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Moyal representation of the string field star product in the presence of a *B* background

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In this paper we show that in the presence of an antisymmetric tensor B background, Witten's star algebra for open string fields persists to possess the structure of a direct product of commuting Moyal pairs. The interplay between the noncommutativity due to three-string overlap and that due to the B background is our main concern. In each pair of noncommutative directions parallel to the B background, the Moyal pairs mix string modes in the two directions and are labeled, in addition to a continuous parameter, by *two* discrete values as well. However, the Moyal parameters are B dependent only for discrete pairs. We have also demonstrated the large-B contraction of the star algebra, with one of the discrete Moyal pairs dropping out and with the other giving rise to the center-of-mass noncommutative function algebra.

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I. INTRODUCTION

Coordinate noncommutativity, which generates noncommutativity of functions, lies at the heart of noncommutative geometry. Physically, this implies non-locality of interactions.

In string theory there are two kinds of noncommutativity. The first one is that proposed by Witten [1] in his cubic open string field theory (OSFT). Witten's star product of string fields describes a noncommutative geometry for the state space of string theory. The noncommutativity is due to the gluing procedure that defines the star product, in which the right half of the first string is glued with the left half of the second string, with the resulting (third) string composed of the left half of the first and the right half of the second. This three-string vertex involves δ-functional overlap interactions. It is the *non-local nature* of the latter that gives rise to the noncommutativity of Witten's star product. (Though δ-functional overlap interactions may look quite formal, the three-string vertex has a precise oscillator formulation, developed in Refs. [2,3].) The second kind of noncommutativity is that of the end points of an open string in the presence of an anti-symmetric B background [4–7], which can be viewed as a descendant of the noncommutativity in matrix theory [8,9]. In open string theory, an antisymmetric tensor B background can be traded off with a gauge field acting on the string ends. In an appropriate double scaling limit that decouples the effects of gravity [7], an open string behaves like a dipole in the lowest Landau level of a magnetic field [10]. It is the non-locality due to the finite size of a dipole in the lowest Landau level that makes the resulting theory in the decoupling limit a noncommutative gauge theory. Mathematically this gauge theory can be obtained [7,11,12] by deforming the ordinary product of functions into the Moyal star product, a procedure familiar in the deformation quantization scheme for quantum mechanics.

One may wonder whether there is a close relation or an interplay between these two kinds of noncommutativity in string theory. Recently important progress has been made on a better understanding of this issue. First, Bars [13] has succeeded in identifying Witten's star product of string fields with Moyal's star product between canonical pairs of the appropriate linear combination of string modes. Second, Rastelli, Sen and Zwiebach [14] (RSZ) have successfully solved the spectrum, both eigenvalues and eigenvectors, of the Neumann coefficient matrices that appear in the oscillator formulation of the three-string vertex in the zero-momentum matter sector. This breakthrough makes it possible to diagonalize the three-string vertex in oscillator formalism. Indeed, soon after Ref. [14], a new basis was found by Douglas *et al.* [15] in the single string Hilbert space that diagonalizes Witten's star product into a continuous tensor product of mutually commuting two-dimensional Moyal star products. The noncommutativity parameter $\theta(\kappa)$ for each of the Moyal products is given as a function of the eigenvalue $\lambda(\kappa)$, where $\kappa\in[0,\infty)$. The Moyal coordinates $x(\kappa)$ and $y(\kappa)$ for a single string are constructed with the twist-even and twist-odd RSZ eigenvectors, respectively. Shortly in Ref. [16], the diagonal representation of the open string star product was generalized to include the zero modes (center-of-mass coordinates) as well.

It would be of great interest to see how the noncommutativity due to a B background in target spacetime would intertwine with that of the string field star product to form a bigger structure, Moyal or not. The string field star product has been studied in, e.g. Refs. [17–20]. Also the spectrum (eigenvalues and eigenvectors) of the Neumann coefficient matrices in a B background in two spatial directions has been solved recently in Refs. [21,22]. Based on these results, in this paper we study the full structure of the open string field algebra, with both zero modes and a background B-field included. It is shown that in the presence of an anti-symmetric tensor B background, Witten's star algebra for open string fields persists to possess the structure of a direct product of commuting Moyal pairs. Assume that the B background is block diagonal with 2×2 blocks. Then in each pair of noncommutative directions along which the B field has nonzero components, the Moyal pairs mix string modes in the two directions and are labeled, in addition to a continuous parameter, by *two* discrete values as well. However, the presence of

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a B field affects only the Moyal parameters for the pairs with a discrete label, while the Moyal parameters for the pairs with a continuous label, even in the noncommutative directions, remain unaffected at all. Moreover, we have also demonstrated the large-B contraction of the star algebra, noticing a singular behavior of one of the discrete Moyal pairs: It drops out the contraction, while the other pair gives rise to the center-of-mass noncommutative function algebra. Previously the large-B limit was discussed in the literature [23,24] in terms of the vertex operators or string oscillators; here we examine the contraction directly in terms of the Moyal modes in the three-string vertex. The singular behavior we found for the large-B limit may require that some caution be taken when an argument with the help of the large-B contraction is made. (For recent papers on other aspects of the continuous Moyal product in OSFT, see Refs. [25–29].)

The paper is organized as follows. In Sec. II and Sec. III, using the eigenvectors of the Neumann matrices, we introduce two sets of new oscillators and rewrite the three-string vertex, first with zero modes and then in a B background, explicitly in terms of them. In Sec. IV we identify the Moyal structure for the open string star product, and present the explicit expression for noncommutativity parameters as well as for the Moyal coordinates. In Sec. V we demonstrate the large-B contraction of the star algebra. Finally a summary of our results and discussions are given in Sec. VI.

II. THREE-STRING VERTEX WITH ZERO MODES

Witten's star product of string fields is defined in terms of a (non-local) half-string overlap. In the oscillator representation [2], this product can be formulated using a three-string vertex involving a quadratic form of string modes, whose coefficients called Neumann matrices. In this section, we will express the three-string vertex, in the absence of a B background but with zero modes included, in terms of a set of new oscillators labeled by the eigenvalues of the Neumann matrices. The three-string vertex in this form has been obtained in Ref. [16]; our brief review here presents a slightly different approach, and serves to set up our notations and introduction to proceeding to the more complicated case with a nonvanishing B background.

The three-string vertex, with zero modes included, that defines Witten's star product is given

$$|V_3'\rangle=\exp\left[-\frac{1}{2}\sum_{r,s}\sum_{m,n\geq 0}a_m^{M(r)\dagger}V'^{rs}_{mn}G_{MN}a_n^{N(s)\dagger}\right]|0\rangle. \tag{2.1}$$

Here the superscripts $r,s=1,2,3$ label different strings, the subscripts $m,n=0,1,2,\ldots$ are the string modes, $G^{MN}=\eta^{MN}=\mathrm{diag}(-1,1,\ldots,1)$ is the metric of target space, and $a_n^{M(r)\dagger}$ is the creation operator of the corresponding string mode. The explicit form of the Neumann matrices V'^{rs} has been given in Refs. [2,30,31], which we do not use in this paper.

The eigenproblem for the Neumann matrices $M'^{rs}=CV'^{rs}$ has been solved in Refs. [20,16], where $C_{mn}=(-1)^n\delta_{mn}$ is the twist matrix. Let us first review the main results of Ref. [20] for our purposes. What we need is the knowledge—eigenvalues and degeneracy—about the spectrum. It is known that the zero momentum sector of the Neumann matrix M^{rs} has a continuous spectrum [14]. After including the zero modes, however, the spectrum will have a continuous branch labeled by a continuous parameter $\kappa\in(-\infty,+\infty)$ and a discrete branch consisting of a single point. The eigenequation reads

$$\sum_{n=0}^{\infty}M'^{rs}_{mn}u_n(\kappa)=\rho^{rs}(\kappa)u_m(\kappa)\qquad\text{for the continuous spectrum} \tag{2.2}$$

$$\sum_{n=0}^{\infty}M'^{rs}_{mn}\phi_n=\rho_0^{rs}\phi_m\qquad\text{for the discrete spectrum.} \tag{2.3}$$

The continuous spectrum consists of the interval[1] $[-\frac{1}{3},0)$. The expression for the continuous eigenvalues of $M'\equiv M'^{11}$ is given by

$$\rho(\kappa)\equiv\rho^{11}(\kappa)=\frac{-1}{1+2\cosh\frac{\pi\kappa}{2}}. \tag{2.4}$$

The eigenvalues of M'^{12} and M'^{21} can be obtained using properties (A14) in Appendix A; they are

$$\rho^{12}(\kappa)=\frac{1}{2}\mathrm{sgn}(\kappa)\sqrt{[1+3\rho(\kappa)][1-\rho(\kappa)]}+\frac{1}{2}[1-\rho(\kappa)],$$

$$\rho^{21}(\kappa)=-\frac{1}{2}\mathrm{sgn}(\kappa)\sqrt{[1+3\rho(\kappa)][1-\rho(\kappa)]}+\frac{1}{2}[1-\rho(\kappa)], \tag{2.5}$$

and other eigenvalues are given by $\rho^{23}=\rho^{31}=\rho^{12}$, $\rho^{32}=\rho^{13}=\rho^{21}$ and $\rho^{22}=\rho^{33}=\rho^{11}$.

The eigenvectors $u_n(\kappa)$ have the following property:

$$u_{2n}(-\kappa)=u_{2n}(\kappa),\qquad u_{2n+1}(-\kappa)=-u_{2n+1}(\kappa). \tag{2.6}$$

From Eq. (2.2) we know that, if $u(\kappa)$ is an eigenvector with eigenvalue $\rho^{rs}(\kappa)$ then $u(-\kappa)$ is an eigenvector with eigenvalue $\rho^{rs}(-\kappa)$. It is easy to see that these two eigenvectors

[1] Although in the following we are going to quote many results from Ref. [20], our terminology for the continuous and discrete spectra is different from theirs. According to the definition in Ref. [20], some points between $-\frac{1}{3}$ and 0, at which the general expression given in that paper for the eigenvectors in the continuous spectrum takes the form 0/0, are said to belong to the discrete spectrum. However, a careful study of the limit shows that the eigenvectors are in fact continuous at these points. So we say these points still belong to the continuous spectrum.

are degenerate for M'^{rr} but not for M'^{rs} with $r\neq s$. We can restrict to $\kappa\geqslant 0$ and construct the following two vectors:

$$(u_+)_n=(u_{2n}(\kappa),u_{2n+1}(\kappa))^t,$$
$$(u_-)_n=(u_{2n}(\kappa),-u_{2n+1}(\kappa))^t. \tag{2.7}$$

After writing matrices M'^{rs} in the following 2×2 block form:

$$\begin{pmatrix} M'^{rs}_{ee} & M'^{rs}_{eo} \\ M'^{rs}_{oe} & M'^{rs}_{oo} \end{pmatrix}$$

where $e(o)$ indicates even (odd) modes, the eigenequation (2.2) now reads

$$M'^{rs}u_\pm(\kappa)=\rho^{rs}_\pm(\kappa)u_\pm(\kappa), \tag{2.8}$$

where the summation over n is implied (we will use this notation hereafter), and

$$\rho^{rs}_\pm(\kappa)=\rho^{rs}(\pm\kappa)|_{\kappa\geqslant 0}. \tag{2.9}$$

The discrete spectrum consists of a single point in (0,1). The eigenvalue $\rho_0\equiv\rho_0^{11}$ is the solution of the following equation:

$$2h(x)=b-4[\gamma+\log(4)], \tag{2.10}$$

where γ is the Euler constant, b a gauge parameter [30], and

$$h(x)=\psi(-g(x))+\psi(1+g(x)),$$
$$g(x)=\frac{i}{2\pi}\text{arcsech}\left(-\frac{2x}{1+x}\right),$$

with $\psi(x)$ the digamma function. The solution to Eq. (2.10) in the interval (0,1) depends on the gauge parameter b; there always exists only one solution no matter what b is. The discrete eigenvalue for M'^{rs} with $r\neq s$ is given by

$$\rho^{12}_{0,+}=\rho^{21}_{0,-}=\frac{1}{2}\sqrt{(1+3\rho_0)(1-\rho_0)}+\frac{1}{2}(1-\rho_0),$$
$$\rho^{12}_{0,-}=\rho^{21}_{0,+}=-\frac{1}{2}\sqrt{(1+3\rho_0)(1-\rho_0)}+\frac{1}{2}(1-\rho_0). \tag{2.11}$$

Denote the eigenvector of M' at this discrete eigenvalue by ϕ_n. We can construct the following two vectors the same way as in the continuous spectrum:

$$(\phi_+)_n=(\phi_{2n},\ \phi_{2n+1})^t,\quad (\phi_-)_n=(\phi_{2n},\ -\phi_{2n+1})^t, \tag{2.12}$$

and the eigenequation (2.3) will be rewritten as

$$M'^{rs}\phi_\pm=\rho^{rs}_\pm\phi_\pm. \tag{2.13}$$

In this paper we will use properly normalized eigenvectors. So we can write down the following completeness and ortho-normal relations (with $a,b=+,-$):

$$u_a^\dagger(\kappa)u_b(\kappa')=\delta_{ab}\delta(\kappa-\kappa'),\quad \phi_a^\dagger\phi_b=\delta_{ab},\quad u_a^\dagger\phi_b=0, \tag{2.14}$$

$$\sum_{a=+,-}\left(\int_0^\infty d\kappa u_a(\kappa)u_a^\dagger(\kappa)+\phi_a\phi_a^\dagger\right)=Id. \tag{2.15}$$

In these relations $\kappa,\kappa'\geqslant 0$.

