A contribution to the many-fermion problem

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In this work we reexamine the many-fermion problem in arbitrary dimensions. It is shown that in two dimensions or higher, the Hamiltonian of interacting fermions can be separated into individual nonintersecting sectors labeled by the wave-vector $\vec{q}$. Within each sector the Hamiltonian maps onto a generalized version of the one-dimensional Luttinger model that resembles a boson *string*. These are chain-like quadratic forms in boson operators that are readily diagonalized in the absence of "exchange" corrections. Moreover, in a simple example involving SU(2) fermions, that of the Hubbard model, we show that it can be possible also to incorporate exchange terms and express them entirely within an enlarged set of string variables.
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In Honor of Elliott Lieb's 80th Birthday.

INTRODUCTION

The many-fermion problem, a fundamental concern of modern physics that includes the charged electron gas in metals and semiconductors as well as the Fermi liquid He^3, has been a perennial preoccupation of condensed matter theorists. Its systematic study over the past half-century created a canon based on diagrammatic perturbation theory sharing a vocabulary and iconography with quantum field theory and axiomatic statistical mechanics and involving such concepts as an "interaction vertex," "irreducible diagrams," "proper mass," etc.[1] Advanced textbooks[1,2] on the topic have allowed us to glean the following, in summary:

> A Fermi surface (FS) exists; fermions live within it and their internal two-body interactions are screened by their own collective motions – whether these are identified as plasmons or phonons. Some of the collective properties are expressed in the complex frequency – and wave-vector – dependent dielectric function $\varepsilon(\vec{q}, \omega)$, the same function as describes the response of the particles to external forces. This function $\varepsilon(\vec{q}, \omega)$ depends on density of the fermions and on their dispersion near the FS. Landau's semi-empirical "Fermi Liquid" theory,[3] originally devised for He^3, explains the conjoint behavior of the fermionic *quasiparticles*. These androgynous[4] (also known as Majorana) fermions are quasi-holes just below the FS and quasi-particles just above it, fluctuating at a rate $\propto \frac{1}{\tau_k}$, their lifetimes depending on distance from the FS as[5] $\tau_k \propto 1/|e(k) - e(k_F)|^n$.

Much of this lore dates back a half century after the semi-empirical "*Random Phase Approximation*" (RPA) had first been conceived and applied systematically to the study of metals.[6] Optical sum rules and other formulas illuminated collective properties of the charged electron gas and the spectra of the individual fermions' propagators (Green functions.)[1,2] Although diagrammatical many-body perturbation theory obsoleted the RPA, the latter remained in current usage because of its simplicity.

The present paper represents an alternate way to analyze the system of interacting particles that retains some of the simplicity of the RPA. It is based on recent progress in one-dimensional physics.

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A half-century ago a two-branched, dispersionless model of spinless fermions in *one* dimension was proposed by Luttinger.[7] *Backward scattering* was prohibited in this model. Although the original solution proposed by Luttinger ultimately proved defective, his model survived and was subsequently solved exactly by Elliot Lieb and the present author[8] for *arbitrary* interactions $V(q)$.[9] The algebra involved only a single set of boson destruction and creation operators $a(q)$ and $a^\dagger(q)$, linear combinations of an infinite number of elementary excitations of the fermions.

GENERALIZED LUTTINGER MODEL

Recently, the present author introduced[10] a generalized one-dimensional Luttinger model (GLM) in which the fermions' kinetic energy exhibits dispersion at or near the Fermi level. Diagonalizing a GLM requires constructing a *string* $a_j(q)$ and $a_j^\dagger(q)$ of bosons. In each channel $a_0(q)$ is the basic operator $a(q)$ of the original Luttinger model and $a_j(q)$'s for $j \geq 1$ are derived from it by an iterative procedure similar to the Lanczös decomposition of an arbitrary matrix into a tridiagonal one. By construction, the $a_j(q)$ and $a_j^\dagger(q)$ interact *only* with nearest-neighbors $a_{j\pm1}(q)$ and $a_{j\pm1}^\dagger(q)$ (linearly at that !) Thus, the GLM maps onto an equivalent "harmonic string" of masses m_j connected by springs K_j. Once the masses and spring constants are known, *this* model, too, is reduced to quadrature by elementary means.

THIS PAPER

The present paper proposes to apply the GLM and its string-theory solution to higher dimensions. This method bypasses the many-body perturbation theory and its baggage of mathematical complications and paradoxes, in favor of constructing the ground state and thermodynamic properties of the interacting Fermi gas explicitly and in closed form. Our method should remain valid over the entire weak-coupling range of parameters, provided the FS continues to exist and that all interactions reaching deep within the Fermi sea can be deemed insignificant – presumably, the same weak-coupling range over which perturbation theory is admissible.

We examine a gas of fermions subject to arbitrary two-body interactions in arbitrary dimensions $d \geq 2$ and show that their Hamiltonian decouples into sectors, $H = \sum_{\vec{q}} H(\vec{q})$, with each $H(\vec{q})$ a one-dimensional GLM isomorphic to its own harmonic string.

