^C$ and $\eta \in V$, we have $I(T,\eta) \neq 0$.*

Proof. Assume by contradiction that there exists a U such that for every neighborhood V of η_Φ^-, there exist $T \in U^C$ and $\eta \in V$ such that $I(T,\eta) = 0$.

Let V_i be an infinite sequence of nested neighborhoods of η_Φ^- such that $V_i \supset V_{i+1}$ and $\cap V_i = \eta_\Phi^-$. Then by assumption, there exist $T_i \in U^C$ and $\eta_i \in V_i$ such that $I(T_i,\eta_i) = 0$. Since $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ is compact, after passing to a subsequence, $T_i \to T$, for $T \neq T_\Phi^+$. Also $\eta_i \to \eta_\Phi^-$. Since the support of η_Φ^- gives a birecurrent lamination, by Lemma 6.8.2, $I(T,\eta_\Phi^-) = 0$, which contradicts Lemma 6.8.4. □

Lemma 7.2 (VU-pair). *For every neighborhood V of η_Φ^- in $\mathbb{P}\mathcal{RC}(\mathcal{A})$, there exists a neighborhood U of T_Φ^+ in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F},\mathcal{A})}$ such that for every $\eta \in V^C$ and $T \in U$, we have $I(T,\eta) \neq 0$.*

Proof. Same as for Lemma 7.1. □

Lemma 7.3. *There exist nested sequences $U_0 \supset U_1 \supset U_2 \supset U_3 \ldots \supset U_{2N}$ and $V_1 \supset V_2 \supset V_3 \ldots \supset V_{2N}$ of neighborhoods of T_Φ^+ and η_Φ^-, respectively, such that the following are true:*

- *$\exists\, k > 0$ such that for every i, $\Phi^k(U_i) \subset U_{i+1}$ and $\Phi^{-k}(V_i) \subset V_{i+1}$.*

- *(U_i, V_{i+1}) form a UV-pair for all $i \geq 0$.*

- *(V_i, U_i) form a VU-pair for all $i \geq 1$.*

Proof. Start with U_0 to build a sequence as in the statement of the lemma. Then there exists V_1 such that (U_0, V_1) form a UV-pair. Next there exists a U_1 such that (V_1, U_1) form a VU-pair. If $U_1 \not\subseteq U_0$, then replace U_1 by a smaller open set in $U_1 \cap U_0$.

Let $r_i = \min\{\mathrm{p} \mid \Phi^p(U_i) \subset U_{i+1}\}$ for $0 \leq i \leq 2N$ and let $s_i = \min\{\mathrm{p} \mid \Phi^{-p}(V_i) \subset V_{i+1}\}$ for $0 < i < 2N$. The numbers r_i and s_i exist because we have uniform north-south dynamics. Now define k to be the maximum of the numbers $\{r_i\}_{i=0}^{2N}, \{s_i\}_{i=1}^{2N}$. $\square$

Theorem A. *Let $\mathcal{A}$ be a nonexceptional free factor system and let $\Phi \in \mathrm{Out}(\mathbb{F}, \mathcal{A})$. Then Φ acts loxodromically on $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ if and only if Φ is fully irreducible relative to $\mathcal{A}$.*

Proof. Let $\mathcal{D} \in \mathcal{FF}(\mathbb{F}, \mathcal{A})$ be a free factor system. Let $T_{\mathcal{D}} \in \mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})$ be a simplicial tree such that its set of vertex stabilizers is equal to $\mathcal{D}$. Let $\eta_{\mathcal{D}}$ be a relative current with support contained in $\partial^2\mathcal{D}$. Consider nested neighborhoods $U_0 \supset U_1 \supset \ldots \supset U_{2N}$ of T_Φ^+ and $V_1 \supset V_2 \supset \ldots \supset V_{2N}$ of η_Φ^- and constant k as in Lemma 7.3 such that $T_{\mathcal{D}} \in U_0 \cap U_1^C$ and $\eta_{\mathcal{D}} \in V_1^C$. See **Figure 7.1**. By Lemma 7.1 and 7.2, the following holds:

- If $T \in U_i^C$ and $I(T, \eta) = 0$, then $\eta \in V_{i+1}^C$.

- If $\eta \in V_i^C$ and $I(T, \eta) = 0$, then $T \in U_i^C$.

We have $T_{\mathcal{D}}\Phi^{ik} \in U_i$ and $\Phi^{-ik}\eta_{\mathcal{D}} \in V_i$. If $\mathcal{D}$ is the set of vertex stabilizers of $T_{\mathcal{D}}$, then $\Phi^{-2ik}(\mathcal{D})$ is the set of vertex stabilizers of $T_{\mathcal{D}}\Phi^{2ik}$.