Now we can rewrite the Neumann matrices in the following diagonal form:

$$M'^{rs}=\sum_{a=+,-}\left(\int_0^\infty d\kappa\rho_a^{rs}(\kappa)_a u_a(\kappa)u_a^\dagger(\kappa)+\rho^{rs}_{0,a}\phi_a\phi_a^\dagger\right). \tag{2.16}$$

Substituting this expression into Eq. (2.1), we can introduce the following two sets of new oscillators to rewrite the three string vertex:

$$e^{M\dagger}_\kappa=\sqrt{2}\sum_{n=0}^\infty u_{2n}(\kappa)a^{M\dagger}_{2n},$$
$$o^{M\dagger}_\kappa=-i\sqrt{2}\sum_{n=0}^\infty u_{2n+1}(\kappa)a^{M\dagger}_{2n+1},$$
$$\tilde{e}^{M\dagger}=\sqrt{2}\sum_{n=0}^\infty \phi_{2n}a^{M\dagger}_{2n},$$
$$\tilde{o}^{M\dagger}=-i\sqrt{2}\sum_{n=0}^\infty \phi_{2n+1}a^{M\dagger}_{2n+1}. \tag{2.17}$$

Here, we have used the tilde to label quantities associated with the discrete eigenvalue, and suppressed the string index r in the above equations. These new oscillators satisfy the commutation relations

$$[e^M_\kappa,e^{N\dagger}_{\kappa'}]=[o^M_\kappa,o^{N\dagger}_{\kappa'}]=G^{MN}\delta(\kappa-\kappa'),$$
$$[e^M_\kappa,o^{N\dagger}_{\kappa'}]=[o^M_\kappa,e^{N\dagger}_{\kappa'}]=0,$$
$$[\tilde{e}^M,\tilde{e}^{N\dagger}]=[\tilde{o}^M,\tilde{o}^{N\dagger}]=G^{MN},$$
$$[\tilde{e}^M,\tilde{o}^{N\dagger}]=[\tilde{o}^M,\tilde{e}^{N\dagger}]=0, \tag{2.18}$$

and the BPZ conjugation

$$bpz(e^M_\kappa)=-e^{M\dagger}_\kappa,\quad bpz(o^M_\kappa)=-o^{M\dagger}_\kappa,$$
$$bpz(\tilde{e}^M)=-\tilde{e}^{M\dagger},\quad bpz(\tilde{o}^M)=-\tilde{o}^{M\dagger}. \tag{2.19}$$

The inverse transformation of Eqs. (2.17) is given by

$$a^{M\dagger}_{2n}=\sqrt{2}\left(\int_0^\infty d\kappa u_{2n}(\kappa)e^{M\dagger}_\kappa+\phi_{2n}\tilde{e}^{M\dagger}\right),$$
$$a^{M\dagger}_{2n+1}=i\sqrt{2}\left(\int_0^\infty d\kappa u_{2n+1}(\kappa)o^{M\dagger}_\kappa+\phi_{2n+1}\tilde{o}^{M\dagger}\right). \tag{2.20}$$

Finally, we obtain the diagonal form of the three-string vertex $|V_3'\rangle$:

$$|V_3'\rangle=\exp\Bigg[\int_0^\infty d\kappa G_{MN}\Bigg\{-\frac{1}{2}\rho(\kappa)(e_\kappa^{M(1)\dagger}e_\kappa^{N(1)\dagger}+o_\kappa^{M(1)\dagger}o_\kappa^{N(1)\dagger}+\text{cycl.})-\rho'(\kappa)(e_\kappa^{M(1)\dagger}e_\kappa^{N(2)\dagger}+o_\kappa^{M(1)\dagger}o_\kappa^{N(2)\dagger}+\text{cycl.})-i\rho''(\kappa)(e_\kappa^{M(1)\dagger}o_\kappa^{N(2)\dagger}-o_\kappa^{N(1)\dagger}e_\kappa^{M(2)\dagger}+\text{cycl.})\Bigg\}+G_{MN}\Bigg\{-\frac{1}{2}\rho_0(\tilde{e}^{M(1)\dagger}\tilde{e}^{N(1)\dagger}+\tilde{o}^{M(1)\dagger}\tilde{o}^{N(1)\dagger}+\text{cycl.})-\rho_0'(\tilde{e}^{M(1)\dagger}\tilde{e}^{N(2)\dagger}+\tilde{o}^{M(1)\dagger}\tilde{o}^{N(2)\dagger}+\text{cycl.})-i\rho_0''(\tilde{e}^{M(1)\dagger}\tilde{o}^{N(2)\dagger}-\tilde{o}^{N(1)\dagger}\tilde{e}^{M(2)\dagger}+\text{cycl.})\Bigg\}\Bigg]|0\rangle, \tag{2.21}$$

where

$$\rho'(\kappa)=\frac{1}{2}[\rho_+^{12}(\kappa)+\rho_-^{12}(\kappa)],$$

$$\rho''(\kappa)=\frac{1}{2}[\rho_+^{12}(\kappa)-\rho_-^{12}(\kappa)],$$

$$\rho_0'=\frac{1}{2}[\rho_{0,+}^{12}+\rho_{0,-}^{12}(\kappa)],\qquad \rho_0''=\frac{1}{2}[\rho_{0,+}^{12}-\rho_{0,-}^{12}(\kappa)].$$

III. THREE-STRING VERTEX IN A B BACKGROUND

Let us turn on a non-vanishing anti-symmetric B background in the first and second spatial dimensions. It is well known that the effective open string metric G^{MN} and the effective anti-symmetric noncommutativity parameter $\theta^{\alpha\beta}$ between the first and second coordinates of the string endpoints are, respectively,

$$G^{MN}=\begin{cases} G^{\mu\nu}=\eta^{\mu\nu} & \text{for } \mu,\nu=0,3,\ldots,25,\\ G^{\alpha\beta}=\frac{1}{\xi}\delta^{\alpha\beta}=\frac{1}{\xi}\text{diag}\{1,1\} & \text{for } \alpha,\beta=1,2,\end{cases} \tag{3.1}$$

$$\theta^{\alpha\beta}=-\frac{(2\pi\alpha')^2B}{\xi}\epsilon^{\alpha\beta},$$

where $\epsilon^{\alpha\beta}$ is the two-by-two anti-symmetric tensor with $\epsilon^{12}=1$ and $\xi=1+(2\pi\alpha' B)^2$.

The three-string vertex will decompose into

$$|V_3\rangle=|V_{3,\parallel}\rangle\otimes|V_{3,\perp}\rangle, \tag{3.2}$$

where $\parallel$ denotes the parallel directions $\alpha,\beta=1,2$, and $\perp$ the transverse directions $\mu,\nu=0,3,\ldots,25$. The transverse part $|V_{3,\perp}\rangle$ will be of the same form as given in Sec. II. Here we focus on the parallel directions. The oscillator representation for $|V_{3,\parallel}\rangle$ can be written as

$$|V_{3,\parallel}\rangle=\exp\left[-\frac{1}{2}\sum_{r,s}\sum_{m,n\geq 0}a_m^{\alpha(r)\dagger}\mathcal{V}_{mn,\alpha\beta}^{rs}a_n^{\beta(s)\dagger}\right]|\tilde{0}\rangle, \tag{3.3}$$

where the Neumann matrices $\mathcal{V}_{nm}^{\alpha\beta,rs}$ have been given in [17–19]. We list the properties of $\mathcal{V}_{nm}^{\alpha\beta,rs}$ and $\mathcal{M}^{\alpha\beta,rs}\equiv C\mathcal{V}^{\alpha\beta,rs}$ in Appendix A.

We write the matrices $\mathcal{M}^{rs}$ in the following 4×4 form:

$$\begin{pmatrix}\Gamma^{rs,11} & \Gamma^{rs,12}\\ \Gamma^{rs,21} & \Gamma^{rs,22}\end{pmatrix} \tag{3.4}$$

where

$$\Gamma^{rs,\alpha\beta}=\begin{pmatrix}\mathcal{M}_{ee}^{rs,\alpha\beta} & \mathcal{M}_{eo}^{rs,\alpha\beta}\\ \mathcal{M}_{oe}^{rs,\alpha\beta} & \mathcal{M}_{oo}^{rs,\alpha\beta}\end{pmatrix}. \tag{3.5}$$

In the next section we will review the results obtained in Ref. [21]. Then in Sec. III B we will construct the eigenvectors of $\mathcal{M}^{rs}(r\neq s)$, and use the results obtained to rewrite the three-string vertex in Sec. III C.

A. The spectrum of the Neumann matrix $\mathcal{M}^{11}$

The eigenvalue problem for the Neumann matrix $\mathcal{M}\equiv\mathcal{M}^{11}$ has been solved in Refs. [21,22]. The spectrum also has two branches: a continuous branch, labeled by a continuous parameter $\kappa\in(-\infty,+\infty)$, and a discrete one labeled by a discrete parameter $j=1,2$. The eigenequation is

$$\sum_{n=0}^{\infty}\sum_{\beta=1}^{2}\mathcal{M}_{mn}^{\alpha\beta}v_n^\beta(\omega)=\lambda(\omega)v_m^\alpha(\omega), \tag{3.6}$$

where $\omega=\kappa,j$. For our purpose, we write the eigenvector as a 4-row vector as

$$X(\omega)_n=(v_{2n}^1(\omega),\ v_{2n+1}^1(\omega),\ v_{2n}^2(\omega),\ v_{2n+1}^2(\omega))^t, \tag{3.7}$$

and the eigenequation as

$$\mathcal{M}X(\omega)=\lambda(\omega)X(\omega). \tag{3.8}$$

For the continuous spectrum, the expression for the eigenvalues of $\mathcal{M}$ is only a rescaling of that in the absence of the B field, given by

$$\lambda(\kappa)=\frac{1}{\xi}\frac{-1}{1+2\cosh\frac{\pi\kappa}{2}} \tag{3.9}$$

and the two degenerate eigenvectors are

$$X(\kappa)_n=(v_{2n}^1(\kappa),\ v_{2n+1}^1(\kappa),\ v_{2n}^2(\kappa),\ v_{2n+1}^2(\kappa))^t, \tag{3.10}$$

$$Y'(\kappa)_n=(-v_{2n}^2(\kappa),\ -v_{2n+1}^2(\kappa),\ v_{2n}^1(\kappa),\ v_{2n+1}^1(\kappa))^t. \tag{3.11}$$

The components of these two degenerate vectors have the following properties.

(1) Complex conjugation is given by

$$v_n^{1*}(\kappa)=v_n^1(\kappa),\quad v_n^{2*}(\kappa)=-v_n^2(\kappa). \tag{3.12}$$

(2) Under $\kappa\to-\kappa$, we have

$$v_{2n}^1(-\kappa)=v_{2n}^1(\kappa),\quad v_{2n+1}^1(-\kappa)=-v_{2n+1}^1(\kappa),$$
$$v_{2n}^2(-\kappa)=-v_{2n}^2(\kappa),\quad v_{2n+1}^2(-\kappa)=v_{2n+1}^2(\kappa). \tag{3.13}$$

These two degenerate eigenvectors are not orthogonal to each other. By the Gram-Schmidt procedure, the eigenvector which is orthogonal to $X(\kappa)$ is given by $Y(\kappa)=Y'(\kappa)-X^\dagger(\kappa)Y'(\kappa)X(\kappa)$. After normalization, we write $Y(\kappa)$ in the form of a 4-row vector as

$$Y(\kappa)_n=(\tilde{v}_{2n}^1(\kappa),\ \tilde{v}_{2n+1}^1(\kappa),\ \tilde{v}_{2n}^2(\kappa),\ \tilde{v}_{2n+1}^2(\kappa))^t. \tag{3.14}$$

It is easy to check that $\tilde{v}_n^\alpha(\kappa)$ has the following properties.