It is only when spin and exchange "forces" are deemed important that the model becomes nontrivial in the language of the string operators. Nevertheless we will show how to incorporate exchange forces among SU(2) fermions in dimensions $d > 1$ in one relatively simple example, that of a modified Hubbard model in which backward scatterings and *Umklapp* are suppressed. The algebra of exchange corrections to more general two-body interactions appears somewhat too involved for an introductory essay and is therefore relegated to a future paper.[11]

THE "A" IN RPA

The basic operators (or diagrams) of RPA are the elementary excitations,

$$\tilde{a}(\vec{k}, \vec{q}) = c^\dagger(\vec{k} - \vec{q}/2)c(\vec{k} + \vec{q}/2). \tag{1}$$

Here[12] $c(k)$ is a fermion annihilation operator at k and $c^\dagger(k')$ a fermion creation operator at k' that satisfy *anticommutation* relations, $\{c(k), c^\dagger(k')\} = \delta_{k,k'}$. The tilde used over these symbols is to be a reminder that $\tilde{a}(\vec{k}, \vec{q})$ cannot, and should not be mistaken for a boson operator *even though* it is bilinear in fermion operators. The reason is that commutators such as $C_{k,q,q'} \equiv [\tilde{a}(\vec{k}, \vec{q}), \tilde{a}^\dagger(\vec{k} + \frac{\vec{q}-\vec{q}'}{2}, \vec{q}')]$ are not 1 or 0, but are themselves dynamical operators that explicitly differs from 1 or 0.

Even in *special cases* $\vec{q} = \vec{q}'$ where it is diagonal in occupation number representation, $C_{k,q,q}$ still has *three distinct* eigenvalues: 1, 0, –1.

Even if the commutator bracket of two such *similar* operators in distinct sectors *can* vanish at some specified values of the arguments – as it always should, were $\tilde{a}(\vec{k}, \vec{q})$ a genuine boson – there exist myriad special points $\vec{k}' = \vec{k} + \frac{\vec{q}+\vec{q}'}{2}$ where such commutators yield nonvanishing operators in

yet a *third* sector, e.g., $[\tilde{a}(\vec{k},\vec{q}),\tilde{a}(\vec{k}',\vec{q}')]=\tilde{a}(\vec{k}'',\vec{q}+\vec{q}')$. We conclude that the algebra of "tilded" operators in Eq. (1) is not that of bosons. This discrepancy stymies the hope – originally embodied in the RPA – of a separation of variables into sectors. The failure of trans-sector commutators to vanish vitiates the notion of separating the Hamiltonian into distinct sectors $\vec{q}$ independent of one another.

THE NEW KINEMATICS

Let us, instead of (1), consider an infinite linear combination of elementary excitations $\tilde{a}(\vec{k},\vec{q})$ that we shall call generically, the "b's":

$$b_n(\vec{q})=\frac{1}{\sqrt{Vol}}\sum_k \theta(\vec{k}\cdot\vec{q})\Phi_n(\vec{k})\tilde{a}(\vec{k},\vec{q}). \tag{2}$$

In this expression, $\theta(x)=1$ if $x>0$ and $\theta(x)=0$ otherwise; it is the canonical Heaviside function and limits the sum to only those operators that *lower* the total kinetic energy.[13] The presence in the denominator of $Vol=L^d$ (in d dimensions) is what allows proceeding to the thermodynamic limit in the evaluation of some expressions below.

We require the b_n operators to be normalized. Based on fermion anticommutator relations

$$\begin{aligned}1=[b_n(\vec{q}),b_n^\dagger(\vec{q})]=\\ \frac{1}{Vol}\sum_k\sum_{k'}\theta(\vec{k}\cdot\vec{q})\theta(\vec{k}'\cdot\vec{q})\Phi_n^*(\vec{k}')\Phi_n(\vec{k})[c^\dagger(\vec{k}'-\vec{q}/2)c(\vec{k}'+\vec{q}/2),c^\dagger(\vec{k}+\vec{q}/2)c(\vec{k}-\vec{q}/2)]\\ =\frac{1}{Vol}\sum_k\theta(\vec{k}\cdot\vec{q})\Phi_n^*(\vec{k})\Phi_n(\vec{k})\{\hat{n}(\vec{k}-\vec{q}/2)-\hat{n}(\vec{k}+\vec{q}/2)\}\\ =\frac{1}{(2\pi)^d}\int d^d\vec{k}\,\theta(\vec{k}\cdot\vec{q})\Phi_n^*(\vec{k})\Phi_n(\vec{k})\{\hat{n}(\vec{k}-\vec{q}/2)-\hat{n}(\vec{k}+\vec{q}/2)\}\end{aligned} \tag{3}$$

defining $\hat{n}(\vec{k})\equiv c^\dagger(\vec{k})c(\vec{k})$ and using $\theta^2=\theta$. Upon proceeding to the thermodynamic limit in the last line, the dependence on Vol completely disappears. The curly brackets in (3) exhibit the near-cancellation of two integrals over hemispheres centered a distance $\vec{q}$ apart. Only a crescent of width q, shown in Fig. 1 as the shaded region, contributes to the integral. At fixed $\vec{q}$ the "wave-functions" Φ_n are a set of linearly independent functions of k that need to be orthogonalized to one another and normalized over the area (in 2D) or volume (in 3D) of this crescent. To maintain the normalization, the Φ_n's also have to be proportional to $1/\sqrt{|\vec{q}|}$ in the limit of small momentum transfers $|\vec{q}|\ll k_F$.

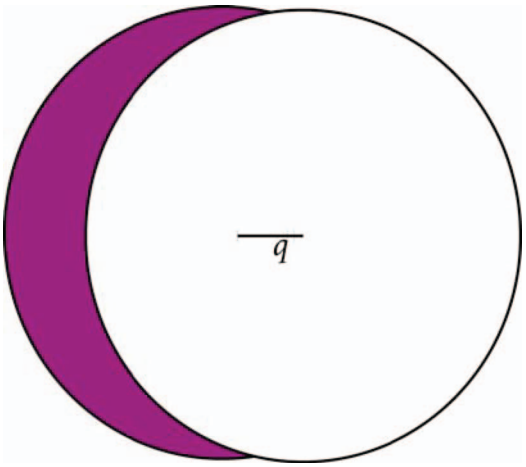

FIG. 1. Two overlapping spheres in 3D (or *circles* in 2D) of radius k_F, their centers displaced by q. The shaded area shows the volume $O(qk_F{}^2)$ (*area* $O(qk_F)$ in 2D) that is "extruded."