We claim that $d_{\mathcal{FF}(\mathbb{F},\mathcal{A})}(\mathcal{D}, \Phi^{-2Nk}\mathcal{D}) > 2N$ and $d_{\mathcal{FF}(\mathbb{F},\mathcal{A})}(\mathcal{D}, \Phi^{2Nk}\mathcal{D}) > 2N$. For simplicity, let's first consider the case when $N = 1$ and for contradiction, assume that the distance $d_{\mathcal{FF}(\mathbb{F},\mathcal{A})}(\mathcal{D}, \Phi^{-2k}\mathcal{D})$ is equal to 2. Let $\mathcal{E}$ be a free factor system distance one from both $\mathcal{D}$ and $\Phi^{-2k}\mathcal{D}$. There are two cases to consider:

(a) $\mathcal{E} \sqsubset \mathcal{D}$ and $\mathcal{E} \sqsubset \Phi^{-2k}\mathcal{D}$: let $T_{\mathcal{E}}$ be a simplicial tree whose set of vertex stabilizers is given by $\mathcal{E}$. Choose η such that $I(T_{\mathcal{E}}, \eta) = 0$. Then $I(T_{\mathcal{D}}, \eta) = 0$. Since $T_{\mathcal{D}} \in U_1^C$, we get $\eta \in V_2^C$. Also $I(T_{\mathcal{D}}\Phi^{2k}, \eta) = 0$ and since $\eta \in V_2^C$, we get $T_{\mathcal{D}}\Phi^{2k} \in U_2^C$. But that is a contradiction since $T_{\mathcal{D}}\Phi^{2k} \in U_2$.

(b) $\mathcal{E} \sqsupset \mathcal{D}$ and $\mathcal{E} \sqsupset \Phi^{-2k}\mathcal{D}$: we have $I(T_{\mathcal{E}}, \eta_{\mathcal{D}}) = 0$. Since $\eta_{\mathcal{D}} \in V_1^C$, we get $T_{\mathcal{E}} \in U_1^C$. Also $I(T_{\mathcal{E}}, \Phi^{2k}\eta_{\mathcal{D}}) = 0$. Since $T_{\mathcal{E}} \in U_1^C$, we get $\Phi^{-2k}\eta_{\mathcal{D}} \in V_2^C$, which is a contradiction.

The above proof in particular also shows that $d_{\mathcal{FF}(\mathbb{F},\mathcal{A})}(\mathcal{D}, \Phi^{-2Nk}(\mathcal{D})) > 2$. For contradiction, suppose that $d_{\mathcal{FF}(\mathbb{F},\mathcal{A})}(\mathcal{D}, \Phi^{-2Nk}\mathcal{D}) \leq 2N$. Consider a geodesic

$$\mathcal{D} = \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \ldots, \mathcal{E}_l, \mathcal{E}_{l+1} = \Phi^{-2Nk}\mathcal{D},$$

$l < 2N$, in $\mathcal{FF}(\mathbb{F}, \mathcal{A})$. Without loss of generality, assume $\mathcal{E}_1 \sqsubset \mathcal{D}$. Then starting with applying the same argument as in (a) for the triple $\mathcal{D}, \mathcal{E}_1, \mathcal{E}_2$, alternatively apply (a) and (b) to reach a contradiction. $\square$

Example 7.4. As an example to exhibit the proof of Theorem A for $N = 3$, consider a geodesic $\mathcal{D} = \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \ldots, \mathcal{E}_5, \mathcal{E}_6 = \Phi^{-6k}\mathcal{D}$ in $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ connecting $\mathcal{D}$ and $\Phi^{-6k}\mathcal{D}$. Without loss of generality, assume $\mathcal{E}_1 \sqsubset \mathcal{D}$. Let T_i be a tree in $\overline{\mathbb{P}\mathcal{O}(\mathbb{F}, \mathcal{A})}$ whose set of vertex stabilizers is given by $\mathcal{E}_i$. We have $T_0 \in U_0 \cap U_1^C$ and thus T_6 is contained in U_6.

- Given T_1, choose η_1 such that $I(T_1, \eta_1) = 0$, which implies that $I(T_{\mathcal{D}}, \eta_1) = 0$ because $\text{supp}(\eta_1) \subset \partial^2\mathcal{E}_1 \subset \partial^2\mathcal{D}$. Also $I(T_2, \eta_1) = 0$ because $\text{supp}(\eta_1) \subset \partial^2\mathcal{E}_1 \subset \partial^2\mathcal{E}_2$.
- Given T_3, choose η_2 such that $I(T_3, \eta_2) = 0$, which implies that $I(T_2, \eta_2) = 0$ because $\text{supp}(\eta_2) \subset \partial^2\mathcal{E}_3 \subset \partial^2\mathcal{E}_2$. Also $I(T_4, \eta_2) = 0$ because $\text{supp}(\eta_2) \subset \partial^2\mathcal{E}_3 \subset \partial^2\mathcal{E}_4$.
- Given T_5, choose η_3 such that $I(T_5, \eta_3) = 0$, which implies that $I(T_4, \eta_3) = 0$ because $\text{supp}(\eta_3) \subset \partial^2\mathcal{E}_5 \subset \partial^2\mathcal{E}_4$. Also $I(T_6, \eta_3) = 0$ because $\text{supp}(\eta_3) \subset \partial^2\mathcal{E}_5 \subset \partial^2\mathcal{E}_6$.

We get the following chain of implications using all of the above information: $T_{\mathcal{D}} \in U_1^C \implies \eta_1 \in V_2^C \implies T_2 \in U_2^C \implies \eta_2 \in V_3^C \implies T_4 \in U_3^C \implies \eta_3 \in V_4^C \implies T_6 \in U_4^C$, which yields a contradiction. See **Figure 7.2**.