(1) Complex conjugation is given by

$$\tilde{v}_n^{1*}(\kappa)=-\tilde{v}_n^1(\kappa),\quad \tilde{v}_n^{2*}(\kappa)=\tilde{v}_n^2(\kappa). \tag{3.15}$$

(2) Under $\kappa\to-\kappa$, we have

$$\tilde{v}_{2n}^1(-\kappa)=-\tilde{v}_{2n}^1(\kappa),\quad \tilde{v}_{2n+1}^1(-\kappa)=\tilde{v}_{2n+1}^1(\kappa),$$
$$\tilde{v}_{2n}^2(-\kappa)=\tilde{v}_{2n}^2(\kappa),\quad \tilde{v}_{2n+1}^2(-\kappa)=-\tilde{v}_{2n+1}^2(\kappa). \tag{3.16}$$

Note that $\lambda(\kappa)=\lambda(-\kappa)$; we have the following fourfold degenerate eigenvectors for $\mathcal{M}$ in the continuous spectrum:

$$(X_e(\kappa))_n=\frac{1}{2}(X(\kappa)_n+X(-\kappa)_n)$$
$$=(v_{2n}^1(\kappa),\ 0,\ 0,\ v_{2n+1}^2(\kappa))^t, \tag{3.17}$$

$$(X_o(\kappa))_n=\frac{1}{2}(X(\kappa)_n-X(\kappa)_n)$$
$$=(0,\ v_{2n+1}^1(\kappa),\ v_{2n}^2(\kappa),\ 0)^t, \tag{3.18}$$

$$(Y_e(\kappa))_n=\frac{1}{2}(Y(\kappa)_n+Y(-\kappa)_n)$$
$$=(0,\ \tilde{v}_{2n+1}^1(\kappa),\ \tilde{v}_{2n}^2(\kappa),\ 0)^t, \tag{3.19}$$

$$(Y_o(\kappa))_n=\frac{1}{2}(Y(\kappa)_n-Y(-\kappa)_n)$$
$$=(\tilde{v}_{2n}^1(\kappa),\ 0,\ 0,\ \tilde{v}_{2n+1}^2(\kappa))^t. \tag{3.20}$$

For a discrete spectrum, we have two points lying in the interval $(0,1/\xi)$, and they are determined by the following equation, respectively, with $x\equiv\xi\lambda$:

$$2h(x)=\mp 4B\pi^2\sqrt{\frac{1-x}{1+3x}}+b-4[\gamma+\log(4)]. \tag{3.21}$$

The double degenerate eigenvectors at each point ($j=1,2$) are

$$(X_e(j))_n=(v_{2n}^1(j),\ 0,\ 0,\ v_{2n+1}^2(j))^t, \tag{3.22}$$

$$(X_o(j))_n=(0,\ iv_{2n+1}^2(j),\ -iv_{2n}^1(j),\ 0)^t. \tag{3.23}$$

They are obtained by setting ($D_1=i$, $D_2=0$) and ($D_1=0$, $D_2=-1$) in Eq. (6.1) of Ref. [21]. The components satisfy $v_{2n}^{1*}(j)=v_{2n}^1(j)$, $v_{2n+1}^{2*}(j)=-v_{2n+1}^2(j)$.

B. The spectrum for $\mathcal{M}^{rs}$ ($r\neq s$)

In this section we will construct the eigenvectors for $\mathcal{M}^{rs}$ ($r\neq s$) explicitly. The continuous eigenvalues of $\mathcal{M}^{rs}$ ($r\neq s$) can be obtained in the same way as in Sec. II. Using the properties (A14) in Appendix A, we have

$$\lambda^{12}(\kappa)=\frac{1}{2}\mathrm{sgn}(\kappa)\sqrt{\left(\frac{1}{\xi}+3\lambda(\kappa)\right)\left(\frac{1}{\xi}-\lambda(\kappa)\right)}+\frac{1}{2}\left(\frac{1}{\xi}-\lambda(\kappa)\right),$$
$$\lambda^{21}(\kappa)=-\frac{1}{2}\mathrm{sgn}(\kappa)\sqrt{\left(\frac{1}{\xi}+3\lambda(\kappa)\right)\left(\frac{1}{\xi}-\lambda(\kappa)\right)}+\frac{1}{2}\left(\frac{1}{\xi}-\lambda(\kappa)\right). \tag{3.24}$$

Here, $\mathrm{sgn}(\kappa)$ appears because of the requirement that the expressions should recover (2.5) as $B\to 0$.

The corresponding eigenvectors can be constructed as follows. We set (for $\kappa\geq 0$)

$$X_+(\kappa)=X_e(\kappa)+X_o(\kappa),\quad X_-(\kappa)=X_e(\kappa)-X_o(\kappa),$$
$$Y_+(\kappa)=Y_e(\kappa)+Y_o(\kappa),\quad Y_-(\kappa)=Y_e(\kappa)-Y_o(\kappa). \tag{3.25}$$

It is easy to see that $X_+(\kappa)[Y_+(\kappa)]$ is just the $X(\kappa)[Y(\kappa)]$ restricted to $\kappa\geq 0$, and $X_-(\kappa)[Y_-(\kappa)]$ the $X(-\kappa)[Y(-\kappa)]$ restricted to $\kappa\geq 0$. For the Neumann matrix $\mathcal{M}^{11}$, it is known that $X_+(\kappa)[Y_+(\kappa)]$ and $X_-(\kappa)[Y_-(\kappa)]$ are degenerate since $\lambda(-\kappa)=\lambda(\kappa)$. However, the last property is no longer true for eigenvalues of the Neumann matrices $\mathcal{M}^{rs}$ ($r\neq s$), i.e. $[X_+(\kappa),Y_+(\kappa)]$ and $[X_-(\kappa),Y_-(\kappa)]$ are no longer degenerate for $\mathcal{M}^{rs}$ ($r\neq s$).

We can rewrite the eigenequation (3.8) as

$$\mathcal{M}^{rs}\Psi_\pm(\kappa)=\lambda_\pm^{rs}(\kappa)\Psi_\pm(\kappa), \tag{3.26}$$

where $\Psi=X,Y$, and

$$\lambda_\pm^{rs}(\kappa)=\lambda^{rs}(\pm\kappa)|_{\kappa\geq 0}. \tag{3.27}$$

For the discrete spectrum, it can be shown that the eigenvectors of $\mathcal{M}^{rs}$ with $r\neq s$ are a linear combination of $X_e(j)$ and $X_o(j)$, i.e.

$$\mathcal{M}^{rs}[X_e(j)+X_o(j)]=\lambda^{rs}_{j,+}[X_e(j)+X_o(j)], \quad (3.28)$$

$$\mathcal{M}^{rs}[X_e(j)-X_o(j)]=\lambda^{rs}_{j,-}[X_e(j)-X_o(j)]. \quad (3.29)$$

The proof is left to Appendix C. From now on we will denote $X_\pm(j)\equiv X_e(j)\pm X_o(j)$. The independent eigenvalues are given by

$$\lambda^{12}_+(j)=\frac{1}{2}\sqrt{\left(\frac{1}{\xi}+3\lambda(j)\right)\left(\frac{1}{\xi}-\lambda(j)\right)}+\frac{1}{2}\left(\frac{1}{\xi}-\lambda(j)\right), \quad (3.30)$$

$$\lambda^{12}_-(j)=-\frac{1}{2}\sqrt{\left(\frac{1}{\xi}+3\lambda(j)\right)\left(\frac{1}{\xi}-\lambda(j)\right)}+\frac{1}{2}\left(\frac{1}{\xi}-\lambda(j)\right), \quad (3.31)$$

$$\lambda^{21}_+(j)=\lambda^{12}_-(j), \quad \lambda^{21}_-(j)=\lambda^{12}_+(j). \quad (3.32)$$

The ortho-normal and completeness relations are expressed as (with $a,b=+,-$)

$$X_a(\kappa)^\dagger X_b(\kappa')=Y_a(\kappa)^\dagger Y_b(\kappa')=\delta_{ab}\delta(\kappa-\kappa'),$$
$$X_a(i)^\dagger X_b(j)=\delta_{ab}\delta_{ij},$$
$$X_a(\kappa)^\dagger Y_b(\kappa')=X_a(i)^\dagger X_b(\kappa)=X_a(i)^\dagger Y_b(\kappa)=0, \quad (3.33)$$

$$\sum_{a=+,-}\left\{\int_0^\infty d\kappa[X_a(\kappa)X_a^\dagger(\kappa)+Y_a(\kappa)Y_a^\dagger(\kappa)]+\sum_{j=1,2}[X_a(j)X_a^\dagger(j)]\right\}=Id.$$

Again, in these relations $\kappa\geq 0$.

C. Diagonal representation of the three-string vertex in a *B* background

The three-string vertex is $|V_{3,\parallel}\rangle=\exp\{-\frac{1}{2}V_\parallel\}|\tilde{0}\rangle$, where

$$V_\parallel=\sum_{r,s}\sum_{m,n\geq 0}a_m^{\alpha(r)\dagger}(C\mathcal{M}^{rs})_{mn,\alpha\beta}a_n^{\beta(s)\dagger}. \quad (3.34)$$

Using the results obtained in the preceding section, we can rewrite the Neumann matrices $\mathcal{M}^{rs}$ in the following form:

$$\mathcal{M}^{rs}=\sum_{a=+,-}\left\{\int_0^\infty d\kappa\lambda_a^{rs}(\kappa)[X_a(\kappa)X_a^\dagger(\kappa)+Y_a(\kappa)Y_a^\dagger(\kappa)]+\sum_{j=1,2}\lambda^{rs}_{j,a}X_a(j)X_a^\dagger(j)\right\}. \quad (3.35)$$

Define $(A^{\dagger(r)}|=(a_0^{\dagger 1(r)},\ldots,a_m^{\dagger 1(r)},\ldots,a_0^{\dagger 2(r)},\ldots,a_m^{\dagger 2(r)},\ldots)=|A^{\dagger(r)})^t$, then

$$\begin{aligned}V_\parallel&=(A^{\dagger(r)}|C\mathcal{M}^{rs}|A^{\dagger(s)})\\&=\Bigg(A^{\dagger(r)}\Bigg|C\Bigg\{\int_0^\infty d\kappa\{\xi^2[\lambda_+^{rs}(\kappa)+\lambda_-^{rs}(\kappa)][X_e(\kappa)X_e^\dagger(\kappa)+X_o(\kappa)X_o^\dagger(\kappa)+Y_e(\kappa)Y_e^\dagger(\kappa)+Y_o(\kappa)Y_o^\dagger(\kappa)]\\&\quad+\xi^2[\lambda_+^{rs}(\kappa)-\lambda_-^{rs}(\kappa)][X_e(\kappa)X_o^\dagger(\kappa)+X_o(\kappa)X_e^\dagger(\kappa)+Y_e(\kappa)Y_o^\dagger(\kappa)+Y_o(\kappa)Y_e^\dagger(\kappa)]\}\\&\quad+\sum_{j=1,2}\{\xi^2(\lambda^{rs}_{j,+}+\lambda^{rs}_{j,-})[X_e(j)X_e^\dagger(j)+X_o(j)X_o^\dagger(j)]+\xi^2(\lambda^{rs}_{j,+}-\lambda^{rs}_{j,-})[X_e(j)X_o^\dagger(j)+X_o(j)X_e^\dagger(j)]\}\Bigg\}\Bigg|A^{\dagger(s)}\Bigg).\end{aligned} \quad (3.36)$$

Note that

$$[CX_e(\omega)]^t=X_e^\dagger(\omega), \quad [CX_o(\omega)]^t=-X_o^\dagger(\omega), \quad \omega=\kappa,j$$
$$[CY_e(\kappa)]^t=Y_e^\dagger(\kappa), \quad [CY_o(\kappa)]^t=-Y_o^\dagger(\kappa). \quad (3.37)$$