In evaluating norms and other quantities it is necessary to use definite values for the occupation number operators that appear under an integral sign. Assuming their fluctuations average out, we replace them by corresponding Fermi functions $\hat{n}(\vec{k}) \Rightarrow \langle c^\dagger(\vec{k})c(\vec{k})\rangle = f(\vec{k})$ that include the rounding effects of *both* temperature T *and* of the interactions. Explicit justification and related calculations are provided in Ref. 11. For present purposes of demonstration suffice it to say that, in weak-coupling and at low T, the distribution function f is usefully approximated in these integrals by the unperturbed values: 1 below the FS and zero above it.

For the b's to be good bosons their commutator brackets must also satisfy known conditions. For example, two boson annihilation operators must commute. Here,

$$[b_n(\vec{q}), b_m(\vec{q}')] = \left(\frac{l_{n,m}(\vec{q},\vec{q}')}{L}\right)^{d/2} b_p(\vec{q}+\vec{q}'). \tag{4}$$

Like the norm in (3), the quantities on the rhs of (4) have been calculated explicitly[10] using the underlying fermions' anticommutator algebra. Here the operator on the rhs does not contain terms with nonvanishing expectation value such as f. What appears instead is an operator $b_p(\vec{q}+\vec{q}')$ of similar genus as the b's on the lhs. In the thermodynamic limit $L \to \infty$ however, the multiplicative factor $(l_{n,m}/L)^{d/2}$ vanishes, hence so does the commutator (4) of any two b's constructed by our method.

Similarly $[b_n(\vec{q}), b_m^\dagger(\vec{q}')] = 0$ whenever either or both $n \neq m$ or $\vec{q} \neq \vec{q}'$. On the other hand, when $n = m$ and $q = q'$ the normalization – defined as in Eq. (3) – ensures the corresponding commutator has the correct value, $[b_n(\vec{q}), b_n^\dagger(\vec{q})] = 1$.

It follows that sectors labeled by distinct q's are, in effect, decoupled from each other.

EQUATIONS OF MOTION DRIVEN BY KINETIC ENERGY ALONE

We show in this section how the fermions' kinetic energy operator generates a *string* of coupled bosons at each $\vec{q}$, a fact that is not self-evident. Consider b operators, the amplitudes of which $\Phi(\vec{k}) = \frac{C(\vec{q})}{\sqrt{q}}$ are independent of $\vec{k}$ in each sector. *In this special case* the bosons, chosen for the seminal role they play in the two-body interactions, are denoted $a_0(\vec{q})$,

$$a_0(\vec{q}) \equiv \frac{C(\vec{q})}{\sqrt{q \times Vol}} \sum_k \theta(\vec{k}\cdot\vec{q}) c^\dagger(\vec{k}-\vec{q}/2)c(\vec{k}+\vec{q}/2). \tag{5}$$

Normalization determines $C(q)$, which is found to be approximately constant at small q,

$$\begin{aligned} 1 &= \frac{C^2(\vec{q})}{Vol} \sum_k \frac{\theta(\vec{k}\cdot\vec{q})}{|q|}(\hat{n}(\vec{k}-\vec{q}/2) - \hat{n}(\vec{k}+\vec{q}/2)) \\ &= \frac{C^2(\vec{q})}{|q|(2\pi)^3} \int d^3k \theta(\vec{k}\cdot\vec{q})(\hat{n}(\vec{k}-\vec{q}/2) - \hat{n}(\vec{k}+\vec{q}/2)). \end{aligned} \tag{6}$$

Initially the fermions' kinetic energy operator $H_1 = \sum_k e(\vec{k})c^\dagger(\vec{k})c(\vec{k})$ is expressed in fermions. We now show how to reformulate it in bosons, $H_1 = \sum_{\vec{q}} H_1(\vec{q})$, such that each $H_1(\vec{q})$ is a quadratic form in a set of bosons $a_0(\vec{q}), a_1(\vec{q}), a_2(\vec{q}), \ldots$ and their Hermitian conjugates. The coefficients are not universal but *depend explicitly* on the functional form of $e(\vec{k})$. To calculate them we shall use "commutator equations of motion" with H_1 in its original form, $H_1 = \sum_k e(\vec{k})c^\dagger(\vec{k})c(\vec{k})$, and $a_0(\vec{q})$ as given explicitly in Eq. (5):

$$[a_0(\vec{q}), H_1] = \frac{C(\vec{q})}{\sqrt{Vol}} \sum_k \frac{\theta(\vec{k}\cdot\vec{q})}{\sqrt{q}} \Big(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)\Big) c^\dagger(\vec{k}-\vec{q}/2)c(\vec{k}+\vec{q}/2). \tag{7a}$$

In the original Luttinger model $e(k) = ck$ for right-hand goers, so the parenthesis on the rhs of Eq. (7a) is $\Big(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)\Big) = cq$. In that case $[a_0(\vec{q}), H_1] = cqa_0(\vec{q})$; then the equations of motion converge at the first iteration and no further operators $a_1(\vec{q}), a_2(\vec{q}), \ldots$ are required beyond $a_0(\vec{q})$. This simple solution *never* works if $e(\vec{k})$ exhibits *any* discernible dispersion. This explains

why the *original* Luttinger model[7,8] does not generalize to $d > 1$ or to *any* model exhibiting a *range* of energies at fixed $\vec{q}$.