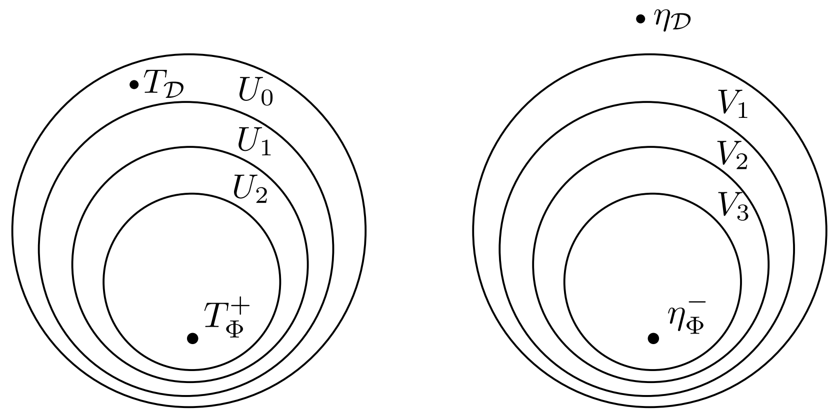

Figure 7.1. Nested sets

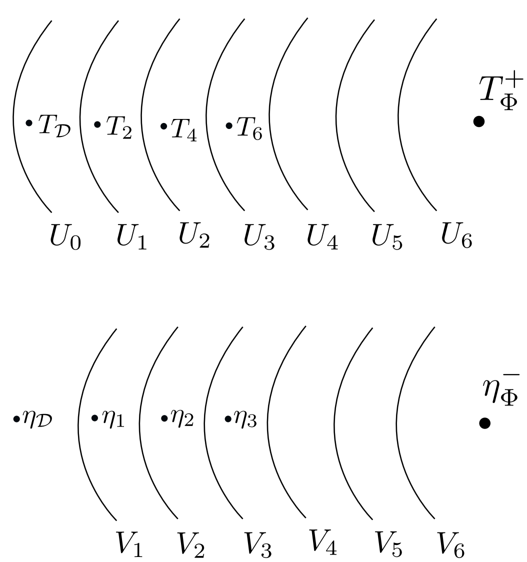

Figure 7.2. Nested sets for Example 7.4

APPENDIX A

EXTENDING RELATIVE CURRENTS

In this appendix, we will prove Lemma 4.2.14, which says that given a relative current η_0, there exists a signed measured current η which is a k-extension of η_0. We will first show that we can extend η_0 to a signed measured current η which may or may not be nonnegative on all words of length less than or equal to k. We then show how to modify η to get a k-extension of η_0.

Throughout this appemdix, we will assume that $\mathcal{A}$ has only one free factor A_0. When $\mathcal{A}$ has more than one free factor in it, then the same process can be repeated for all the free factors independently of each other.

Notation A.1.

- Let $\mathfrak{B}_{\mathcal{A}}$ be a relative basis of $\mathbb{F}$. Let s be the rank of the free factor A_0. Denote the generators of A_0 by a_i, $1 \leq i \leq s$. Also let $A := \{a_1^{\pm}, \ldots, a_s^{\pm}\}$.
- Let S_k be the set of words in A_0 of length k with respect to $\mathfrak{B}_{\mathcal{A}}$. Let $\#S_k$ denote the cardinality of S_k.
- Let S_k^0 be a subset of S_k (chosen once and for all) such that for every $w \in S_k$, exactly one of w or $\overline{w}$ appears in S_k^0.
- We will use letters e, x, y, z to denote the elements of $\mathfrak{B}_{\mathcal{A}}$.
- Whenever we write a forward (backward) extension of a word w by $e \in \mathfrak{B}_{\mathcal{A}}$ as we (ew), it is to be understood that e is not the inverse of the last (first) letter of w.

For every $k > 0$, define a signed measured current η on words in A_0 of length $(k-1)$ and use those values together with the additivity laws satisfied by η to define η on words of length k. To start with words of length one, choose arbitrary values for $\eta(a_i)$ for all $1 \leq i \leq s$. By induction, assume $\eta(v)$ is defined for all words v of length less than or equal to $(k-1)$. The following holds for all $v \in S_{k-1}^0$ by additivity:

$$\eta(v) = \sum_{e \in A} \eta(ve) + \sum_{e \notin A} \eta_0(ve),$$
$$\eta(\overline{v}) = \sum_{e \in A} \eta(\overline{v}e) + \sum_{e \notin A} \eta_0(\overline{v}e).$$

Since η is invariant under taking inverses, the equation obtained from forward extension of $\overline{v}$ is the same as the equation obtained from backward extension of v.

Rearranging the equations to have the unknown terms on the left-hand side, we get

$$\sum_{e \in A} \eta(ve) = \eta(v) - \sum_{e \notin A} \eta_0(ve) =: c_v,$$
$$\sum_{e \in A} \eta(\overline{v}e) = \eta(\overline{v}) - \sum_{e \notin A} \eta_0(\overline{v}e) =: c_{\overline{v}}.$$

Thus there are $\#S_{k-1}$ equations in $\#S_k^0$ variables and the number of variables are more than the number of equations. Denote this system of equations by E^1_{k-1}, that is, equations obtained from one edge extensions of length $(k-1)$ words. Similarly, we can look at the system E^i_{k-i}.

