# Phonons and Phase Transitions of Helium*

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We find a new collective shear-wave–like mode above $T_\lambda$, merging with the longitudinal acoustic mode at a higher critical temperature. The speed of the new sound wave is predicted to vanish near $T_\lambda$ as $s \propto (T-T_\lambda)^{\frac{1}{2}}$.

Whereas the ideal Bose-Einstein gas sheds little light on the complex properties of liquid helium, beyond exhibiting a third-order phase transition to a condensed phase[1] (at, however, the wrong temperature), by contrast the weakly nonideal fluid displays a wealth of dynamical and thermodynamical structure. By studying collective modes we have been able to identify a gas phase, a normal fluid phase, and a superfluid phase. The present study concentrates on the higher-temperature phases, corresponding to He I at and above $T_\lambda$, and the transition to a gas phase at $T_b$. In the fluid phase we obtain two solutions corresponding to a longitudinal and a transverse mode of acoustic propagation. The latter seems related to the anomalies at the superfluid transition temperature, for its speed of propagation vanishes at $T_\lambda$ and fits the law

$$s \propto (T-T_\lambda)^{1/2} \quad (T \gtrsim T_\lambda). \qquad (1)$$

Thus at $T_\lambda$ the system becomes unstable against an arbitrary number of long-wavelength excitations of this shear mode, at little cost in energy. Interestingly enough, this mode appears not to have been detected experimentally, but the reasons for this may be related either to the difficulties in observing an ultraslow dispersive wave following a relatively fast nondispersive longitudinal signal, or to the lack of any previous theoretical guidelines concerning the existence of this mode, rather than to the perverse nonexistence of such a phenomenon.

We study the usual Hamiltonian,

$$\mathcal{H} = \mu N + \sum (k^2/2m - \mu) n_k + \sum V_q a_{k'-q}^\dagger a_{k+q}^\dagger a_k a_{k'} \quad (\hbar = 1), \quad (2)$$

merely to lowest nontrivial order in the interaction strength $V_q$.

In the random-phase approximation,[2] one has

$$[a_{k+q}^\dagger a_k, \mathcal{H}] = (\epsilon_k - \epsilon_{k+q}) a_{k+q}^\dagger a_k + 2V_q \langle n_{k+q} - n_k \rangle \sum_{k'} a_{k'+q}^\dagger a_{k'}, \quad (3)$$

where $\epsilon_k = \epsilon_0 + k^2/2m^* + O(k^4)$. Formulas for the self-consistent evaluation of $m^*$ and of $\langle n \rangle$ are given in the recent book of Fetter and Walecka.[3] The important result is

$$\langle n_k \rangle = f(\epsilon_k) = [\exp(\beta\epsilon_0)\exp(\beta k^2/2m^*) - 1]^{-1}, \quad (4)$$

where $\beta \equiv 1/kT$. The quantity $\beta\epsilon_0(T)$ is then a parameter, required to satisfy conservation of particles:

$$N = N_0 + \frac{\Omega}{(2\pi)^3} 4\pi \left(\frac{2m^* kT}{\hbar^2}\right)^{3/2} \times \int_0^\infty dx\, x^2 [e^{\beta\epsilon_0} e^{x^2} - 1]^{-1}. \quad (5)$$

We note merely that $\beta\epsilon_0$ vanishes below $T_\lambda$ and is an increasing function of $T$ above this temperature. Conversely, $N_0$ has a maximum value ($N$) at $T = 0$ and decreases to zero at $T_\lambda$, remaining zero above $T_\lambda$. In our work no consideration is given to the Bogoliubov pairs nor to their effect on the depletion of the condensate $N_0$ below $T_\lambda$, matters which have received considerable attention in the literature.[4] We are presently mainly concerned with the dynamic properties of the fluid above $T_\lambda$.

For two-particle properties we study the Green's functions[5]:

$$G_{kk',q}(\omega) \equiv \langle\langle a_{k+q}^\dagger a_k | a_{k'}^\dagger a_{k'+q} \rangle\rangle. \quad (6)$$

Using (3) we can solve their equation of motion,

$$2\pi G_{kk',q}(\omega) = \delta_{k,k'} G_{k,q}^{(0)} + G_{k,q}^{(0)} T_q G_{k',q}^{(0)}, \quad (7)$$

in which

$$G_{k,q}^{(0)}(\omega) \equiv \frac{\langle n_{k+q} - n_k \rangle}{\omega - \epsilon_k + \epsilon_{k+q}}, \quad (8)$$

and we define $T_q$ and the associated quantity $S_q$ via

$$T_q \equiv \frac{2V_q}{1 - 2V_q \sum_k G_{k,q}^{(0)}} \equiv \frac{2V_q}{1 - 2V_q S_q(\omega)}. \quad (9)$$