But in all generality, the rhs of (7a) *can* be decomposed into a part proportional to the initial $a_0(\vec{q})$ plus a part proportional to some new operator denoted $a_1(\vec{q})$, as remainder. $a_1(\vec{q})$ is normalized as in Eq. (3) and is "orthogonal" to $a_0(\vec{q})$ by construction. (By this is meant that $a_1(\vec{q})$ and its conjugate $a_1^\dagger(\vec{q})$ both commute with $a_0(\vec{q})$.) Then (7a) takes the form:

$$[a_0(\vec{q}), H_1] = A_0^0(\vec{q})a_0(\vec{q}) + A_0^1(\vec{q})a_1(\vec{q}). \tag{7b}$$

The A's are the coefficients, real numbers that must be calculated explicitly. For example, we obtain $A_0^0(\vec{q})$ by taking the commutator bracket of Eq. (7b) with respect to $a_0^\dagger(\vec{q})$, using the boson property, $[a_n(\vec{q}), a_0^\dagger(\vec{q})] = \delta_{n,0}$:

$$\begin{aligned} A_0^0(\vec{q}) &= [[a_0(\vec{q}), H_1], a_0^\dagger(\vec{q})] \\ &= \frac{C^2(\vec{q})}{|q|Vol} \sum_k \theta(\vec{k}\cdot\vec{q}) \left(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)\right) \left(\hat{n}(\vec{k}-\vec{q}/2) - \hat{n}(\vec{k}+\vec{q}/2)\right). \end{aligned} \tag{8}$$

Once we know $A_0^0(\vec{q})$ it is a simple matter to obtain $A_0^1(\vec{q})$, as we shall see below. But it helps to first deal with the notation, as we do next.

AVERAGES

The notation is simplified if, as in Eq. (8), we define *averages* of functions of k over the phase space, as in the following:

$$\langle G(\vec{k})\rangle \equiv \frac{C^2(\vec{q})}{|q|Vol} \sum_k \theta(\vec{k}\cdot\vec{q}) \left(G(\vec{k})\right) \left(\hat{n}(\vec{k}-\vec{q}/2) - \hat{n}(\vec{k}+\vec{q}/2)\right). \tag{9}$$

This definition implies the existence of a normalized distribution function $P_{\vec{q}}(\vec{k}) = \frac{C^2(\vec{q})}{|q|Vol}\theta(\vec{k}\cdot\vec{q}) \left(\hat{n}(\vec{k}-\vec{q}/2) - \hat{n}(\vec{k}+\vec{q}/2)\right)$ over which to perform the averages in each sector. This ensures that Eq. (6) is an identity $1 = \langle 1\rangle$. This same notation also allows Eq. (8) to be expressed succinctly, $A_0^0(\vec{q}) = \langle e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)\rangle$.

To obtain $A_0^1(\vec{q})$, rewrite (7b) as follows: $[a_0(\vec{q}), H_1] - A_0^0(\vec{q})a_0(\vec{q}) = A_0^1(\vec{q})a_1(\vec{q})$, i.e.,

$$\begin{aligned} \frac{C(\vec{q})}{\sqrt{|q|Vol}} \sum_k \theta(\vec{k}\cdot\vec{q}) \left(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2) - A_0^0(\vec{q})\right) c^\dagger(\vec{k}-\vec{q}/2)c(\vec{k}+\vec{q}/2) \\ = A_0^1(\vec{q})a_1(\vec{q}). \end{aligned} \tag{10a}$$

Without knowing a_1 explicitly we can still use (10a) to evaluate the square of the rhs: $[A_0^1(\vec{q})a_1(\vec{q}), A_0^1(\vec{q})^* a_1^\dagger(\vec{q})] = \left|A_0^1(\vec{q})\right|^2$. The quantity $A_0^1(\vec{q})$ is then identified as the *variance* in the distribution of the elementary excitations:

$$A_0^1(\vec{q}) = \sqrt{\langle\left(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)\right)^2\rangle - \langle\left(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)\right)\rangle^2}. \tag{10b}$$

Dividing (10a) by (10b) allows explicit evaluation of the new operator $a_1(q)$,

$$\begin{aligned} a_1(\vec{q}) &= \frac{A_0^1(\vec{q})a_1(\vec{q})}{A_0^1(\vec{q})} \\ &= \frac{\frac{C(\vec{q})}{\sqrt{|q|Vol}} \sum_k \theta(\vec{k}\cdot\vec{q}) \left(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2) - A_0^0(\vec{q})\right) c^\dagger(\vec{k}-\vec{q}/2)c(\vec{k}+\vec{q}/2)}{\sqrt{\langle\left(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)\right)^2\rangle - \langle\left(e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)\right)\rangle^2}}. \end{aligned} \tag{11}$$

COUNTING RADIAL NODES

Note that in the numerator of (11) some elementary excitations having energy $e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)$ lie lower than the average (which, according to (8), is $A_0^0(\vec{q})$), and some lie higher. From this one concludes that a_1 possesses one extra "radial" node relative to a_0 which had none. Next, a_2 has two radial nodes upon being constructed to be orthogonal to a_0 and a_1. Higher a_j's have j radial nodes.