Consider the augmented matrix $[M|c]$ for the system of equations E^1_{k-1} with rows labeled by $v \in S_{k-1}$ and columns by $w \in S^0_k$. Then $M_{v,w} = 1$ if $w = ve$ or $\overline{w} = ve$ for some $e \in A$ and 0 otherwise. Denote a row vector of M by r_v corresponding to $v \in S_{k-1}$. Here are some observations about the matrix M.

- Each column has exactly two ones. Indeed, $M_{v,w}$ is 1 exactly when v is a prefix of w or $\overline{w}$.

- There are $(2s-1)$ nonzero entries in each row because there are $(2s-1)$ possible extensions of v by $e \in A$.

- Any two distinct rows can be the same in at most one column. Let w be common to two distinct rows r_{v_1} and r_{v_2}. Then

  $$w = v_1 e_1 \text{ or } \overline{e_1}\,\overline{v_1} \quad \text{and} \quad w = v_2 e_2 \text{ or } \overline{e_2}\,\overline{v_2}$$

  for some $e_1, e_2 \in A$. Then it must be true that v_1 begins with $\overline{e_2}$ and v_2 begins with $\overline{e_1}$. Thus w is uniquely determined.

Lemma A.2. *(a) For every $i \geq 1$, an equation in the system E^{i+1}_{k-i-1} is a linear combination of equations in the system E^i_{k-i}. Thus it is sufficient to look at the system E^1_{k-1} to obtain all constraints satisfied by $\eta(w)$ for all $w \in S^0_k$.*

(b) Let $u \in S_{k-2}$. Then we have

$$\sum_{x\in A} r_{xu} = \sum_{x\in A} r_{x\overline{u}}.$$

(c) The set of relations $\sum_{x\in A} r_{xu} = \sum_{x\in A} r_{x\overline{u}}$ for every $u \in S_{k-2}$ generate any other relation among the rows of M.

(d) We also have that

$$\sum_{x\in A} c_{xu} = \sum_{x\in A} c_{x\overline{u}}$$

where c_v is the constant term of the equation determined by $v \in S_{k-1}$.

(e) The system of equations E^1_{k-1} is consistent and hence has a solution. Thus we can define η on words of length k.

Proof. (a) Let $u \in S_{k-i-1}$. Then

$$\eta(u) = \sum_{x\in A} \eta(ux) + \sum_{x\notin A} \eta(ux).$$

By equations in E^i_{k-i}, we have

$$\eta(ux) = \sum_{y\in\mathbb{F},|y|=i} \eta(uxy).$$

Adding all these equations over $x \in \mathfrak{B}_{\mathcal{A}}$ we get

$$\eta(u) = \sum_{x,y\in\mathbb{F},|x|=1,|y|=i} \eta(uxy) = \sum_{z\in\mathbb{F},|z|=i+1} \eta(uz)$$

Thus we recovered an equation in E^{i+1}_{k-i-1} by a combination of equations in E^i_{k-i}.

(b) For every $x \in A$, $M_{xu,w} \neq 0$ exactly when $w = x\,u\,\overline{y}$ or $w = y\,\overline{u}\,\overline{x}$ for some $y \in A$. Therefore, if $M_{xu,w} \neq 0$, then $M_{y\overline{u},w} \neq 0$ for some $y \in A$.

(c) Consider a minimal relation R given by $\sum_{v\in S_{k-1}} d_v r_v = 0$ where $d_v \in \mathbb{R}$. The equation can be rescaled such that coefficient of at least one row, say r_{xu} for some $x \in A$ and $u \in S_{k-2}$, is 1.

For every $y \in A$ and $w = xu\overline{y}$, we have $M_{xu,w} = M_{y\overline{u},w} = 1$. Thus r_{xu} and $r_{y\overline{u}}$ share exactly one common entry w and no other row has a nonzero entry in w. Thus $d_{y\overline{u}} = -1$. Now consider $y \in A$. For any $z \in A$ and $w = y\overline{u}z$, we have $M_{y\overline{u},w} = M_{\overline{z}u,w} = 1$. Thus $d_{\overline{z}u} = 1$. Hence our minimal relation is just $\sum_{x\in A} r_{xu} - \sum_{y\in A} r_{y\overline{u}} = 0$.

(d) We have

$$\begin{aligned}\sum_{x\in A} c_{xu} &= \sum_{x\in A}\eta(xu) - \sum_{x\in A, y\notin A}\eta(xuy)\\ &= \eta(u) - \sum_{x\notin A}\eta(xu) - \sum_{x\in A, y\notin A}\eta(xuy)\\ &= \eta(u) - \sum_{x\notin A, y\in \mathfrak{B}_{\mathcal{A}}}\eta(xuy) - \sum_{x\in A, y\notin A}\eta(xuy)\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{x\in A} c_{x\overline{u}} &= \eta(u) - \sum_{x\notin A, y\in \mathfrak{B}_{\mathcal{A}}}\eta(x\overline{u}y) - \sum_{x\in A, y\notin A}\eta(x\overline{u}y)\\ &= \eta(u) - \sum_{x\notin A, y\in \mathfrak{B}_{\mathcal{A}}}\eta(\overline{y}u\overline{x}) - \sum_{x\in A, y\notin A}\eta(\overline{y}u\overline{x})\end{aligned}$$