Let us introduce the following new oscillators:

$$e_\kappa^{1\dagger}=\sqrt{2}X_e^\dagger(\kappa)|A^\dagger)=\sqrt{2}\sum_{n=0}^\infty[v_{2n}^1(\kappa)a_{2n}^{1\dagger}-v_{2n+1}^2(\kappa)a_{2n+1}^{2\dagger}], \quad (3.38)$$

$$o_\kappa^{1\dagger}=-i\sqrt{2}X_o^\dagger(\kappa)|A^\dagger)=-i\sqrt{2}\sum_{n=0}^\infty[v_{2n+1}^1(\kappa)a_{2n+1}^{1\dagger}-v_{2n}^2(\kappa)a_{2n}^{2\dagger}], \quad (3.39)$$

$$e_\kappa^{2\dagger}=\sqrt{2}Y_e^\dagger(\kappa)|A^\dagger)=\sqrt{2}\sum_{n=0}[\tilde{v}_{2n}^2(\kappa)a_{2n}^{2\dagger}-\tilde{v}_{2n+1}^1(\kappa)a_{2n+1}^{1\dagger}], \quad (3.40)$$

$$o_\kappa^{2\dagger}=-i\sqrt{2}Y_o^\dagger(\kappa)|A^\dagger\rangle$$
$$=-i\sqrt{2}\sum_{n=0}[\tilde{v}_{2n+1}^2(\kappa)a_{2n+1}^{2\dagger}-\tilde{v}_{2n}^1(\kappa)a_{2n}^{1\dagger}], \tag{3.41}$$

$$\tilde{e}^{j\dagger}=\sqrt{2}X_e^\dagger(j)|A^\dagger\rangle=\sqrt{2}\sum_{n=0}^{\infty}[v_{2n}^1(j)a_{2n}^{1\dagger}-v_{2n+1}^2(j)a_{2n+1}^{2\dagger}], \tag{3.42}$$

$$\tilde{o}^{j\dagger}=-i\sqrt{2}X_o^\dagger(j)|A^\dagger\rangle$$
$$=-i\sqrt{2}\sum_{n=0}^{\infty}[iv_{2n+1}^2(j)a_{2n+1}^{1\dagger}+iv_{2n}^1(j)a_{2n}^{2\dagger}]. \tag{3.43}$$

They satisfy the commutation relations

$$[e_\kappa^{\bar\alpha},\ e_{\kappa'}^{\bar\beta\dagger}]=[o_\kappa^{\bar\alpha},\ o_{\kappa'}^{\bar\beta\dagger}]=G^{\bar\alpha\bar\beta}\delta(\kappa-\kappa'),$$
$$[e_\kappa^{\bar\alpha},\ o_{\kappa'}^{\bar\beta\dagger}]=[o_\kappa^{\bar\alpha},\ e_{\kappa'}^{\bar\beta\dagger}]=0$$
$$[\tilde{e}^{\bar i},\ \tilde{e}^{\bar j\dagger}]=[\tilde{o}^{\bar i},\ \tilde{o}^{\bar j\dagger}]=G^{\bar i\bar j},\quad [\tilde{e}^{\bar i},\ \tilde{o}^{\bar j\dagger}]=[\tilde{o}^{\bar i},\ \tilde{e}^{\bar j\dagger}]=0, \tag{3.44}$$

and the Belavin-Polyakov-Zamolodchikov (BPZ) conjugation

$$bpz(e_\kappa^{\bar\alpha})=-e_\kappa^{\bar\alpha\dagger},\quad bpz(o_\kappa^{\bar\alpha})=-o_\kappa^{\bar\alpha\dagger},$$
$$bpz(\tilde{e}^{\bar i})=-\tilde{e}^{\bar i\dagger},\quad bpz(\tilde{o}^{\bar i})=-\tilde{o}^{\bar i\dagger}. \tag{3.45}$$

The inverse transformation of Eqs. (3.38)–(3.43) is

$$\sqrt{2}|A^\dagger\rangle=\int_0^\infty d\kappa[X_e(\kappa)e_\kappa^{1\dagger}+iX_o(\kappa)o_\kappa^{1\dagger}+Y_e(\kappa)e_\kappa^{2\dagger}$$
$$+iY_o(\kappa)o_\kappa^{2\dagger}]+\sum_{j=1,2}[X_e(j)\tilde{e}^{j\dagger}+iX_o(j)\tilde{o}^{j\dagger}]. \tag{3.46}$$

Finally, the operator $V_\parallel$ can be written in the following diagonal form:

$$|V_{3,\parallel}\rangle=\exp\Bigg[\int_0^\infty d\kappa G_{\bar\alpha\bar\beta}\Bigg\{-\frac{1}{2}\xi\lambda(\kappa)(e_\kappa^{\bar\alpha(1)\dagger}e_\kappa^{\bar\beta(1)\dagger}+o_\kappa^{\bar\alpha(1)\dagger}o_\kappa^{\bar\beta(1)\dagger}+\text{cycl.})-\xi\lambda'(\kappa)(e_\kappa^{\bar\alpha(1)\dagger}e_\kappa^{\bar\beta(2)\dagger}+o_\kappa^{\bar\alpha(1)\dagger}o_\kappa^{\bar\beta(2)\dagger}+\text{cycl.})$$
$$-i\xi\lambda''(\kappa)(e_\kappa^{\bar\alpha(1)\dagger}o_\kappa^{\bar\beta(2)\dagger}-o_\kappa^{\bar\beta(1)\dagger}e_\kappa^{\bar\alpha(2)\dagger}+\text{cycl.})\Bigg\}+G_{\bar i\bar j}\Bigg\{-\frac{1}{2}\xi\lambda_i(\tilde{e}^{\bar i(1)\dagger}\tilde{e}^{\bar j(1)\dagger}+\tilde{o}^{\bar i(1)\dagger}\tilde{o}^{\bar j(1)\dagger}+\text{cycl.})$$
$$-\xi\lambda_i'(\tilde{e}^{\bar i(1)\dagger}\tilde{e}^{\bar j(2)\dagger}+\tilde{o}^{\bar i(1)\dagger}\tilde{o}^{\bar j(2)\dagger}+\text{cycl.})-i\xi\lambda_i''(\tilde{e}^{\bar i(1)\dagger}\tilde{o}^{\bar j(2)\dagger}-\tilde{o}^{\bar j(1)\dagger}\tilde{e}^{\bar i(2)\dagger}+\text{cycl.})\Bigg\}\Bigg]|\tilde{0}\rangle. \tag{3.47}$$

Here

$$\lambda'(\kappa)=\frac{1}{2}[\lambda_+^{12}(\kappa)+\lambda_-^{12}(\kappa)],$$
$$\lambda''(\kappa)=\frac{1}{2}[\lambda_+^{12}(\kappa)-\lambda_-^{12}(\kappa)],$$
$$\lambda_i'=\frac{1}{2}[\lambda_{i,+}^{12}+\lambda_{i,-}^{12}(\kappa)],\quad \lambda_i''=\frac{1}{2}[\lambda_{i,+}^{12}-\lambda_{i,-}^{12}(\kappa)].$$

The explicit form of the three-string vertex in full 26 dimensions in the presence of a B background can be constructed as follows: If in Eq. (2.21) we restrict the indices M,N to the transverse directions $\mu\nu$, we get the vertex $|V_{3,\perp}\rangle$; then $|V_3\rangle=|V_{3,\parallel}\rangle\otimes|V_{3,\perp}\rangle$. It is easy to see from Eqs. (3.47) and (2.21) that it is of the same form as Eq. (3.21) in Ref. [15], which examined only the zero-momentum sector with $B=0$, except that now we have two sets of oscillators corresponding to the two types of spectra, continuous and discrete.

IV. IDENTIFICATION OF THE FULL MOYAL STRUCTURE

In this section we will present the explicit Moyal structure of the three-string vertex, including both the Moyal coordinates and the corresponding noncommutativity parameters. In Ref. [15] the oscillator form of the three-vertex has been identified with the Moyal multiplication in an infinite commuting set of two-dimensional noncommutative subspaces. Generalizing their result, in our present case we will obtain the Moyal structure for the three-string vertex, which is of the following form:

$$[x^M(\omega),y^N(\omega')]_\star=i\theta_M(\omega)G^{MN}\delta(\omega-\omega'),(M,N)$$
$$=(\mu,\nu)\text{ or }(\bar\alpha,\bar\beta)\text{ or }(\bar i,\bar j) \tag{4.1}$$

where $\theta_M(\omega)$ is the noncommutativity parameter in a given sector, and the space-time metric is the modified (diagonal) one given in Eq. (3.1).

According to Ref. [15], if to each pair of Moyal coordinates (with M fixed), we associate one pair of oscillators:

$$(x^{M(r)},y^{M(r)})\leftrightarrow(a^{M(r)\dagger},b^{M(r)\dagger}),\quad r=1,2,3 \tag{4.2}$$

then the three-string vertex will be of the following form (no summation over M):

$$\begin{aligned}|V_3(\theta)\rangle\sim\exp\Bigg[&-\frac{1}{2}\left(\frac{-4+\theta^2}{12+\theta^2}\right)G^{MM}(a_M^{(1)\dagger}a_M^{(1)\dagger}+b_M^{(1)\dagger}b_M^{(1)\dagger}\\&+\text{cycl.})-\left(\frac{8}{12+\theta^2}\right)G^{MM}(a_M^{(1)\dagger}a_M^{(2)\dagger}+b_M^{(1)\dagger}b_M^{(2)\dagger}\\&+\text{cycl.})-\left(\frac{4i\theta}{12+\theta^2}\right)G^{MM}(a_M^{(1)\dagger}b_M^{(2)\dagger}-b_M^{(1)\dagger}a_M^{(2)\dagger}\\&+\text{cycl.})\Bigg]|0\rangle.\end{aligned} \tag{4.3}$$

Here we have suppressed the index M in the noncommutativity parameter θ_M between (x^M,y^M).

Comparing the exponents in Eqs. (2.21), (3.47) with those in Eq. (4.3), we see that in the presence of the B background one should identify

$$(a^{\mu\dagger},\ b^{\mu\dagger})\leftrightarrow(e_\kappa^{\mu\dagger},\ o_\kappa^{\mu\dagger})\ \text{or}\ (a^{\mu\dagger},\ b^{\mu\dagger})\leftrightarrow(\tilde{e}^{\mu\dagger},\ \tilde{o}^{\mu\dagger}), \tag{4.4}$$

$$(a^{\bar{\alpha}\dagger},\ b^{\bar{\alpha}\dagger})\leftrightarrow(e_\kappa^{\bar{\alpha}\dagger},\ o_\kappa^{\bar{\alpha}\dagger})\ \text{or}\ (a^{\bar{i}\dagger},\ b^{\bar{i}\dagger})\leftrightarrow(\tilde{e}^{\bar{i}\dagger},\ \tilde{o}^{\bar{i}\dagger}), \tag{4.5}$$

by requiring the following conditions. For the transverse directions $M=\mu$,

$$\begin{aligned}\rho(\kappa)&=\frac{-4+\theta_\perp^2(\kappa)}{12+\theta_\perp^2(\kappa)},\quad \rho_0=\frac{-4+\theta_\perp^2}{12+\theta_\perp^2},\\ \rho'(\kappa)&=\frac{8}{12+\theta_\perp^2(\kappa)},\quad \rho_0'=\frac{8}{12+\theta_\perp^2},\\ \rho''(\kappa)&=\frac{4\theta_\perp(\kappa)}{12+\theta_\perp^2(\kappa)},\quad \rho_0''=\frac{4\theta_\perp}{12+\theta_\perp^2}.\end{aligned} \tag{4.6}$$

For the parallel directions $M=\alpha$,

$$\begin{aligned}\xi\lambda(\kappa)&=\frac{-4+\theta_\parallel^2(\kappa)}{12+\theta_\parallel^2(\kappa)},\quad \xi\lambda_i=\frac{-4+\theta_{\parallel,i}^2}{12+\theta_{\parallel,i}^2},\\ \xi\lambda'(\kappa)&=\frac{8}{12+\theta_\parallel^2(\kappa)},\quad \xi\lambda_i'=\frac{8}{12+\theta_{\parallel,i}^2},\\ \xi\lambda''(\kappa)&=\frac{4\theta_\parallel(\kappa)}{12+\theta_\parallel^2(\kappa)},\quad \xi\lambda_i''=\frac{4\theta_{\parallel,i}}{12+\theta_{\parallel,i}^2}.\end{aligned} \tag{4.7}$$

This identification also requires the consistency conditions

$$\xi\lambda+\xi\lambda^{12}+\xi\lambda^{21}=1=\rho+\rho^{12}+\rho^{21}. \tag{4.8}$$

Indeed, these conditions are satisfied because of the properties (A11) in Appendix A.

We thus identify the two types of noncommutativity parameters as follows:

$$\theta_\parallel(\kappa)=\theta_\perp(\kappa)=2\tanh\left(\frac{\pi\kappa}{4}\right), \tag{4.9}$$

$$\begin{aligned}\theta_\perp&=2\sqrt{\frac{1+3\rho_0}{1-\rho_0}},\quad \theta_{\parallel,1}=2\sqrt{\frac{1+3\xi\lambda_1}{1-\xi\lambda_1}},\\ \theta_{\parallel,2}&=-2\sqrt{\frac{1+3\xi\lambda_2}{1-\xi\lambda_2}},\end{aligned} \tag{4.10}$$

where ρ_0 is determined by Eq. (2.10) and λ_j is determined by Eq. (3.21). We see that the two types of noncommutativity parameters belong to non-overlapping regions: $\theta_\perp(\kappa)$, $\theta_\parallel(\kappa)\in(0,2]$, while $\theta_\perp$, $\theta_{\parallel,1}\in(2,\infty)$, $\theta_{\parallel,2}\in(-\infty,-2)$. Note that only $\theta_{\parallel,j}$ depends on the B background.