SECOND EQUATION OF MOTION AND BEYOND

Consider successive commutator brackets, $[a_1(\vec{q}), H_1]$ and beyond. Because H_1 is Hermitian and real, this next iteration produces three objects. First, $A_0^1 a_0 = A_1^0 a_0$. Second, a multiple of $a_1(\vec{q})$ itself. Last, a residue that we can denote $A_1^2 a_2(\vec{q})$. A_1^2 is a new constant and $a_2(\vec{q})$ a boson orthogonal to a_0 and a_1 that remain to be calculated and normalized. Symbolically:

$$[a_1(\vec{q}), H_1] = A_0^1(\vec{q})a_0(\vec{q}) + A_1^1(\vec{q})a_1(\vec{q}) + A_1^2(\vec{q})a_2(\vec{q}). \tag{12}$$

A_0^1 is known, $a_1(\vec{q})$ is known, and $A_1^1 = [[a_1(\vec{q}), H_1], a_1^\dagger(\vec{q})]$. The square of $A_1^2(\vec{q})$ is calculated directly from the nested commutators:

$$[[a_1(\vec{q}), H_1] - A_0^1(\vec{q})a_0(\vec{q}) - A_1^1(\vec{q})a_1(\vec{q}), [H_1, a_1^\dagger(\vec{q})] - A_0^1(\vec{q})a_0^\dagger(\vec{q}) - A_1^1(\vec{q})a_1^\dagger(\vec{q})] = (A_1^2(\vec{q}))^2.$$

Knowing $A_1^2(\vec{q})$ one obtains the new, normalized, operator a_2 explicitly by rewriting (12) in the form, $\left([a_1(\vec{q}), H_1] - A_0^1(\vec{q})a_0(\vec{q}) - A_1^1(\vec{q})a_1(\vec{q})\right)/A_1^2(\vec{q}) = a_2(\vec{q})$. As remarked previously, a_2 has 2 radial nodes.

One calculates successive A_n^m coefficients at each turn by similar iterations. It is only required to specify $e(k)$ and $e(k_F)$ to build the coefficients on those that precede it.

Thus the generalization of (12) to all n is,

$$[a_n(\vec{q}), H_1] = A_{n-1}^n(\vec{q})a_{n-1}(\vec{q}) + A_n^n(\vec{q})a_n(\vec{q}) + A_n^{n+1}a_{n+1}(\vec{q}), \tag{13}$$

where $A_0^{-1} = A_{-1}^0 \equiv 0$ serves as the initial condition. Also,

$$[H_1, a_n^\dagger(\vec{q})] = A_{n-1}^n(\vec{q})a_{n-1}^\dagger(\vec{q}) + A_n^n(\vec{q})a_n^\dagger(\vec{q}) + A_n^{n+1}(\vec{q})a_{n+1}^\dagger(\vec{q}). \tag{14}$$

This shows that the coefficients are all real and symmetric: $A_n^m = A_m^n$. Next we interpret Eqs. (13) and (14) in terms of a string of harmonic oscillators.

HARMONIC STRING

Consider the tridiagonal quadratic form in *boson* operators that we are denoting a harmonic "string" $\hat{A}(\vec{q})$:

$$\hat{A}(\vec{q}) = \sum_{n=0}^{\infty} A_n^n(\vec{q})a_n^\dagger(\vec{q})a_n(\vec{q}) + \sum_{n=0}^{\infty}\left(A_n^{n+1}(\vec{q})a_n^\dagger(\vec{q})a_{n+1}(\vec{q}) + H.c.\right). \tag{15a}$$

The bosons $a_0(\vec{q}), a_1(\vec{q}), \ldots, a_n(\vec{q})$ (and their conjugates) have the same equations of motion with $\hat{A}(\vec{q})$ as their fermionic representations did with $H_1 = \sum_k e(\vec{k})c^\dagger(\vec{k})c(\vec{k})$.

Now, we know[1,2] the exact excitation spectrum of the *original* fermion H_1 at fixed $\vec{q}$; it consists of the set of energies $\Delta e = e(\vec{k}+\vec{q}/2) - e(\vec{k}-\vec{q}/2)$ in the half-space $\vec{k}\cdot\vec{q} > 0$. At a given q this forms a *continuum* within a range: $0 \le \omega_{\min}(q) \le \Delta e \le \omega_{\max}(q)$. The boundaries $\omega_{\min/\max}$ are readily calculated and typically consist of parallel parabolas.

On the other hand, upon diagonalizing the quadratic form in (15a) we obtain at each $\vec{q}$,

$$\hat{A}_{diag}(\vec{q}) \Rightarrow \sum_{j=0}^{\infty} \omega_j(\vec{q})\hat{a}_j^\dagger(\vec{q})\hat{a}_j(\vec{q}). \tag{15b}$$

The boson operators in this diagonal representation, the $\hat{a}$'s, are appropriately normalized linear combinations of the original a_j's. The eigenvalue spectrum of (15) is given by the $\omega_j(\vec{q})$'s. When

plotted as functions of $\vec{q}$, these eigenvalues $\omega_j(\vec{q})$ define "trajectories" within the aforementioned bounds, $0 \leq \omega_{\min}(q) \leq \omega_j(\vec{q}) \leq \omega_{\max}(q)$.

MORE BOSONS

Because a_0 has an amplitude Φ that is a constant independent of $\vec{k}$, each of the a_n's that follow it in the string, ($a_1(\vec{q})$ up to to arbitrary $a_n(\vec{q})$), by virtue of being connected to the preceding by the equations of motion, have their n nodes *in the radial direction* only – defined as the direction orthogonal to surfaces of constant $\Delta e = e(\vec{k} + \vec{q}/2) - e(\vec{k} - \vec{q}/2)$. Given their lack of nodes *within* or *upon* the surfaces of constant Δe, $a_0(\vec{q})$ and its descendants $a_1(\vec{q}), \ldots$, are all denoted "*s*-wave bosons."

Other families of bosons *can be* constructed from the equations of motion with the kinetic energy operator, possessing any integer number of "angular" nodes *along* the surfaces of constant Δe in addition to the radial nodes. These families would be denoted *p*-, *d*-, *f*-, etc., in an obvious notation. None of them appears in the string Eqs. (7) and *ff.* emanating from the *s*-wave operator $a_0(\vec{q})$. However, such *p*-, *d*-, ... operators allow the total weight of bosonic frequencies to have the same measure as the spectrum of elementary excitations of the original fermions.