We see that

$$\sum_{x\notin A, y\in \mathfrak{B}_{\mathcal{A}}}\eta(xuy) + \sum_{x\in A, y\notin A}\eta(xuy) = \sum_{x\notin A, y\in \mathfrak{B}_{\mathcal{A}}}\eta(\overline{y}u\overline{x}) + \sum_{x\in A, y\notin A}\eta(\overline{y}u\overline{x}).$$

Geometrically, we are looking at the same subset of $\partial^2\mathbb{F}$ as a union of cylinder sets in two different ways. See **Figure A.1** when $\mathbb{F} = \langle a, b, c, d\rangle$.

(e) Since the relations which generate all other relations among the rows of M are consistent, $[M|c]$ has a solution. □

Proof of Lemma 4.2.14. Given a relative current η_0, by Lemma A.2, get a signed measured current η such that $\eta_0(w) = \eta(w)$ for all $w \in \mathbb{F}\setminus\mathcal{A}$. This extension need not be nonnegative on all words of length less than or equal to k. Let $-M$ for $M > 0$ be the smallest value attained by $\eta(w)$ for a word $w \in \mathcal{A}$ with $|w| \le k$. Consider a signed measured current $\eta_{\mathcal{A},C}$ defined as follows:

$$\eta_{\mathcal{A},C}(w) = \frac{C}{(2s-1)^{|w|-1}} \text{ for } w\in\mathcal{A} \text{ and } 0 \text{ otherwise.}$$

For $C = M(2s-1)^{k-1}$, $\eta + \eta_{\mathcal{A},C}$ is nonnegative on words of length less than or equal to k. □

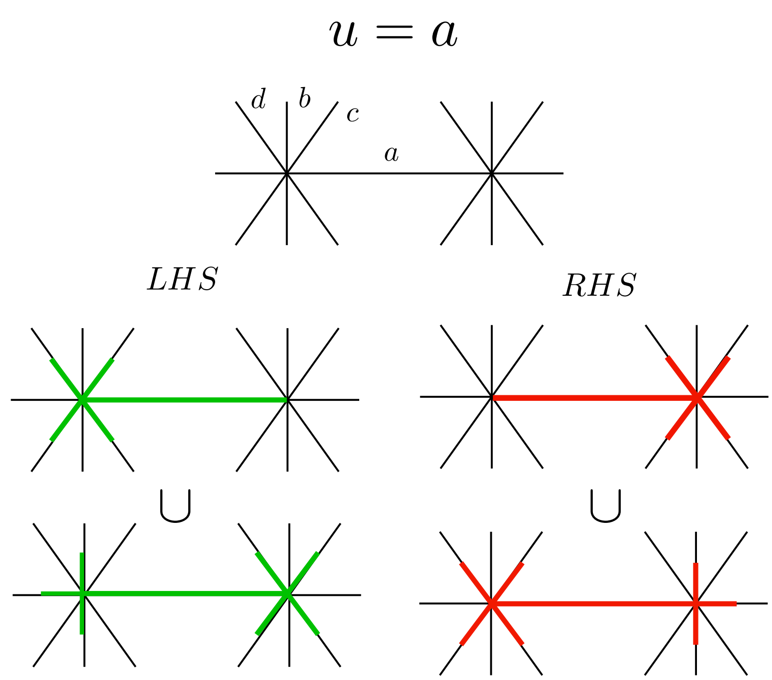

Figure A.1. Example for proof of Lemma A.2(d)

APPENDIX B

ANOTHER DEFINITION OF RELATIVE CURRENTS

In this section, we give another formulation of relative currents. We show that this new space of relative currents is homeomorphic to $\mathcal{RC}(\mathcal{A})$.

Let $\mathcal{SMC}(\mathbb{F})$ be the space of $\mathbb{F}$-invariant, locally finite, signed measured currents (Definition 4.2.11) on $\partial^2\mathbb{F}$. It is a vector space and comes equipped with a weak-$*$ topology, that is, a sequence $\eta_i \in \mathcal{SMC}(\mathbb{F})$ converges to η iff $\int f\, d\eta_i \to \int f\, d\eta$ for all compactly supported continuous functions f on $\partial^2\mathbb{F}$. Let

$$\mathcal{SMC}(\mathbb{F})^+ := \{\eta \in \mathcal{SMC}(\mathbb{F}) | \eta(w) \geq 0 \text{ for all } w \in \mathbb{F} \setminus \mathcal{A}\}.$$

Define an equivalence relation on $\mathcal{SMC}(\mathbb{F})^+$ as follows : $\eta_1 \sim \eta_2$ if $\eta_1|_{\mathbf{Y}} = \eta_2|_{\mathbf{Y}}$, that is, $\eta_1(w) = \eta_2(w)$ for all $w \in \mathbb{F} \setminus \mathcal{A}$. Denote the equivalence class of $\eta \in \mathcal{SMC}(\mathbb{F})^+$ by $[\eta]$. Note that all currents supported on $\partial^2\mathcal{A}$ are in a single equivalence class, denoted $[\eta_{\mathcal{A}}]$. A sequence $[\eta_i]$ converges to $[\eta]$ iff there exist signed measured currents $\mu_i \in [\eta_{\mathcal{A}}]$ such that $\eta_i(w) + \mu_i(w) \to \eta(w)$ for all $w \in \mathbb{F}$.