Having obtained the noncommutativity parameters, we now consider the Moyal coordinates. Define the following coordinate and momentum operators in terms of our new oscillators by

$$\begin{aligned}\hat{x}&=\frac{i}{\sqrt{2}}(e-e^\dagger),\quad \hat{q}_x=\frac{1}{\sqrt{2}}(e+e^\dagger),\\ \hat{y}&=\frac{i}{\sqrt{2}}(o-o^\dagger),\quad \hat{q}_y=\frac{1}{\sqrt{2}}(o+o^\dagger).\end{aligned} \tag{4.11}$$

Recall that the string mode expansion is (choosing $\alpha'=\frac{1}{2}$)

$$\begin{aligned}\hat{X}^\mu(\sigma)&=\hat{x}_0^\alpha+\sqrt{2}\sum_{n=1}^{\infty}\hat{x}_n^\alpha\cos n\sigma\\ \hat{X}^\alpha(\sigma)&=\hat{x}_0^\alpha+\frac{1}{\pi}\theta^{\alpha\beta}\left(\sigma-\frac{\pi}{2}\right)\hat{p}_{0,\beta}\\&\quad+\sqrt{2}\sum_{n=1}^{\infty}\left[\hat{x}_n^\alpha\cos n\sigma+\frac{1}{\pi}\theta^{\alpha\beta}\hat{p}_{n,\beta}\sin n\sigma\right],\\ \pi\hat{P}^M(\sigma)&=\hat{p}_0^M+\sqrt{2}\sum_{n=1}^{\infty}\hat{p}_n^M\cos n\sigma,\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}\hat{x}_n^M&=\frac{i}{\sqrt{2n}}(a_n^M-a_n^{M\dagger}),\quad \hat{p}_n^M=\sqrt{\frac{n}{2}}(a_n^M+a_n^{M\dagger}),\quad n>0,\\ \hat{x}_0^M&=i\frac{\sqrt{b}}{2}(a_o^M-a_0^{M\dagger}),\quad \hat{p}_0^M=\sqrt{\frac{1}{b}}(a_0^M+a_0^{M\dagger}).\end{aligned} \tag{4.13}$$

Making use of the transformations (2.20) and (3.46), we can express the coordinate operators $\{\hat{x}_\kappa^\mu,\hat{y}_\kappa^\mu;\hat{\hat{x}}^\mu,\hat{\hat{y}}^\mu\}$ and $\{\hat{x}_\kappa^{\bar{\alpha}},\hat{y}_\kappa^{\bar{\alpha}};\hat{\hat{x}}^{\bar{\alpha}},\hat{\hat{y}}^{\bar{\alpha}}\}$ in terms of the original $\hat{x}_n^M,\hat{p}_n^M$:

$$\hat{x}^{\mu}_{\kappa}=\sqrt{2}\sum_{n=0}^{\infty}u_{2n}(\kappa)\sqrt{2n}\hat{x}^{\mu}_{2n},$$

$$\hat{y}^{\mu}_{\kappa}=-\sqrt{2}\sum_{n=0}^{\infty}\frac{u_{2n+1}(\kappa)}{\sqrt{2n+1}}\hat{p}^{\mu}_{2n+1},\tag{4.14}$$

$$\hat{\tilde{x}}^{\mu}=\sqrt{2}\sum_{n=0}^{\infty}\phi_{2n}\sqrt{2n}\hat{x}^{\mu}_{2n},\quad \hat{\tilde{y}}^{\mu}=-\sqrt{2}\sum_{n=0}^{\infty}\frac{\phi_{2n+1}}{\sqrt{2n+1}}\hat{p}^{\mu}_{2n+1},\tag{4.15}$$

$$\hat{x}^{\bar{1}}_{\kappa}=\sqrt{2}\sum_{n=0}^{\infty}\left(v^{1}_{2n}(\kappa)\sqrt{2n}\hat{x}^{1}_{2n}+i\frac{v^{2}_{2n+1}(\kappa)}{\sqrt{2n+1}}\hat{p}^{2}_{2n+1}\right),\tag{4.16}$$

$$\hat{y}^{\bar{1}}_{\kappa}=\sqrt{2}\sum_{n=0}^{\infty}\left(-\frac{v^{1}_{2n+1}(\kappa)}{\sqrt{2n+1}}\hat{p}^{1}_{2n+1}+iv^{2}_{2n}(\kappa)\sqrt{2n}\hat{x}^{2}_{2n}\right),\tag{4.17}$$

$$\hat{x}^{\bar{2}}_{\kappa}=\sqrt{2}\sum_{n=0}^{\infty}\left(\tilde{v}^{2}_{2n}(\kappa)\sqrt{2n}\hat{x}^{2}_{2n}+i\frac{\tilde{v}^{1}_{2n+1}(\kappa)}{\sqrt{2n+1}}\hat{p}^{1}_{2n+1}\right),\tag{4.18}$$

$$\hat{y}^{\bar{2}}_{\kappa}=\sqrt{2}\sum_{n=0}^{\infty}\left(-\frac{\tilde{v}^{2}_{2n+1}(\kappa)}{\sqrt{2n+1}}\hat{p}^{2}_{2n+1}+i\tilde{v}^{1}_{2n}(\kappa)\sqrt{2n}\hat{x}^{1}_{2n}\right),\tag{4.19}$$

$$\hat{\tilde{x}}^{\bar{j}}=\sqrt{2}\sum_{n=0}^{\infty}\left(v^{1}_{2n}(j)\sqrt{2n}\hat{x}^{1}_{2n}+i\frac{v^{2}_{2n+1}(j)}{\sqrt{2n+1}}\hat{p}^{2}_{2n+1}\right),\tag{4.20}$$

$$\hat{\tilde{y}}^{\bar{j}}=\sqrt{2}\sum_{n=0}^{\infty}\left(-i\frac{v^{2}_{2n+1}(j)}{\sqrt{2n+1}}\hat{p}^{1}_{2n+1}+v^{1}_{2n}(j)\sqrt{2n}\hat{x}^{2}_{2n}\right),\tag{4.21}$$

where $\bar{n}=1/b$ for $n=0$. We immediately see that in the directions parallel to the background B field, the string modes in different spatial directions get mixed with each other.

The Moyal coordinates $\{x^{\mu}(\kappa),y^{\mu}(\kappa);\tilde{x}^{\mu},\tilde{y}^{\mu}\}$ and $\{x^{\bar{\alpha}}(\kappa),y^{\bar{\alpha}}(\kappa);\tilde{x}^{\bar{\alpha}},\tilde{y}^{\bar{\alpha}}\}$ are the eigenvalues of operators $\{\hat{x}^{\mu}_{\kappa},\hat{y}^{\mu}_{\kappa};\hat{\tilde{x}}^{\mu},\hat{\tilde{y}}^{\mu}\}$ and $\{\hat{x}^{\bar{\alpha}}_{\kappa},\hat{y}^{\bar{\alpha}}_{\kappa};\hat{\tilde{x}}^{\bar{\alpha}},\hat{\tilde{y}}^{\bar{\alpha}}\}$, respectively. Their eigenvectors can be constructed by starting from the eigenvectors of $\hat{x}^{M}_{n},\hat{p}^{M}_{n}$:

$$\prod_{n}|x^{M}_{n}\rangle=\exp\left\{-\frac{1}{2}(x|H^{2}|x)-\sqrt{2}i(a^{\dagger}|H|x)+\frac{1}{2}(a^{\dagger}|a^{\dagger})\right\}|0\rangle,\tag{4.22}$$

$$\prod_{n}|p^{M}_{n}\rangle=\exp\left\{-\frac{1}{2}(p|H^{-2}|p)+\sqrt{2}(a^{\dagger}|H^{-1}|p)-\frac{1}{2}(a^{\dagger}|a^{\dagger})\right\}|0\rangle,\tag{4.23}$$

where $H_{mn}=\delta_{mn}\sqrt{n}+\delta_{m0}\delta_{n0}\sqrt{\frac{2}{b}}$ and $(z|f|z)=\sum_{m,n\geq 0}z^{M}_{m}f_{mn}z_{n,M}$ (no summation over M). From Eqs. (4.14)–(4.21) we see that only x_{2n} and p_{2n+1} are involved in the Moyal coordinate operators. Define

$$X^{\mu}(\kappa)=(x^{\mu}(\kappa),\ y^{\mu}(\kappa)),\quad E^{\mu}_{\kappa}=(e^{\mu\dagger}_{\kappa},\ o^{\mu\dagger}_{\kappa}),\tag{4.24}$$

$$\tilde{X}^{\mu}=(\tilde{x}^{\mu},\ \tilde{y}^{\mu}),\quad \tilde{E}^{\mu}=(\tilde{e}^{\mu\dagger},\ \tilde{o}^{\mu\dagger}),\tag{4.25}$$

$$X^{\bar{\alpha}}(\kappa)=(x^{\bar{\alpha}}(\kappa),\ y^{\bar{\alpha}}(\kappa)),\quad E^{\bar{\alpha}}_{\kappa}=(e^{\bar{\alpha}\dagger}_{\kappa},\ o^{\bar{\alpha}\dagger}_{\kappa}),\tag{4.26}$$

$$\tilde{X}^{\bar{\alpha}}=(\tilde{x}^{\bar{\alpha}},\ \tilde{y}^{\bar{\alpha}}),\quad \tilde{E}^{\bar{\alpha}}=(\tilde{e}^{\bar{\alpha}\dagger},\ \tilde{o}^{\bar{\alpha}\dagger}),\tag{4.27}$$

then the eigenstates of the Moyal coordinates are

$$|x^{\mu}(\kappa),y^{\mu}(\kappa)\rangle=\exp\left\{\int_{0}^{\infty}d\kappa\left(-\frac{1}{2}X(\kappa)^{\mu}\cdot X^{t}_{\mu}(\kappa)+\sqrt{2}iE^{\mu}_{\kappa}\cdot X^{t}_{\mu}(\kappa)+\frac{1}{2}E^{\mu}_{\kappa}\cdot E^{t}_{\kappa\mu}\right)\right\}|0\rangle,\tag{4.28}$$

$$|\tilde{x}^{\mu},\tilde{y}^{\mu}\rangle=\exp\left\{\left(-\frac{1}{2}\tilde{X}^{\mu}\cdot\tilde{X}^{t}_{\mu}+\sqrt{2}i\tilde{E}^{\mu}\cdot\tilde{X}^{t}_{\mu}+\frac{1}{2}\tilde{E}^{\mu}\cdot\tilde{E}^{t}_{\mu}\right)\right\}|0\rangle,\tag{4.29}$$

$$|x^{\bar{\alpha}}(\kappa),y^{\bar{\alpha}}(\kappa)\rangle=\exp\left\{\int_{0}^{\infty}d\kappa\left(-\frac{1}{2}X(\kappa)^{\bar{\alpha}}\cdot X^{t}_{\bar{\alpha}}(\kappa)+\sqrt{2}iE^{\bar{\alpha}}_{\kappa}\cdot X^{t}_{\bar{\alpha}}(\kappa)+\frac{1}{2}E^{\bar{\alpha}}_{\kappa}\cdot E^{t}_{\kappa\bar{\alpha}}\right)\right\}|0\rangle,\tag{4.30}$$

$$|\tilde{x}^{\bar{\alpha}},\tilde{y}^{\bar{\alpha}}\rangle=\exp\left\{\left(-\frac{1}{2}\tilde{X}^{\bar{\alpha}}\cdot\tilde{X}^{t}_{\bar{\alpha}}+\sqrt{2}i\tilde{E}^{\bar{\alpha}}\cdot\tilde{X}^{t}_{\bar{\alpha}}+\frac{1}{2}\tilde{E}^{\bar{\alpha}}\cdot\tilde{E}^{t}_{\bar{\alpha}}\right)\right\}|0\rangle.\tag{4.31}$$

Finally we are in a position to write down the commutation relations for the Moyal coordinates (for $\mu,\nu=0,3,\ldots,25$; $\alpha,\beta=1,2$; and $\kappa>0$):

$$[x^{\mu}(\kappa),\ y^{\nu}(\kappa')]_{*}=i2\tanh\frac{\pi\kappa}{4}G^{\mu\nu}\delta(\kappa-\kappa'),\tag{4.32}$$

$$[\tilde{x}^{\mu},\ \tilde{y}^{\nu}]_{*}=i2\sqrt{\frac{1+3\rho_0}{1-\rho_0}}G^{\mu\nu}, \tag{4.33}$$

$$[x^{\bar{\alpha}}(\kappa),\ y^{\bar{\beta}}(\kappa')]_{*}=i2\tanh\frac{\pi\kappa}{4}G^{\bar{\alpha}\bar{\beta}}\delta(\kappa-\kappa'), \tag{4.34}$$

$$[\tilde{x}^{\bar{1}},\ \tilde{y}^{\bar{1}}]_{*}=i2\sqrt{\frac{1+3\xi\lambda_1}{1-\xi\lambda_1}}G^{\bar{1}\bar{1}}, \tag{4.35}$$

$$[\tilde{x}^{\bar{2}},\ \tilde{y}^{\bar{2}}]_{*}=-i2\sqrt{\frac{1+3\xi\lambda_2}{1-\xi\lambda_2}}G^{\bar{2}\bar{2}}. \tag{4.36}$$

Let us discuss our results at $\kappa=0$. We have already seen from Ref. [15] that in the zero momentum sector without B background only the twist even eigenvectors survive; this makes two noncommutative coordinates collapse to a commutative one, which was interpreted as the momentum carried by half of the string. In our present case, we have the following remarks.