QUASIPARTICLES

For completeness, in order to accurately represent *Landau quasiparticles*, one must use not just the *s*-waves labeled $l = 0$ but also those $a_j^{(l)}(\vec{q})$ belonging to higher l quantum numbers, generated from kinetic energy alone, starting from a initial $a_0^{(l)}(\vec{q})$ (without radial nodes) that does not appear in the interactions H_2.

DIRECT INTERACTIONS

The *direct interaction* Hamiltonian of nonrelativistic SU(2) fermions H_2 is expressed naturally in terms of density fluctuations. In familiar notation,

$$H_2 = \frac{1}{2! Vol} \sum_{\vec{q}} V(\vec{q}) \left\{ \sum_{\sigma'} \rho_{\sigma'}(\vec{q}) \sum_{\sigma} \rho_{\sigma}(-\vec{q}) \right\} \tag{16}$$

in which the subscripts σ label the spin component *up* or *down*. Introducing this subscript into the definition of the $a_0(\vec{q})$ operators in Eq. (5), we rewrite (16) as

$$H_2 = \frac{1}{2!} \sum_{\vec{q}} V(\vec{q}) \frac{|q|}{C^2(q)} \left\{ \sum_{\sigma'} \sum_{\sigma} \left(a^{\dagger}_{0,\sigma}(\vec{q}) + a_{0,-\sigma}(-\vec{q}) \right) a_{0,-\sigma'}(\vec{q}) + H.c. \right\}. \tag{17a}$$

> The function $V(\vec{q})$ is the Fourier transform of the two-body interaction (e.g., $V(\vec{q}) = \frac{4\pi e^2}{q^2}$ if the direct interaction is $\frac{e^2}{r_{i,j}}$ for physical electrons, or it is $V(\vec{q}) = U$, a constant, for the Hubbard model of fermions on a space lattice $\vec{r}_j$). Note that the linear combination, $a_{0,\uparrow}(\vec{q}) - a_{0,\downarrow}(\vec{q})$, having an extra non-radial node, is not present in (17a). The linear combination that *does* occur is $a_{0,\uparrow}(\vec{q}) + a_{0,\downarrow}(\vec{q})$, rewritten as a spinless $a_0(\vec{q})\sqrt{2}$. Thus (17a) simplifies to,

$$H_2 = \sum_{\vec{q}} V(\vec{q}) \frac{|q|}{C^2(q)} \left\{ \left(a^{\dagger}_0(\vec{q}) + a_0(-\vec{q}) \right) a_0(\vec{q}) + H.c. \right\}. \tag{17b}$$

Like the kinetic energy, H_2 clumps into nonoverlapping sectors, $H_2 = \sum_{q_z > 0} \hat{B}(q)$, where

$$\hat{B}(q) = \frac{V(q)|q|}{C^2(q)} \left(a^{\dagger}_0(\vec{q}) a_0(\vec{q}) + a^{\dagger}_0(-\vec{q}) a_0(-\vec{q}) + 2\left(a_0(\vec{q}) a_0(-\vec{q})\right) + H.C.) \right) \tag{18}$$

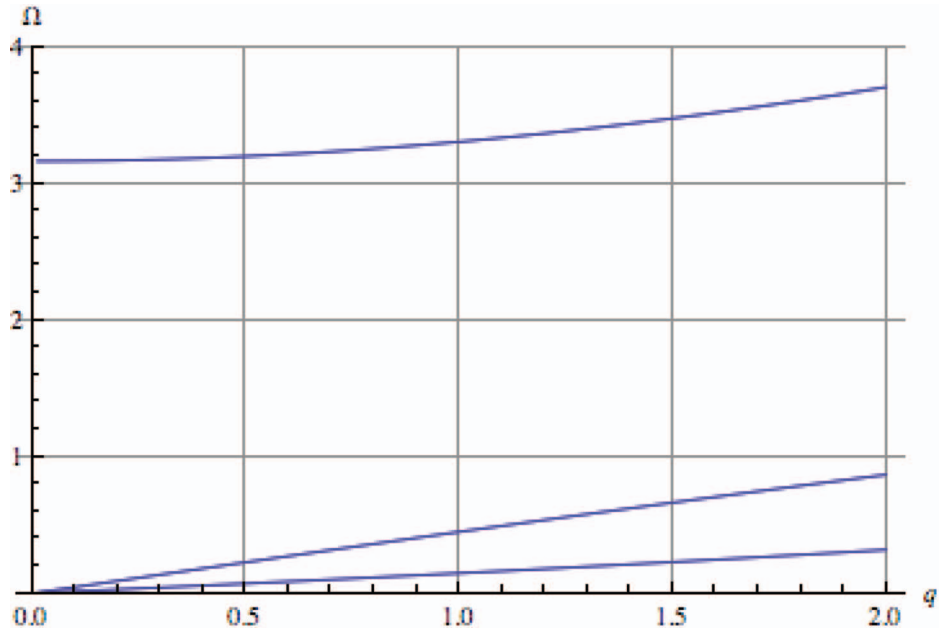

FIG. 2. First 3 eigenvalues of string equations of motion for Coulomb interaction (using $V(q) = \mathrm{K}/q^2$) with $\mathrm{K} = 10$ and representative values of the other quantities) plotted (schematically) as function of q. Top curve is the *plasmon* rising above the continuum, bottom two are discrete modes that are part of the family of "*s*-wave" strings that are affected by the two-body interactions.