Armed with the above, we can compute the internal energy $U = \langle \mathcal{H} \rangle$, and thereby all the thermodynamic functions. After some elementary manipulations, we find

$$U = \mu N + \sum_k \epsilon_k f(\epsilon_k) - 2\sum_{kk'} V_{k-k'} f(\epsilon_k)[\tfrac{1}{2} + f(\epsilon_{k'})] + U_{\text{correl}}, \quad (10a)$$

$$U_{\text{correl}} \equiv \frac{1}{2\pi} \sum_q \int d\omega f(\omega) \times \operatorname{Im}\left[\frac{1}{1 - 2V_q S_q(\omega - i0^+)}\right]. \quad (10b)$$

To the extent that $T_q$ has sharp resonances or poles, the correlation energy cannot be expanded in powers of $V_q$. We shall find that at a critical temperature $T_b > T_\lambda$ these resonances disappear, and the correlation energy could then, in principle, be expanded in powers of $V_q$ above $T_b$. For this reason we have tentatively identified $T_b$, the point at which the analytic properties of $U_{\text{correl}}$ changes, as the boiling point. Owing to the finite value of $\operatorname{Im}(S_q)$ in our theory there will be no discontinuity in $U_{\text{correl}}$ at $T_b$ but only a "bump" in the specific heat. This is because $T_q$ has no sharp poles on the real axis, but only resonances. We can only conjucture that, were we able to introduce weak Van der Waals-like attractive forces into the theory, the "bump" in specific heat might sharpen into a veritable latent heat, and the curve $p(T_b)$, the locus of this first-order singularity, would properly terminate at a critical point $T_c$ (where $kT_c$ might be a measure of the depth of the attractive potential). At present there is no convenient way to introduce attractive forces into the Green's-function formalism. The calculation we now perform, for weak repulsive potentials, can therefore be considered suggestive at best in its thermodynamic implications.

The vanishing of the real part of $T_q^{-1}$ locates the singularity. Accordingly we study the equation

$$1 = 2V_q \operatorname{Re}(S_q), \quad (11)$$

which takes the form

$$1 = 4V_0N_0q^2\left[\omega_q^{\,2} - \left(\frac{q^2}{2m^*}\right)^2\right]^{-1}(2m^*)^{-1} + 2m^*\Omega V_q\frac{1}{(2\pi)^2}\int_0^\infty dk\,k^2 f(\epsilon_k)\left\{\frac{1}{kq}\ln\left[\frac{(m^*\omega_q)^2 - q^2(k-\frac{1}{2}q)^2}{(m^*\omega_q)^2 - q^2(k+\frac{1}{2}q)^2}\right]\right\}. \tag{12}$$

Let us set

$$\omega_q^{\,2} = s_q^{\,2}q^2 + q^4/4m^{*2} \tag{13}$$

so that the speed of sound $s$ is given as $\lim(q\to 0)s_q$. Evaluating (12) in this limit, under the assumption that $\lim(q\to 0)V_q = V_0$ is finite, we obtain

$$1 = 4V_0N_0/2m^*s^2 + m^*\Omega V_0\pi^{-2}\int_0^\infty dk\,k^2 f(\epsilon_k)[(m^*s)^2 - k^2]^{-1}. \tag{14}$$

Above temperature $T_b$ this equation has no solution but it is still appropriate to enquire where $T_q$ has a maximum. At high temperature this is tantamount to finding the value of $s$ which maximizes

$$\int_0^\infty dk\,k^2\exp(-\beta k^2/2m^*)[(m^*s)^2 - k^2]^{-1}. \tag{15}$$

After some calculation, we obtain for this

$$s_{\rm gas} = (\tfrac{5}{3}\gamma kT/m^*)^{1/2}, \quad \gamma = \tfrac{26}{25}, \tag{16}$$

which differs little from the usual value ($\gamma = 1$) for the propagation of sound in quasi-ideal gases, particularly as $m^*\to m$ when $N\to 0$.

In this and subsequent singular integrals, the principal part must be computed by numerical means. It has proved convenient to introduce a dimensionless coupling strength parameter $A = \frac{1}{2}m^*s_\lambda^{\,2}/kT_\lambda$ in these equations. Using Eq. (5) for $N$, evaluated at $T_\lambda$, we obtain for (14) at $T_\lambda$

$$J(A, T_\lambda) \equiv \int_0^\infty dx\,x^2\{(A - x^2)[\exp(x^2) - 1]\}^{-1} = \alpha kT_\lambda/NV_0, \tag{17}$$

in which $\alpha \equiv \pi^{1/2}\zeta(\frac{3}{2})/4 = 1.157\,58$. [$J(A, T_\lambda)$ is well approximated by $\alpha A^{-1}$ for $A > 1$.] At $T > T_\lambda$ the corresponding equation is

$$J(A\tilde{s}^2, T) \equiv t^{1/2}\int_0^\infty dx\,x^2\{[(A\tilde{s}^2/t) - x^2][\exp(\beta\epsilon_0 + x^2) - 1]\}^{-1} = \alpha kT_\lambda/NV_0, \tag{18}$$