(1) For Neumann matrices with zero modes, according to Ref. [20], when $\kappa=0$ the determinant defined at this point is zero, so the eigenvector should be modified. They found two eigenvectors which are denoted as $u_{\pm,-(1/3)}$. However, it was pointed out in Ref. [16] that the double degeneracy at $-1/3$ in Ref. [20] is due to an improper normalization. At $-1/3$, the degeneracy of the eigenvector should be one which is $u_{+,-(1/3)}$. The explicit form of $u_{+,-(1/3)}$ shows that in the perpendicular dimensions, only $\hat{x}^{\mu}$ does not vanish; so only one Moyal coordinate for each perpendicular dimension will survive. The physical meaning of this coordinate is that it is proportional to the midpoint coordinate of string $X^{\mu}(\pi/2)$ [15,16].

(2) With a B background turned on, we have

$$v^{1}_{2n}(0)=0,\quad v^{2}_{2n+1}(0)=0, \tag{4.37}$$

at $\kappa=0$. Thus, in the parallel dimensions, only $\hat{y}^{\bar{\alpha}}$ survives, resulting in two commuting coordinates because of two spatial dimensions.

V. THE LARGE-B CONTRACTION

Witten [23] has pointed out that the study of noncommutative tachyon condensation is considerably simplified in the large-B limit. This is because in this limit the string field star algebra factorizes into two commuting sub-algebras as $\mathcal{A}=\mathcal{A}_0\otimes\mathcal{A}_1$. Here $\mathcal{A}_1$ is the algebra of functions in the noncommutative directions that acts only on the string center of mass, while $\mathcal{A}_0$ is the string field algebra in the zero-momentum sector if the B-field has the maximal rank. If B has less than maximal rank, then $\mathcal{A}_0$ may carry momentum, but only in commutative directions. Previously this statement has been proved in terms of either vertex operators [23] or oscillator modes [24]. Here we demonstrate this large-B contraction explicitly in our Moyal representation (4.32)–(4.36) of the string field algebra.

Suppose the first and second spatial dimensions are the only noncommutative directions. The key issue for the large-B contraction is the fate of the *two* discrete Moyal pairs in the two directions. (The commutation relations of continuous Moyal pairs are always the same as in the zero-momentum sector.) Certainly only one discrete Moyal pair gives rise to the center-of-mass noncommutative function algebra. We need to identify this pair and examine what happens to the other discrete Moyal pair. We will show that the latter simply drops out the contraction because of the singular behavior of the eigenvectors of the Neumann matrices at the corresponding eigenvalue: They simply vanish in the large-B limit.

Suppose the B field does not vanish only in the first and second spatial dimensions; we rewrite the string mode expansion in these two directions [Eq. (4.12)] as

$$\hat{X}^{\alpha}(\sigma)=\hat{x}^{\alpha}+\frac{\sigma}{\pi}\theta^{\alpha\beta}\hat{p}_{\beta}+\sqrt{2}\sum_{n=1}^{\infty}\left[\hat{x}^{\alpha}_{n}\cos n\sigma+\frac{1}{\pi}\theta^{\alpha\beta}\hat{p}_{n,\beta}\sin n\sigma\right], \tag{5.1}$$

where

$$\hat{x}^{\alpha}=\hat{x}^{\alpha}_{0}-\frac{1}{2}\theta^{\alpha\beta}\hat{p}_{0,\beta},\qquad \hat{p}^{\alpha}=\hat{p}^{\alpha}_{0}. \tag{5.2}$$

The commutation relations for the Fock space generators are

$$[x^{\alpha}_{m},\ p^{\beta}_{n}]=iG^{\alpha\beta}\delta_{mn},\quad m,n\geqslant 1$$
$$[\hat{x}^{\alpha},\ \hat{p}^{\beta}]=iG^{\alpha\beta},$$
$$[\hat{x}^{\alpha},\ \hat{x}^{\beta}]=i\theta^{\alpha\beta}. \tag{5.3}$$

Set $B=tB_0$ and take the limit $t\to\infty$. The open string parameters $G^{\alpha\beta},\theta^{\alpha\beta}$ scale like

$$G^{\alpha\beta}\sim G^{\alpha\beta}_{0}t^{-2},\quad \theta^{\alpha\beta}\sim\theta^{\alpha\beta}_{0}t^{-1}. \tag{5.4}$$

In order to get the contraction of the string field algebra, we should rescale the Fock space generators to make the commutation relations (5.3) have a definite limit as $B\to\infty$; so one chooses [24]

$$\hat{x}^{\alpha}_{m}\to\hat{x}^{\alpha}_{m}t^{-1},\quad \hat{p}^{\alpha}_{m}\to\hat{p}^{\alpha}_{m}t^{-1},\quad m\geqslant 1$$
$$\hat{x}^{\alpha}\to\hat{x}^{\alpha}t^{-1/2},\quad \hat{p}^{\alpha}\to\hat{p}^{\alpha}t^{-3/2}. \tag{5.5}$$

Note that our Moyal coordinates are transformed from string modes using eigenvectors of Neumann matrices; we should also investigate the behavior of these eigenvectors as $B\to\infty$. For the eigenvectors in the continuous spectrum,

$$v^{1}_{0}(\kappa)\sim t^{-4},\quad v^{1}_{n}(\kappa)\sim|\kappa\rangle_{n},$$

$$v^{2}_{0}(\kappa)\sim t^{-3},\quad v^{2}_{n}(\kappa)\sim t^{-3}\frac{1}{\rho(\kappa)-M}(|v_e\rangle+|v_o\rangle)_{n},$$

$$\tilde{v}_0^1(\kappa)\sim t^{-3}, \quad \tilde{v}_n^1(\kappa)\sim t^{-3}\frac{1}{\rho(\kappa)-M}(|v_e\rangle+|v_o\rangle)_n,$$

$$\tilde{v}_0^2(\kappa)\sim t^{-4}, \quad v_n^2(\kappa)\sim|\kappa\rangle_n, \tag{5.6}$$

where $|\kappa\rangle$ is the continuous eigenvector of the matrix M [14] and the notations $|v_{e,o}\rangle$ are given in Appendix A. For the eigenvectors in the discrete spectrum, there are some subtleties. It is easy to show that as $B\to\infty$, one eigenvalue approaches unity, while the other approaches zero as e^{-e^B}. It can be shown that the eigenvector with the eigenvalue approaching zero will become vanishing as $B\to\infty$. We leave the proof to Appendix B 2. The eigenvector with the eigenvalue approaching unity scales like

$$v_0^1(1)\sim 1, \quad v_n^1(1)\sim t^{-2}\frac{1}{1-M}|v_e\rangle_n,$$

$$v_0^2(1)=0, \quad v_n^2(1)\sim t^{-1}\frac{1}{1-M}|v_o\rangle_n. \tag{5.7}$$

Thus only one Moyal pair of coordinates in the discrete spectrum survives the large-B contraction.

Keeping only the leading order in t, Eqs. (4.16)–(4.21) change to

$$\hat{x}_\kappa^1=t^{-1}\sqrt{2}\sum_{n=1}^{\infty}|\kappa\rangle_{2n}\sqrt{2n}\hat{x}_{2n}^1,$$

$$\hat{y}_\kappa^1=-t^{-1}\sqrt{2}\sum_{n=1}^{\infty}\frac{|\kappa\rangle_{2n+1}}{\sqrt{2n+1}}\hat{p}_{2n+1}^1, \tag{5.8}$$

$$\hat{x}_\kappa^2=t^{-1}\sqrt{2}\sum_{n=1}^{\infty}|\kappa\rangle_{2n}\sqrt{2n}\hat{x}_{2n}^2,$$

$$\hat{y}_\kappa^2=-t^{-1}\sqrt{2}\sum_{n=1}^{\infty}\frac{|\kappa\rangle_{2n+1}}{\sqrt{2n+1}}\hat{p}_{2n+1}^2, \tag{5.9}$$

$$\hat{\tilde{x}}^1=t^{-1/2}\frac{2}{\sqrt{b}}\hat{x}_0^1, \quad \hat{\tilde{y}}^1=t^{-1/2}\frac{2}{\sqrt{b}}\hat{x}_0^2. \tag{5.10}$$

Now, we are ready to read off the large B contraction of our Moyal representation of string field algebra (4.32)–(4.36) (for $\mu,\nu=0,3,\ldots,25$; $\alpha,\beta=1,2$; and $\kappa>0$):

$$[x^\mu(\kappa),\ y^\nu(\kappa')]_*=i2\tanh\frac{\pi\kappa}{4}G^{\mu\nu}\delta(\kappa-\kappa'), \tag{5.11}$$

$$[\tilde{x}^\mu,\ \tilde{y}^\nu]_*=i2\sqrt{\frac{1+3\rho_0}{1-\rho_0}}G^{\mu\nu}, \tag{5.12}$$

$$[x^\alpha(\kappa),\ y^\beta(\kappa')]_*=i2\tanh\frac{\pi\kappa}{4}G_0^{\alpha\beta}\delta(\kappa-\kappa'), \tag{5.13}$$

$$[x_0^1,\ x_0^2]_*=i\frac{1}{2}. \tag{5.14}$$

The commutation relations (5.11)–(5.13) are those for the sub-algebra $\mathcal{A}_0$ and the relation (5.14) is for the sub-algebra $\mathcal{A}_1$. We note that after recovering string tension, the noncommutativity parameter for the center-of-mass function algebra is nothing but α'.

VI. SUMMARY OF RESULTS AND DISCUSSIONS

In this paper we have found that the commuting Moyal-pair structure of Witten's star algebra for open string fields persists in the presence of a constant B background. We have worked out explicitly a new basis in which Witten's three-string vertex for string fields is diagonalized, and identified the commuting Moyal pairs and the corresponding noncommutativity parameters. The full set of commutation relations for Witten's star algebra in the Moyal representations are summarized by Eqs. (4.32)–(4.36).

A central issue is the interplay between the noncommutativity due to three-string overlap and that due to a background B field. It is known [15] that in the zero-momentum sector, i.e. without string zero modes, the commuting Moyal pairs for Witten's star algebra are labeled by a continuous parameter $\kappa\in[0,\infty]$. If the strong zero modes are included, then besides those commuting ones labeled by κ there is an extra Moyal pair commuting with them [16]. In our present case, in the presence of a B background, the situation in the commutative directions is the same as before. But in each pair of noncommutative directions in which the block-diagonal B field has non-vanishing components, besides the commuting Moyal pairs labeled by κ there are *two* extra Moyal pairs commuting with them and with each other. As for noncommutativity parameters, only those between the discrete Moyal pairs are B dependent, while the Moyal parameters between the continuous pairs are the same as the case in the zero momentum sector. However, we note that the transformations, Eqs. (4.16)–(4.21), from the oscillator modes to the Moyal pairs are B dependent both for the continuous and discrete ones.

Moreover, we have studied the large-B contraction for Witten's star algebra. Indeed, we have confirmed Witten's statement [23], in the Moyal representation, that the large-B contraction consists of two commuting sub-algebras: one is the ordinary noncommutative function algebra for the center of mass of the string in the noncommutative directions while the other commuting sub-algebra consists of the star algebra for the zero momentum sector in the noncommutative direction and that for momenta in commutative directions. Our contribution to this topic is that we have clarified the fate of the other discrete Moyal pair in the noncommutative directions: It drops out the contraction because of the singular behavior of the corresponding eigenvalue of the Neumann matrices: It simply moves out the spectrum; in other words, the corresponding eigenvectors become vanishing in this limit.

We would like to emphasize the following points concerning some of the details.

(1) The θ spectrum (noncommutativity) for the continuous Moyal pairs is positive and always bounded from above (less than 2). In contrast, the noncommutativity parameters for the two discrete pairs have opposite sign, one of them is positive and bounded from below (larger than 2), while the other is negative and bounded from above (less than -2). Presumably this is related to the fact that the two end points of the open string are "oppositely charged."