Proposition B.1. *(a) There exists a continuous injective map $f_{rest} : \mathcal{SMC}^+(\mathbb{F})/\sim \to \mathcal{RC}(\mathcal{A})$.*

(b) There exists a continuous injective map $f_{ext} : \mathcal{RC}(\mathcal{A}) \to \mathcal{SMC}^+(\mathbb{F})/\sim$.

Proof. (a) Given $[\eta] \in \mathcal{SMC}^+(\mathbb{F})/\sim$, $\eta(w)$ for $w \in \mathbb{F} \setminus \mathcal{A}$ is well defined. Thus $f_{rest}([\eta]) := \eta|_{\mathbf{Y}}$. The function is injective since two different classes $[\eta_1], [\eta_2]$ differ on some $w \in \mathbb{F} \setminus \mathcal{A}$ giving different relative currents in the image. Consider a sequence $[\eta_i]$ converging to $[\eta]$. Then $\eta_i(w)$ converges to $\eta(w)$ for all $w \in \mathbb{F} \setminus \mathcal{A}$.

(b) Given $\eta \in \mathcal{RC}(\mathcal{A})$, let $\eta' \in \mathcal{SMC}(\mathbb{F})^+$ be an extension of η given by Lemma 4.2.14. Define $f_{ext}(\eta) := [\eta']$. This function is well defined because any two extensions of η differ only by values on $w \in \mathcal{A}$. This map is injective since two distinct relative currents differ on some $w \in \mathbb{F} \setminus \mathcal{A}$ and hence the equivalence classes of the extensions are also distinct.

To establish continuity of the map f_{ext}, consider a sequence $\eta_i \to \eta \in \mathcal{RC}(\mathcal{A})$ and an extension η' of η. We will show that there exist extensions η'_i of η_i such that $\eta'_i(w)$ converges to $\eta'(w)$ for all $w \in \mathbb{F}$. The convergence is clear for $w \in \mathbb{F} \setminus \mathcal{A}$.

Let $\mathfrak{B}_{\mathcal{A}}$ be a relative basis of $\mathbb{F}$ and let $|w|$ be the length of $w \in \mathbb{F}$ with respect to $\mathfrak{B}_{\mathcal{A}}$. Given $\epsilon > 0$ and $n > 0$, there exists $M > 0$ such that $|\eta_i(v) - \eta(v)| \leq \epsilon$ for all $i \geq M$ and $v \in \mathbb{F} \setminus \mathcal{A}$ with $|v| \leq n$. Let N be the rank of the cofactor of $\mathcal{A}$. Since the extension process (Appendix A) can be done for each free factor in $\mathcal{A}$ independently of one another, we may assume that $\mathcal{A}$ has only one free factor A of rank s. Starting with words of length one in A, set $\eta'_i(e)$ equal to $\eta'(e)$. We claim that for all $i \geq M$, η'_i can be chosen such that for all words $w \in A$ such that $|w| \leq n$, we have

$$|\eta'_i(w) - \eta'(w)| \leq \frac{2N\epsilon(1 + \ldots + q^{l-2})}{q^{l-1}} \leq 2N\epsilon$$

where $q = 2s - 1$ and $l = |w|$.

Let the augmented matrix representing the system of equations for extension of η_i to words of length l be $[M|c^{i,l}]$ and let the corresponding matrix for η be $[M|c^l]$. For a length $l-1$ word v, let $c_v^{i,l}$ represent the corresponding entry of the vector $c^{i,l}$. We have

$$c_v^{i,l} = \eta'_i(v) - \sum_{e \notin A} \eta_i(ve)$$

Thus

$$|c_v^{i,l} - c_v^l| \leq |\eta'_i(v) - \eta'(v)| + \left| \sum_{e \notin A} \eta_i(ve) - \sum_{e \notin A} \eta(ve) \right| \leq |\eta'_i(v) - \eta'(v)| + 2N\epsilon,$$

Consider the base case $l = 2$ for induction. We have $c_v^2 - 2N\epsilon \leq c_v^{i,2} \leq c_v^2 + 2N\epsilon$ since $\eta'_i(e) = \eta'(e)$ for $e \in A$ and $|e| = 1$. Since the sum of every row of M is $q = 2s-1$, find $\eta'_i(w)$ such that

$$\eta'(w) - \frac{2N\epsilon}{q} \leq \eta'_i(w) \leq \eta'(w) + \frac{2N\epsilon}{q}.$$

Now by induction on length, we have

$$|c_v^{i,l} - c_v^l| \leq |\eta'_i(v) - \eta'(v)| + 2N\epsilon \leq \frac{2N\epsilon(1 + \ldots + q^{l-3})}{q^{l-2}} + 2N\epsilon = \frac{2N\epsilon(1 + \ldots + q^{l-2})}{q^{l-2}}.$$

Again using the fact that row sum is q, we get

$$|\eta_i'(w) - \eta'(w)| \leq \frac{2N\epsilon(1 + \ldots + q^{l-2})}{q^{l-1}} \leq 2N\epsilon$$

for $|w| = l$.