(inserting $q_z > 0$ in the sum to resolve duplication). Unlike H_1, the interactions H_2 connect the sectors at $\vec{q}$ and $-\vec{q}$; also the *boson-number* is not conserved. Note that *only* the *s*-wave operators a_0 and $a_0^\dagger$ and none of the subsequent a_j's, $j > 0$, appear in (18).

Thus, when we evaluate the complete equations of motion using *both* H_1 *and* H_2, the string structure remains essentially that of H_1 except for the initial modification at $j = 0$ that doubles the lengths of the strings. That is, each string of $a_j(-\vec{q})$'s is connected to the string of $a_j(\vec{q})$'s, through the interaction operators $(a_0(\vec{q})a_0(-\vec{q})) + H.C.)$ at $j = 0$.

Within each expanded sector the total Hamiltonian takes the form:

$$H(\vec{q}) = \hat{A}(\vec{q}) + \hat{A}(-\vec{q}) + \hat{B}(\vec{q}). \tag{19}$$

The rhs is a quadratic form that can, and needs to be, solved at each value of $\vec{q}$. Fig. 2 illustrates the first three *s-wave* trajectories (the eigenvalues $\Omega_j(q)$ arising out of Eq. (19)) in a calculation based on direct Coulomb interactions. The top curve shows the "plasmon," while the two lower curves are the next two *s*-wave trajectories, slightly modified by the interactions. The additional spectrum of modified and un-modified trajectories forms a continuum over the range $0 \le \omega_{\min}(q) \le \Omega_j(\vec{q}) \le \omega_{\max}(q)$ that was omitted from the figure for the sake of clarity.

EXAMPLE OF EXCHANGE "FORCES"

We next illustrate the dynamics of exchange operators (basically, that of *spin density* operators) with the aid of a Hubbard-like model:

$$H_2 = \frac{U}{Vol} \sum_{\vec{q}, q_z > 0} \left\{ \rho_\uparrow(\vec{q})\rho_\downarrow(-\vec{q}) + \rho_\downarrow(\vec{q})\rho_\uparrow(-\vec{q}) \right\}. \tag{20}$$

Its Ising-like version is a result of the trivial identity,

$$\rho_\uparrow(\vec{q})\rho_\downarrow(-\vec{q}) + \rho_\downarrow(\vec{q})\rho_\uparrow(-\vec{q}) = \frac{1}{2}(\rho_\uparrow(\vec{q}) + \rho_\downarrow(\vec{q}))(\rho_\uparrow(-\vec{q}) + \rho_\downarrow(-\vec{q})) - \frac{1}{2}(\rho_\uparrow(\vec{q}) - \rho_\downarrow(\vec{q}))(\rho_\uparrow(-\vec{q}) - \rho_\downarrow(-\vec{q})).$$

The parenthesis with the $+$'s yields direct interactions. It is analogous to H_2 in Eqs. (17) and (18):

$$H_{2,dir} = U \sum_{\vec{q},q_z>0} \frac{|q|}{C^2(q)} \left\{ a_0^\dagger(\vec{q})a_0(\vec{q}) + a_0^\dagger(-\vec{q})a_0(-\vec{q}) + (a_0(\vec{q})a_0(-\vec{q}) + H.c.) \right\}. \tag{21}$$

The parenthesis with –'s yields

$$H_{2,z} = -U \sum_{\vec{q},q_z>0} \frac{|q|}{C^2(q)} \left\{ z_0^\dagger(\vec{q})z_0(\vec{q}) + z_0^\dagger(-\vec{q})z_0(-\vec{q}) + (z_0(\vec{q})z_0(-\vec{q}) + H.c.) \right\}, \tag{22}$$

in which the operator z_0 is defined, by analogy with a_0 in (5),

$$z_0(\vec{q}) \equiv \frac{C_z(\vec{q})}{\sqrt{2 \times q \times Vol}} \sum_k \theta(\vec{k}\cdot\vec{q}) \left(c_\uparrow^\dagger(\vec{k} - \vec{q}/2)c_\uparrow(\vec{k} + \vec{q}/2) - c_\downarrow^\dagger(\vec{k} - \vec{q}/2)c_\downarrow(\vec{k} + \vec{q}/2) \right) \tag{23}$$

$z_0(\vec{q})$ is a normalized s-wave bosonic annihilation operator. By rotational symmetry we are required to define two additional operators in its same sector,

$$x_0(\vec{q}) \equiv \frac{C_x(\vec{q})}{\sqrt{2 \times q \times Vol}} \sum_k \theta(\vec{k}\cdot\vec{q}) \left(c_\uparrow^\dagger(\vec{k} - \vec{q}/2)c_\downarrow(\vec{k} + \vec{q}/2) + c_\downarrow^\dagger(\vec{k} - \vec{q}/2)c_\uparrow(\vec{k} + \vec{q}/2) \right) \tag{24}$$

and

$$y_0(\vec{q}) \equiv \frac{C_y(\vec{q})}{i\sqrt{2 \times q \times Vol}} \sum_k \theta(\vec{k}\cdot\vec{q}) \left(c_\uparrow^\dagger(\vec{k} - \vec{q}/2)c_\downarrow(\vec{k} + \vec{q}/2) - c_\downarrow^\dagger(\vec{k} - \vec{q}/2)c_\uparrow(\vec{k} + \vec{q}/2) \right) \tag{25}$$

Their algebra is of interest. For example,

$$[z_0(\vec{q}), x_0^\dagger(\vec{q})] = i y_0(\vec{0}) \frac{C(\vec{q})}{\sqrt{2 \times q \times Vol}}, \text{ and cyclic permutations,} \tag{26}$$

assuming $C(\vec{q}) = C_x(\vec{q}) = C_y(\vec{q}) = C_z(\vec{q})$, also by rotational invariance. This commutator is reminiscent of Eq. (4). Here, in the thermodynamic limit, the rhs of Eq. (26) (or of any cyclic permutation) again vanishes *unless* there is a symmetry-breaking spontaneous *ground state magnetization*, with one or all of the following operators assuming macroscopic values, $x_0(0)$, $y_0(0)$, or $z_0(0) \propto \sqrt{Vol}$. This cannot occur in weak-coupling.