in which $t \equiv T/T_\lambda$ is the temperature in dimensionless units, $\tilde{s} \equiv s/s_\lambda$ is the speed of sound in units of the speed of sound at $T_\lambda$, and $\beta\epsilon_0$ satisfies Eq. (5). A plot of our computed values of $J(X, T)$ at various temperatures is shown in Fig. 1, from which one readily deduces that there is no solution to Eqs. (17) and (18) if $\alpha kT_\lambda/NV_0 > 1.294$, i.e., for $NV_0 < 1.12kT_\lambda$. For larger values of $NV_0/kT_\lambda$, there is a critical temperature above which (18) can no longer be satisfied, denoted $T_b$, an increasing function of $A$.

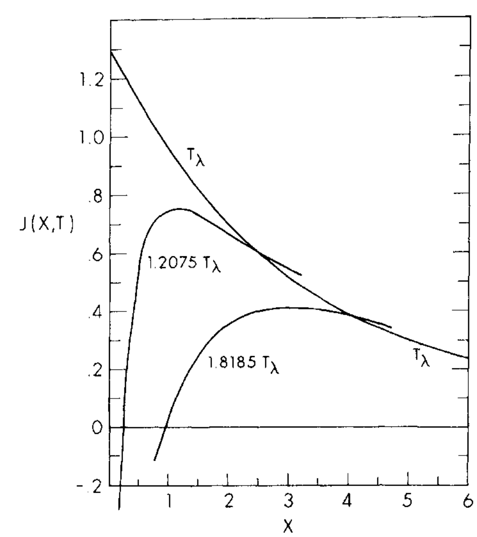


FIG. 1. $J(X, T)$ at various temperatures.

In the temperature range $T_\lambda < T < T_b$ there are two distinct values of $\tilde{s}$ which satisfy (18), as plotted in Fig. 2. The value of $A$ representative of He I near $T_\lambda$ can be deduced using $T_\lambda = 2.18$°K and $N/\Omega = 2.2\times 10^{22}$, which implies $m^*/m = 1.5$, according to (5). Using the experimental speed of sound[6] $s_\lambda = 218$ m/sec, we obtain $A = 7.87$ (Fig. 2 shows a calculation for $A = 7.5$, which is reasonably close). Below $T_\lambda$ the equations with $N_0 = N(1 - t^{3/2})$ give us only one solution. Of academic interest only, this branch shown in Fig. 2 has a positive slope with temperature instead of the experimentally observed negative slope. It is expected that this deficiency will be remedied when the interactions of this branch with the Bogoliubov-pair spectrum is taken into account, a refinement which we are presently introducing

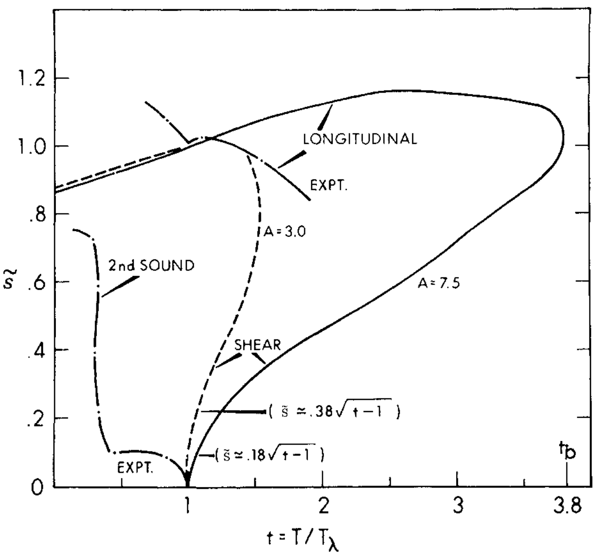


FIG. 2. Speed of sound $\tilde{s}=s/s_\lambda$ as function of temperature $t=T/T_\lambda$. Experimental curves (based on Ref. 6) are at constant pressure, and therefore are not directly comparable with our calculations at constant density [density drops by 50% at experimental critical point $t_c=2.4$ under own vapor pressure (Ref. 6)]. For completeness, we also show the experimental second-sound spectrum for He II, outside the framework of the present theoretical investigation. This figure is suggestive that the shear waves predicted by the present theory are the continuation into He I of the second-sound spectrum of He II.

into the theory.

We deduced that the low-speed collective mode has transverse character, because the vanishing of resonant denominators in our Green's functions requires $k \perp q$ when $s$ is small. We have therefore identified it as a shear wave, at least at short wavelengths. The upper branch requires $k \parallel q$ on the whole, and therefore has a longitudinal character. It can be identified with first sound of hydrodynamic theory, whereas the lower branch has no hydrodynamic analog. Its experimental existence or lack thereof would then be of great interest.

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