Thus we can find signed measured currents η_i' that are extensions of η_i such that $\eta_i'(w) \to \eta'(w)$ for all $w \in \mathbb{F}$. Thus the function f_{ext} is continuous. $\square$

APPENDIX C

EXAMPLES OF TRANSVERSE COVERING

Let Φ be a fully irreducible outer automorphism relative to $\mathcal{A}$. Let $\phi_0' : G' \to G'$ be a relative train track representative of Φ and let $\phi_0 : G \to G$ be an $\mathcal{A}$-train track representative of Φ (see Definition 5.2.2). Let $T_{G'}$ and T_G be the universal covers of G' and G, respectively. In Section 5.5, a transverse covering for T_G was defined. In this appendix, we record some examples of such transverse coverings and study their skeleton (defined below).

Definition C.1 (Skeleton of a transverse covering)**.** Given a transverse covering $\mathcal{Y} = \{Y\}_i$, the skeleton S is a graph obtained as follows: the vertex set is the set $\mathcal{Y} \cup V_0(S)$ where $V_0(S)$ is the set of all intersection points between distinct subtrees in $\mathcal{Y}$. There is an edge between $Y \in \mathcal{Y}$ and $y \in V_0(S)$ whenever $y \in Y$. The skeleton S is in fact a tree with a simplicial action of $\mathbb{F}$.

Remark C.2. In the absolute case of a fully irreducible outer automorphism, the Whitehead graph of a leaf of the attracting lamination at a vertex (see Definition 5.4.1) is connected and the skeleton of the transverse covering corresponding to the attracting lamination is just a point. But in the relative case, there seems to be no relation between the connectivity of the Whitehead graph of $\Lambda^+_\Phi(T_{G'})$ and the skeleton of transverse covering of T_G corresponding to Λ^+_Φ, as can be seen by the examples that follow.

Example C.3. Recall Example 5.5.4. The Whitehead graph of Λ^+_Φ at the vertex of G' was disconnected. The $\mathbb{F}$-quotient of the skeleton for the transverse covering $\mathcal{Y}(\Lambda^+_\Phi)$ of T_G is a graph with two vertices and two edges with one endpoint on each vertex. The vertex stabilizers are $[\langle a, b\rangle]$ and $[\langle c, ad, abd\rangle]$ and the edge stabilizers are conjugates of $\langle b\rangle$. Indeed, there are two orbits of edges in the skeleton since the group element d acts with positive translation length on the skeleton and corresponds to the loop.

Example C.4. Consider the automorphism Φ given by

$$\Phi(a) = ab, \Phi(b) = bab, \Phi(c) = cad, \Phi(d) = dcad.$$

Let $\phi' : G' \to G'$ be a relative train track representative where G' is a rose on four petals. The Whitehead graph of Λ^+_Φ is connected at the vertex of G'. The skeleton of $\mathcal{Y}(\Lambda^+_\Phi)$ is just a point with stabilizer $\mathbb{F}$.

Example C.5. Consider the automorphism Φ given by

$$\Phi(c) = c\sigma\overline{d}\sigma d, \Phi(d) = d\sigma c\sigma\overline{d}\sigma d, \Phi(a) = a, \Phi(b) = b,$$

where $\sigma = ab\overline{a}\overline{b}$. Let $\mathcal{A} = \{[\langle a, b\rangle]\}$. Let $\phi'_0 : G' \to G'$ be a relative train track representative, where G' is a rose on four petals. In this example, the Whitehead graph of Λ^+_Φ is connected at the vertex of G' as shown in **Figure C.1**.

A partial covering of the universal cover of G', which gives the transverse covering of T_G, is shown in **Figure C.2**. Different colors correspond to different equivalence classes. The transverse covering $\mathcal{Y}(\Lambda^+_\Phi)$ of T_G is nontrivial. The stabilizer of a subtree in $\mathcal{Y}(\Lambda^+_\Phi)$ is $[\langle c, \sigma, \overline{d}\sigma d\rangle]$. The $\mathbb{F}$-quotient of the skeleton of the transverse covering has two vertices and two edges with one endpoint on each vertex. The edges are labelled by $\langle\sigma\rangle$ and the loop corresponds to d.

Example C.6. Consider the automorphism Φ given by

$$\Phi(c) = c\sigma_1 d, \Phi(d) = dc\sigma_1 d, \Phi(a) = ab, \Phi(b) = a,$$

where $\sigma_1 = abAB$ is not fixed under Φ. Let $\sigma_i := \Phi^{i-1}(\sigma)$. By iterating d under Φ, get the ray

$$dc\sigma_1 dc\sigma_1 d\sigma_2 dc\sigma_1 dc\sigma_1 dc\sigma_1 d\sigma_2 dc\sigma_1 d\sigma_3 d \ldots.$$

Some subwords that appear in this ray are $d\sigma_i d$ for all i. We claim that the stabilizer of a tree Y in the transverse covering $\mathcal{Y}(\Lambda^+_\Phi)$ of T_G will be infinitely generated such that the set of generators contains the set $\{c, \sigma_1, \sigma_2, \ldots, \}$. Indeed, when we draw a covering of $T_{G'}$ which descends to $\mathcal{Y}(\Lambda^+_\Phi)$, then the only deck transformation of $T_{G'}$ that takes two edges labeled d at the beginning and end of a σ_i is given by σ_i. Moreover, neither a nor b stabilize the subtree Y. For $H = \langle c, \sigma_1, \sigma_2, \ldots\rangle$, the subgroup $\Phi(H)$ is properly contained in H. Also in $S/\mathbb{F}$ (which is finite), there is an edge with infinitely generated stabilizer generated by the σ_i.