(2) In the presence of a background B field, the Moyal pairs mix the string modes in the noncommutative directions parallel to the B field. It is interesting to note that, for example, x^1_{2n} is mixed with p^2_{2n+1}. Suppressing the mode index, this is the correct structure for the guiding center coordinates of a charged particle in a magnetic field.

(3) When the continuous parameter κ approaches zero, only one Moyal coordinate for each "dimension" survives. It is either x^μ or $y^{\bar{\alpha}}$ that gives a commuting coordinate. This feature does not depend on whether the zero modes and a B background are included or not.

(4) As we mentioned above, the large-B contraction is singular in one aspect: Namely one of the discrete Moyal pairs does not survive this contraction. We feel perhaps some caution has to be taken for some arguments that exploit the large-B limit in the open string field theory. In other words, at finite and large B for such arguments to work, one needs to show that the contributions from the dropped-out modes are indeed negligible.

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APPENDIX A: THE PROPERTIES OF NEUMANN MATRICES

In this appendix we list the properties of Neumann matrices $\mathcal{V}^{rs,\alpha\beta}$ and $\mathcal{M}^{rs,\alpha\beta}$, which are quoted from Refs. [19,21] $(m,n>0)$:

$$\mathcal{V}^{rs,\alpha\beta}_{00}=G^{\alpha\beta}\delta^{rs}-\Omega b(G^{\alpha\beta}\phi^{rs}+\Xi\epsilon^{\alpha\beta}\chi^{rs}),$$

$$\mathcal{V}^{rs,\alpha\beta}_{0n}=\Omega\sqrt{b}\sum_{t=1}^{3}(G^{\alpha\beta}\phi^{rt}+\Xi\epsilon^{\alpha\beta}\chi^{rt})V^{ts}_{0n},\tag{A1}$$

$$\mathcal{V}^{rs,\alpha\beta}_{mn}=G^{\alpha\beta}V^{rs}_{mn}-\Omega\sum_{t,v=1}^{3}V^{rv}_{m0}(G^{\alpha\beta}\phi^{vt}+\Xi\epsilon^{\alpha\beta}\chi^{vt})V^{rs}_{0n},$$

where V^{rs}_{mn} are the Neumann matrices in the zero momentum sector, and V^{rs}_{m0} and V^{rs}_{0m} can be written in the following form [30,31]:

$$V^{rr}_{n0}=-\frac{2\sqrt{2}}{3}|v_e\rangle,\quad V^{rr}_{0n}=-\frac{2\sqrt{2}}{3}\langle v_e|,$$

$$V^{21}_{n0}=\frac{\sqrt{2}}{3}|v_e\rangle+\frac{\sqrt{6}}{3}|v_o\rangle,$$

$$V^{12}_{0n}=\frac{\sqrt{2}}{3}\langle v_e|+\frac{\sqrt{6}}{3}\langle v_o|,$$

$$V^{12}_{n0}=\frac{\sqrt{2}}{3}|v_e\rangle-\frac{\sqrt{6}}{3}|v_o\rangle,$$

$$V^{21}_{0n}=\frac{\sqrt{2}}{3}\langle v_e|-\frac{\sqrt{6}}{3}\langle v_o|,\tag{A2}$$

with $|v_{e,o}\rangle$ defined as

$$|v_e\rangle_n=\frac{1}{\sqrt{n}}\frac{[1+(-1)^n]}{2}A_n,\quad |v_o\rangle_n=\frac{1}{\sqrt{n}}\frac{[1-(-1)^n]}{2}A_n,\tag{A3}$$

where A_n is the coefficients of the series expansion

$$\left(\frac{1+ix}{1-ix}\right)^{1/3}=\sum_{n=\text{even}}A_nx^n+i\sum_{n=\text{odd}}A_nx^n.\tag{A4}$$

The other quantities in Eqs. (A1) are

$$\Omega=\frac{2\beta}{4\pi^4\alpha'^4B^2+3\beta^2},\quad \Xi=i\frac{\pi^2\alpha'^2B}{\xi\beta},\quad \beta=\ln\frac{27}{16}+\frac{b}{2},\tag{A5}$$

and

$$\chi^{rs}=\begin{pmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{pmatrix},\quad \phi^{rs}=\begin{pmatrix}1&-\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&1&-\frac{1}{2}\\-\frac{1}{2}&-\frac{1}{2}&1\end{pmatrix}.\tag{A6}$$

We have

$$(\mathcal{V}^{rs,\alpha\beta})^t=\mathcal{V}^{sr,\beta\alpha},\quad (\mathcal{M}^{rs,\alpha\beta})^t=\mathcal{M}^{rs,\beta\alpha},\tag{A7}$$

$$C\mathcal{V}^{rs,\alpha\beta}=\mathcal{V}^{sr,\beta\alpha}C,\quad C\mathcal{M}^{rs,\alpha\beta}=\mathcal{M}^{sr,\beta\alpha}C,\tag{A8}$$

$$[\mathcal{V}^{rs},\mathcal{V}^{r's'}]=0=[\mathcal{M}^{rs},\mathcal{M}^{r's'}],\tag{A9}$$

and

$$\mathcal{M}^{rs}=\mathcal{M}^{r+1,s+1}\quad (r,s\mod 3),\tag{A10}$$

$$\mathcal{M}^{11}+\mathcal{M}^{12}+\mathcal{M}^{21}=\frac{1}{\xi}\mathbf{I},\tag{A11}$$

$$(\mathcal{M}^{11})^2+(\mathcal{M}^{12})^2+(\mathcal{M}^{21})^2=\frac{1}{\xi^2}\mathbf{I},\tag{A12}$$

$$\mathcal{M}^{12}\mathcal{M}^{21}=\mathcal{M}^{11}\left(\mathcal{M}^{11}-\frac{1}{\xi}\mathbf{I}\right), \quad \text{(A13)}$$

where $\mathbf{I}$ stands for $\delta^{\alpha\beta}\delta_{mn}$ and matrix multiplication is understood both for the indices m,n and α,β. It is easy to get the following equations from the above properties:

$$\mathcal{M}^{12}+\mathcal{M}^{21}=\frac{1}{\xi}\mathbf{I}-\mathcal{M}^{11},$$

$$(\mathcal{M}^{12}-\mathcal{M}^{21})^2=\left(\frac{1}{\xi}\mathbf{I}+3\mathcal{M}^{11}\right)\left(\frac{1}{\xi}\mathbf{I}-\mathcal{M}^{11}\right). \quad \text{(A14)}$$

If we set $B=0$, the expressions and the properties of matrices $\mathcal{V}^{rs},\mathcal{M}^{rs}$ will recover those of V'^{rs},M'^{rs}.

APPENDIX B: THE SMALL AND LARGE B LIMIT OF EIGENVECTORS

These two limiting case for the continuous eigenvectors are straightforward; in this appendix, we will concentrate on the discrete spectrum. The two discrete eigenvalues are determined by the following equations, respectively, with $x \equiv \xi\lambda$:

$$2h(x)=\mp 4B\pi^2\sqrt{\frac{1-x}{1+3x}}+b-4[\gamma+\log(4)]. \quad \text{(B1)}$$

x_1 is the solution of Eq. (B1) with the "$-$" sign in it and x_2 is the solution with the "$+$" sign. The eigenvectors for them are (quoted from Ref. [21])

$$X_e(j)=i\left(-\frac{2\sqrt{6b}\Omega\Xi\xi}{3(\xi\lambda_j-1+\Omega b)},\frac{\xi d_{oe}}{\xi\lambda_j-M}|v_e\rangle,0,\frac{\xi d_{oo}-\xi A_{ee}(d_{oo}d_{ee}+d_{oe}^2)}{\xi\lambda_j-M}|v_o\rangle\right)^t,$$

$$X_o(j)=\left(0,-\frac{\xi d_{oo}-\xi A_{ee}(d_{oo}d_{ee}+d_{oe}^2)}{\xi\lambda_j-M}|v_o\rangle,-\frac{2\sqrt{6b}\Omega\Xi\xi}{3(\xi\lambda_j-1+\Omega b)},\frac{\xi d_{oe}}{\xi\lambda_j-M}|v_e\rangle\right)^t,$$

where $(\alpha'=1)$

$$\Omega=\frac{2\beta}{4\pi^4B^2+3\beta^2}, \quad \Xi=i\frac{\pi^2}{\xi\beta}B,$$

$$d_{oo}=-\frac{2\Omega}{3\xi}\left[-3+\frac{4b\Omega\Xi^2\xi^2}{\xi\lambda-1+\Omega b}\right],$$

$$d_{oe}=-\frac{4\Omega\Xi(\xi\lambda-1)}{\sqrt{3}(\xi\lambda-1+\Omega b)},$$

$$d_{ee}=-\frac{2\Omega(x-1)}{\xi(x-1+\Omega b)},$$

$$A_{ee}=-\frac{\xi}{4(x-1)}\{9(x-1)\ln 3+16\ln 2 +(1+3x)[h(x)+2\gamma]\}. \quad \text{(B2)}$$

1. The small-B limit

Let us consider the small-B limit and expand the discrete eigenvalue to first order in B

$$x_i(B)=\xi\lambda_i(B)=(\xi\lambda_i)|_{B=0}+B(\xi\lambda_i)'|_{B=0}+O(B^2) =\rho_0+Bx_i^{(1)}+O(B^2). \quad \text{(B3)}$$

We also expand the function $h(x)$ around ρ_0 to first order in B,

$$h(x)=h(\rho_0)+(x-\rho_0)h'(\rho_0)+\frac{1}{2}(x-\rho_0)^2h''(\rho_0)+\cdots =\frac{1}{2}b-2[\gamma+\log(4)]+Bx^{(1)}h'(\rho_0)+O(B^2). \quad \text{(B4)}$$

Expanding the right-hand side (RHS) of Eq. (B1) to first order in B and comparing it with Eq. (B4), it is easy to get

$$h'(\rho_0)x^{(1)}=\begin{cases}-2\pi^2\sqrt{\dfrac{1-\rho_0}{1+3\rho_0}} & \text{for } \lambda_1,\\ +2\pi^2\sqrt{\dfrac{1-\rho_0}{1+3\rho_0}} & \text{for } \lambda_2.\end{cases} \quad \text{(B5)}$$

Now let us consider the behavior of the eigenvectors in this limit. We have

$$\Omega=\frac{2}{3\beta}+O(B^2), \quad \Xi=i\frac{\pi^2}{\beta}B+O(B^3),$$

$$d_{oo}=\frac{4}{3\beta}+O(B^2),$$

$$d_{oe}=-i\frac{8\pi^2(\rho_0-1)}{\sqrt{3}\beta(3\beta[\rho_0-1)+2b]}B+O(B^2),$$

$$1-d_{ee}A_{ee}=\frac{1+3\rho_0}{3\beta(\rho_0-1)+2b}h'(\rho_0)x^{(1)}B+O(B^2). \quad \text{(B6)}$$

From Eq. (B5) we see that the leading term of $1-d_{ee}A_{ee}$ has an opposite sign for λ_1 and λ_2. Comparing with the expressions of ϕ_n [Eqs. (6.6),(6.7) in Ref. [20]], we get

$$\lim_{B\to 0} v_{2n}^1(1)=\phi_{2n}=\lim_{B\to 0} v_{2n}^1(2),$$

$$\lim_{B\to 0} v_{2n+1}^2(1)=-i\phi_{2n+1}=-\lim_{B\to 0} v_{2n+1}^2(2), \tag{B7}$$

where $n\geqslant 0$.