Commutators other than the normalizations are again all zero. This is not surprising – after all, we are dealing with the classical limit of sums of spin angular momenta of a very large number of excitations.

When assembling all the relevant terms in H_2 of Eq. (20) we find that the interaction disgorges not just $H_{2,dir}$ and $H_{2,z}$ as in the above construction, but additionally $H_{2,x}$ and $H_{2,y}$ resulting from the pairing of fermions of spins "up" with spins "down." They are, obviously,

$$H_{2,x} = -U \sum_{\vec{q},q_z>0} \frac{|q|}{C_x^2(q)} \left\{ x_0^\dagger(\vec{q})x_0(\vec{q}) + x_0^\dagger(-\vec{q})x_0(-\vec{q}) + (x_0(\vec{q})x_0(-\vec{q}) + H.c.) \right\} \text{ with the aid}$$

of Eq. (24), and

$$H_{2,y} = -U \sum_{\vec{q},q_z>0} \frac{|q|}{C_y^2(q)} \left\{ y_0^\dagger(\vec{q})y_0(\vec{q}) + y_0^\dagger(-\vec{q})y_0(-\vec{q}) + (y_0(\vec{q})y_0(-\vec{q}) + H.c.) \right\}.$$

Introducing a more compact notation, we combine these into a single "sigma model,"

$$H_{2,\sigma} = -U \sum_{\vec{q},q_z>0} \frac{|q|}{C^2(q)} \left\{ \vec{\sigma}_0^\dagger(\vec{q}) \cdot \vec{\sigma}_0(\vec{q}) + \vec{\sigma}_0^\dagger(-\vec{q}) \cdot \vec{\sigma}_0(-\vec{q}) + (\vec{\sigma}_0(\vec{q}) \cdot \vec{\sigma}_0(-\vec{q}) + H.c.) \right\} \tag{27}$$

defining a vector spin operator $\vec{\sigma} = (x, y, z)$ the components of which are individually normalized. This interaction Hamiltonian (27) is rotationally invariant.

The kinetic energy strings in the 4 boson fields all have the identical form (15a):

$$\hat{H}_1(\xi|\vec{q}) = \sum_{n=0}^{\infty} A_n^n(\vec{q})\xi_n^\dagger(\vec{q})\xi_n(\vec{q}) + \sum_{n=0}^{\infty}\left(A_n^{n+1}(\vec{q})\xi_n^\dagger(\vec{q})\xi_{n+1}(\vec{q}) + H.c.\right), \tag{28}$$

where ξ stands for any boson a, x, y, or z. The various density or spin components are to be diagonalized separately as they are all unlinked, subject to the basic assumption that in weak-coupling the ground state is unmagnetized.

When (28) is combined with the interaction Hamiltonian given in Eqs. (21) and (27), we find the direct term (in the a's) is proportional to $+U$ and the three magnetic terms (in the $x,y,$ or z operators) to $-U$. Thus it is the latter that encounter the stability boundary. For if U exceeds a critical value U_c (the actual value depending on the kinetic energy parameters), the boson spin Hamiltonians become unstable. In that case one has to re-align about a new, possibly magnetized and symmetry-broken ground state, to recover stability.

CONCLUSION

In this work we uncovered a slew of bosonic excitations of the low temperature fermion gas in weak-coupling. When we consider only the density fluctuations in the interaction Hamiltonian, only density-type bosons are involved. When *exchange* terms were included in a simplified Hubbard model, spin-density bosons came to the fore. In all instances, the strings that describe the Hamiltonian have similar structures that can be diagonalized independently and exactly. The key came from a *harmonic string* representation of the kinetic energy operator.

We identified the presence of Landau quasi-particles, insofar as they involve s-wave components (those affected by the interactions) and the infinite set of unperturbed $l > 1$ components. Fermions can be exponentially reconstructed from bosonic excitations, using an exponential form that has become standard by now in Luttinger-model calculations.[14]

In this paper we have not delved deeply into the actual solutions, as several distinct methods are known for diagonalizing one-dimensional strings: continued fractions, Green functions, etc., and this is a topic deserving of separate, extensive treatment. We have also left to future a publication those knotty issues as come up when studying exchange corrections to the Coulomb potential and other similar, non-separable, two-body potentials $V(q)$.[11]

The p-, d-, ... modes, all unaffected by the interactions, supply an infinite number of missing trajectories (not shown) that makes up the continuum of elementary excitations at a given q.

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[4] I.e., bereft of electrical charge but still capable of short-range (unscreened) interactions, as discussed *inter alia* by G. Rickayzen, *Theory of Superconductivity* (Interscience Publ., New York, 1965), p. 152.
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[12] We omit spin variables, for clarity in this and subsequent sections, until needed.
[13] Assuming the one-body kinetic energies are $e(\vec{k}) = e(|k|) \propto |\vec{k}|$ or $\vec{k}^2$. If the surfaces of constant energy had a different topological structure, e.g., if they were reentrant or if they had a tight-binding form such as $e = W(\cos k_x + \cos k_y + \cos k_z)$, they should replace the argument of the Heaviside function by the more general:$\theta(e(\vec{k} + \vec{q}/2) - e(\vec{k} - \vec{q}/2))$.
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