In this example, the lamination is in fact carried by the infinitely generated subgroup $\langle c, \sigma_1 d, \sigma_2 d, \ldots \rangle$, which is also not Φ-invariant.

Definition C.7. We say a group Γ is *finitely generated relative* to a collection of subgroups $\{H_1, \ldots, H_k\}$ if there exists a finite set $F \subset \Gamma$ such that Γ is generated by $F, H_1, \ldots, H_k$.

Definition C.8 (Finitely supported action [Gui08, Definition 1.13])**.** An action of a countable group Γ on an $\mathbb{R}$-tree T is said to be finitely supported if there is a finite subtree K whose images under Γ cover T.

The following lemma is about the structure of the skeleton.

Lemma C.9. *Let S be the skeleton of the transverse covering $\mathcal{Y}(\Lambda^+_\Phi)$ of T_G.*

(a) $S/\mathbb{F}$ is a finite graph of groups decomposition of $\mathbb{F}$.

(b) The vertex stabilizers of S are finitely generated relative to peripheral subgroups.

(c) There is only one $\mathbb{F}$-orbit of vertices with nonperipheral stabilizer in S.

Proof. (a) Since $\mathbb{F}$ is finitely generated and its action on T_G is minimal, the action on T_G is finitely supported. By [Gui08, Lemma 1.14], the action of $\mathbb{F}$ on S is minimal and finitely supported. Since S is simplicial, $S/\mathbb{F}$ is a finite graph of groups decomposition of $\mathbb{F}$.

(b) By [Gui08, Lemma 1.11], for a finite graph of groups decomposition of a finitely generated group, the vertex groups are finitely generated relative to the edge groups. Since every edge in S is incident to a peripheral subgroup, an edge stabilizer is either trivial, or nontrivial and peripheral. Thus the vertex stabilizers of S are finitely generated relative to peripheral subgroups.

(c) Since each subtree $Y_i \in \mathcal{Y}(\Lambda^+_\Phi)$ contains a generic leaf of a lamination as a line, every orbit of edges in T_G crosses Y_i. Let e, e' be two edges in two different subtrees Y_i and Y_j such that e maps to e' under some deck transformation g. Then by definition of our transverse covering, g in fact takes Y_i to Y_j. Thus up to the action of $\mathbb{F}$, there is only one subtree in $\mathcal{Y}(\Lambda_\Phi^+)$. Therefore, there is only one vertex with nonperipheral vertex stabilizer in $S/\mathbb{F}$. $\square$

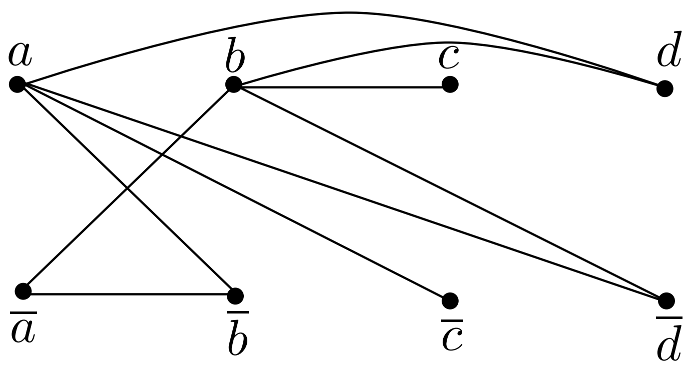

Figure C.1. Whitehead graph for Example C.5

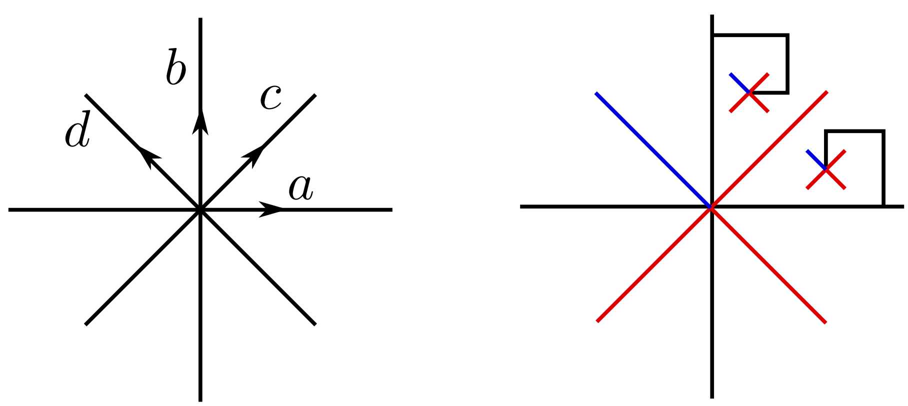

Figure C.2. Transverse covering for Example C.5

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