2. The large-*B* limit

In the large-B limit, x_1 approaches 1 and x_2 approaches 0. For x_1, one has

$$x_1=1-x_1^{(1)}\frac{1}{B^2}+\cdots,$$

$$h(x_1)=h(1)+(x_1-1)h'(1)+\cdots. \tag{B8}$$

Using Eq. (B1), we obtain

$$x_1=1-\frac{b^2}{4\pi^4}\frac{1}{B^2}. \tag{B9}$$

For $x_2\sim 0$, we have $h(x\sim 0)\sim 2\ln[-\ln(|x|)]$; substituting it into Eq. (B1) we get, up to a constant factor,

$$x_2=e^{-e^{\pi^2 B}}. \tag{B10}$$

Now, let us consider the eigenvectors [here we only consider $X_e(j)$, $j=1,2$; the result for $X_o(j)$ can be obtained similarly]. For x_1, it is easy to get

$$X_e(1)=\left(1,\frac{\sqrt{2}b^{3/2}}{2\pi^4}\frac{1}{B^2}\frac{1}{1-M}|v_e\rangle,0,i\frac{\sqrt{6b}}{3\pi^2}\frac{1}{B}\frac{1}{1-M}|v_o\rangle\right)^t. \tag{B11}$$

For $x_2\sim 0$, we need to examine the vector $|V_{e,o}^{x_2}\rangle\equiv[1/(x_2-M)]|v_{e,o}\rangle$ carefully. First, let us calculate the modulus of the vector $|V_o^{x_2}\rangle$:

$$\begin{aligned}I_o(2)&\equiv\langle v_o|\left(\frac{1}{x_2-M}\right)^2|v_o\rangle\\&=\int_{-\infty}^{\infty}\frac{d\kappa}{\mathcal{N}(\kappa)}\langle v_o|\frac{1}{x_2-M}|\kappa\rangle\langle\kappa|\frac{1}{x_2-M}|v_o\rangle\\&=\int_{-\infty}^{\infty}d\kappa\frac{1}{[x_2-M(\kappa)]^2}\frac{1}{\mathcal{N}(\kappa)}\langle v_o|\kappa\rangle^2\\&=\frac{3}{2}\int_{-\infty}^{\infty}d\kappa\frac{\sinh\left(\frac{\pi\kappa}{2}\right)}{\kappa\left[1+x_2+2x_2\cosh\left(\frac{\pi\kappa}{2}\right)\right]^2}\\&=\frac{3}{2}\int_{-\infty}^{\infty}dt\frac{\sinh(t)}{t[1+x_2+2x_2\cosh(t)]^2}.\end{aligned} \tag{B12}$$

Summing over all the residues of poles $t_n=2\pi i[n+\frac{1}{2}\pm i(\eta/2\pi)],n\geqslant 0$ with $\eta=\cosh^{-1}[(1+x_2)/2x_2]$ yields

$$\begin{aligned}I_o(2)&=\frac{3}{2\pi^2}\frac{\eta\sinh(\eta)}{(1-x_2)(1+3x_2)}\times\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{\left[\left(n+\frac{1}{2}\right)^2+(\eta^2/4\pi^2)\right]^2}\\&=\frac{3\eta}{4\pi^2x_2}\sqrt{(1-x_2)(1+3x_2)}S_1(\eta).\end{aligned} \tag{B13}$$

Similarly, we can calculate the modulus of the vector $|V_e^{x_2}\rangle$:

$$\begin{aligned}I_e(2)&\equiv\langle v_e|\left(\frac{1}{x_2-M}\right)^2|v_e\rangle\\&=\frac{1}{2}\int_{-\infty}^{\infty}dt\frac{[\cosh(t)-1]^2}{t\sinh(t)[1+x_2+2x_2\cosh(t)]^2}.\end{aligned} \tag{B14}$$

Summing over all the residues of the poles $t_n=2\pi i(n+\frac{1}{2})$, $2\pi i[n+\frac{1}{2}\pm i(\eta/2\pi)]$, $n\geqslant 0$ yields

$$\begin{aligned}I_e(2)&=\frac{1}{(1-x_2)^2}\left\{-\frac{\eta^2}{2\pi^2}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)\left[\left(n+\frac{1}{2}\right)^2+(\eta^2/4\pi^2)\right]}+\frac{\eta}{4\pi^2x_2}\sqrt{(1-x_2)(1+3x_2)}\times\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{\left[\left(n+\frac{1}{2}\right)^2+(\eta^2/4\pi^2)\right]^2}\right\}\\&=\frac{1}{(1-x_2)^2}\left\{\frac{\eta}{4\pi^2x_2}\sqrt{(1-x_2)(1+3x_2)}S_1(\eta)-\frac{\eta^2}{2\pi^2}S_2(\eta)\right\}.\end{aligned} \tag{B15}$$

Let us estimate the summation $S_1(\eta)$ as follows:

$$\frac{\frac{1}{2}}{\left[\frac{1}{4}+(\eta^2/4\pi^2)\right]^2}<S_1(\eta)<\frac{\frac{1}{2}}{\left[\frac{1}{4}+(\eta^2/4\pi^2)\right]^2}+\int_0^\infty dx\frac{x+\frac{1}{2}}{\left[\left(x+\frac{1}{2}\right)^2+(\eta^2/4\pi^2)\right]^2}$$

$$\frac{8\pi^4}{(\pi^2+\eta^2)^2}<S_1(\eta)<\frac{8\pi^4}{(\pi^2+\eta^2)^2}+\frac{2\pi^2}{\pi^2+\eta^2}, \tag{B16}$$

and the summation $S_2(\eta)$

$$\frac{1}{\frac{1}{2}\left(\frac{1}{4}+(\eta^2/4\pi^2)\right)}<S_2(\eta)<\frac{1}{\frac{1}{2}\left[\frac{1}{4}+(\eta^2/4\pi^2)\right]}+\int_0^\infty dx\frac{1}{\left(x+\frac{1}{2}\right)\left[\left(x+\frac{1}{2}\right)^2+(\eta^2/4\pi^2)\right]}$$

$$\frac{8\pi^2}{(\pi^2+\eta^2)}<S_2(\eta)<\frac{8\pi^2}{(\pi^2+\eta^2)}+\frac{4\pi^2}{\eta^2}\ln\left(1+\frac{\eta^2}{\pi^2}\right). \tag{B17}$$

So, for $x_2=e^{-e^{\pi^2 B}}\sim 0$, we see that

$$I_o(2)\sim e^{e^{\pi^2 B}},\quad I_e(2)\sim e^{e^{\pi^2 B}}. \tag{B18}$$

The normalized eigenvector for x_2 is then

$$X_e(2)\sim\left(0,2\sqrt{3}\,\frac{e^{-(1/2)e^{\pi^2 B}}}{x_2-M}|v_e\rangle,0,i\frac{e^{-(1/2)e^{\pi^2 B}}}{x_2-M}|v_o\rangle\right)^t. \tag{B19}$$

Expanding it in the $|\kappa\rangle$ basis, we have

$$\frac{e^{-(1/2)e^{\pi^2 B}}}{x_2-M}|v_o\rangle=e^{-(1/2)e^{\pi^2 B}}\int_{-\infty}^{\infty}\frac{d\kappa}{\mathcal{N}(\kappa)}|\kappa\rangle\langle\kappa|\frac{1}{x_2-M}|v_o\rangle$$
$$=e^{-(1/2)e^{\pi^2 B}}\int_{-\infty}^{\infty}d\kappa\sqrt{\frac{3\sinh(\pi\kappa/2)}{2\kappa}}\frac{1}{1+x_2+2x_2\cosh(\pi\kappa/2)}|\widetilde{\kappa}\rangle,$$

$$\frac{e^{-\frac{1}{2}e^{\pi^2 B}}}{x_2-M}|v_e\rangle=e^{-(1/2)e^{\pi^2 B}}\int_{-\infty}^{\infty}\frac{d\kappa}{\mathcal{N}(\kappa)}|\kappa\rangle\langle\kappa|\frac{1}{x_2-M}|v_e\rangle$$
$$=e^{-(1/2)e^{\pi^2 B}}\int_{-\infty}^{\infty}d\kappa\tanh(\pi\kappa/4)\sqrt{\frac{\sinh(\pi\kappa/2)}{2\kappa}}\frac{1}{1+x_2+2x_2\cosh(\pi\kappa/2)}|\widetilde{\kappa}\rangle, \tag{B20}$$

where $|\widetilde{\kappa}\rangle$ is the normalized eigenvector of the Neumann matrix M. We see that the larger the κ, the bigger the coefficient in front of the basis vector $|\widetilde{\kappa}\rangle$. However, for fixed κ, the coefficient in front of the basis vector $|\widetilde{\kappa}\rangle$ goes to zero as $B\to\infty$. So we conclude that as $B\to\infty$, the contribution can only come from $\kappa=\infty$, which is, however, known to be not in the spectrum of M [14]. In other words, since the Neumann matrix M has no nontrivial eigenvector with eigenvalue $\kappa=\infty$, the normalized vectors $X_{e,o}(2)$ will move out the string Hilbert space as $B\to\infty$. (The non-normalized vector will have all components vanishing in the limit.) Thus, only the eigenvalue x_1 survives the large-B limit, while the behavior of the eigenvalue x_2 is singular: It moves out the spectrum.

APPENDIX C: THE DISCRETE EIGENVECTORS OF $\mathcal{M}^{12}(\mathcal{M}^{21})$

Let us prove that

$$\mathcal{M}^{rs}X_\pm(j)=\lambda^{rs}_{j,\pm}X_\pm(j), \tag{C1}$$

where $X_\pm(j)=X_e(j)\pm X_o(j)$, $j=1,2$. First, from Eq. (A7), we know

$$\mathcal{M}^{11}(\mathcal{M}^{12}-\mathcal{M}^{21})X_{e,o}(j)=(\mathcal{M}^{12}-\mathcal{M}^{21})\mathcal{M}^{11}X_{e,o}(j)$$
$$=\lambda_j^{11}(\mathcal{M}^{12}-\mathcal{M}^{21})X_{e,o}(j). \tag{C2}$$

So, $(\mathcal{M}^{12}-\mathcal{M}^{21})X_{e,o}(j)$ should be a linear combination of the eigenvectors of $\mathcal{M}^{11}$, i.e.,

$$(\mathcal{M}^{12}-\mathcal{M}^{21})X_e(j)=a_{1j}X_e(j)+b_{1j}X_o(j),$$
$$(\mathcal{M}^{12}-\mathcal{M}^{21})X_o(j)=a_{2j}X_e(j)+b_{2j}X_o(j). \tag{C3}$$

Using $C\mathcal{M}^{rs,\alpha\beta}C=\mathcal{M}^{sr,\beta\alpha}$ and expanding Eq. (C3) explicitly, we can get immediately

$$a_{1j}=b_{2j}=0, \quad a_{2j}=b_{1j}=\eta_j. \tag{C4}$$

Thus, Eqs. (C3) now read

$$(\mathcal{M}^{12}-\mathcal{M}^{21})X_e(j)=\eta_j X_o(j),$$
$$(\mathcal{M}^{12}-\mathcal{M}^{21})X_o(j)=\eta_j X_e(j), \tag{C5}$$

and we can further get

$$(\mathcal{M}^{12}-\mathcal{M}^{21})^2X_{e,o}(j)=\eta_j^2X_{e,o}(j). \tag{C6}$$

On the other hand, we know that $(\mathcal{M}^{12}-\mathcal{M}^{21})^2=[(1/\xi)+3\mathcal{M}^{11}][(1/\xi)-\mathcal{M}^{11}]$, so

$$\eta_j^2=\left(\frac{1}{\xi}+3\lambda_j\right)\left(\frac{1}{\xi}-\lambda_j\right). \tag{C7}$$

Thus,

$$\mathcal{M}^{12}X_+(j)=\left[\frac{1}{2}(\mathcal{M}^{12}-\mathcal{M}^{21})+\frac{1}{2}(\mathcal{M}^{12}+\mathcal{M}^{21})\right]$$
$$\times[X_e(j)+X_o(j)]$$
$$=\left[\frac{1}{2}\eta_j+\frac{1}{2}\left(\frac{1}{\xi}-\lambda_j\right)\right][X_e(j)+X_o(j)]$$
$$=\lambda_{j,+}^{12}X_+(j). \tag{C8}$$

Similarly, we get

$$\mathcal{M}^{12}X_-(j)=\lambda_{j,-}^{12}X_-(j), \tag{C9}$$

$$\mathcal{M}^{21}X_+(j)=\lambda_{j,+}^{21}X_+(j), \tag{C10}$$

$$\mathcal{M}^{21}X_-(j)=\lambda_{j,-}^{21}X_-(j), \tag{C11}$$

where

$$\lambda_{j,\pm}^{12}=\pm\frac{1}{2}\eta_j+\frac{1}{2}\left(\frac{1}{\xi}-\lambda_j\right), \tag{C12}$$

$$\lambda_{j,\pm}^{21}=\mp\frac{1}{2}\eta_j+\frac{1}{2}\left(\frac{1}{\xi}-\lambda_j\right). \tag{C13}$$

In order to determine the sign in front of $\sqrt{\eta_j^2}$, first we take the limit $B\to 0$ in Eqs. (C5):

$$\sum_{n=0}^{\infty} M_{2m,2n+1}^{12}[iv_{2n+1}^2(j)]_{B=0}=[\eta_j v_{2m}^1(j)]_{B=0},$$

$$\sum_{n=0}^{\infty} M_{2m+1,2n}^{12}[v_{2n}^1(j)]_{B=0}=\{\eta_j[iv_{2m+1}^2(j)]\}_{B=0}.$$

Recall Eqs. (B7); we obtain

$$\eta_1|_{B=0}=-\eta_2|_{B=0}. \tag{C14}$$

Then taking the limit $B\to 0$ in Eq. (C12) and comparing it with Eq. (2.11), we get

$$\eta_1=\frac{1}{2}\sqrt{\left(\frac{1}{\xi}+3\lambda_1\right)\left(\frac{1}{\xi}-\lambda_1\right)}. \tag{C15}$$

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