



Quiver Matrix Mechanics for IIB String Theory (II): Generic Dual Tori, Fractional Matrix Membrane and $SL(2, \mathcal{Z})$ Duality

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Abstract

With the deconstruction technique, the geometric information of a torus can be encoded in a sequence of orbifolds. By studying the Matrix Theory on these orbifolds as quiver mechanics, we present a formulation that (de)constructs the torus of *generic shape* on which Matrix Theory is “compactified”. The continuum limit of the quiver mechanics gives rise to a (1+2)-dimensional SYM. A hidden (fourth) dimension, that was introduced before in the Matrix Theory literature to argue for the electric-magnetic duality, can be easily identified in our formalism. We construct membrane wrapping states rigorously in terms of Dunford calculus in the context of matrix regularization. Unwanted degeneracy in the spectrum of the wrapping states is eliminated

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by using $SL(2, \mathcal{Z})$ symmetry and the relations to the FD-string bound states.

The dual IIB circle emerges in the continuum limit, constituting a critical evidence for IIB/M duality.

1 Introduction

The duality between type IIB string theory and M-theory, as an indispensable component of the string/M-theory web, *à la* Schwarz and Aspinwall [1, 2, 3], has two characteristic features: First there is a generalized T-duality between IIB theory on a circle and M-theory on a two-torus. According to this duality, the winding modes of the fundamental/D-string bound states (FD-string) in the IIB spectrum are dualized to Kaluza-Klein (KK) modes in M-theory. Secondly the non-perturbative $SL(2, \mathcal{Z})$ symmetry in IIB theory is geometrized to be the modular group of the two-torus in M-theory. The physical ideas behind this duality are elegant and beautiful, but how to formulate them in an explicit formalism and in a constructive way remains a challenge.

To our knowledge, up to now the only way available to formulate M-theory microscopically is the BFSS Matrix Theory [4]. (For other attempts to formulate the notion of the so-dubbed “protean degrees of freedom” of M-theory, see the review [5].) In this framework the statement for the non-perturbative IIB/M duality is that Matrix Theory compactified on a two-torus, with the size of the torus shrinking to zero, is dual to IIB string theory in a flat background [6, 7, 8]. However, it is highly non-trivial to see how this can come about. Matrix theory has nine transverse dimensions; when two of them are compactified and shrink to vanishing size, only seven dimensions survive. One needs to have the eighth transverse dimension in IIB string theory emerging in this limit. This emergent dimension should have two crucial properties in conformity with the IIB/M duality. First before decompactification the KK modes along it have to be associated with the membrane wrapping



states on the torus in Matrix Theory. Secondly after decompactification it should be on the same footing as other transverse dimensions, sharing an eight dimensional rotational invariance.

Since the compactified Matrix Theory is formulated as a $(1 + 2)$ -dimensional Super Yang-Mills theory (SYM_{1+2}) on the dual torus, IIB/M(atr) duality is addressed in the language of SYM in refs. [6, 7, 8]. On the one hand, the wrapping membranes are argued to correspond to configurations of the Yang-Mills fields, with nonvanishing (abelian) magnetic flux, which gives the wrapping number. On the other hand, the rotational symmetry between the decompactified emergent dimension and other flat transverse ones in the IIB target space, is argued to be related to the (conjectured) electric-magnetic (EM) duality of the $(1 + 3)$ -dimensional SYM, resulting from Matrix Theory compactified on a three-torus. Though these intuitive SYM arguments are compelling, an explicit construction of the wrapping membrane, as well as the definition of its wrapping number, is in demand in either the compactified Matrix Theory or in the dual $(1 + 2)$ -dimensional SYM [9]. Certainly one would prefer a more direct approach without the detour into the EM-duality in the three-dimensional world-volume of SYM. Moreover, IIB/M(atr) duality has been addressed only for the rectangular tori. No serious attempts have been made to formulate the notion of “dual torus” of *generic shape*.

To fill the gap, in our previous work in this series [10], we tried to generalize the definition of the wrapping number for a continuous map between two tori to matrix states wrapping on the compactified torus. We first adopted the deconstruction techniques [11] to approximate the compactified torus by a sequence of orbifolds, that encode the geometric information of a *rectangular* torus. This resulted in a quiver matrix quantum mechanics, whose continuum limit gives rise to SYM_{1+2} . Then the wrapping matrix states were constructed explicitly in the quiver matrix mechanics framework in terms of fractional powers of the 't Hooft clock and shift

matrices. And it has been checked that this construction possesses all properties required by the IIB/M duality. The present paper is sequential to the previous one, to study the case in which the compactified torus is of *generic* shape. Our motivation is to better understand how the *generic* geometry of the *dual torus* is encoded in the discrete setting of deconstruction (based on the orbifolding approach), and how a continuous geometry is restored on the SYM world-volume in the continuum limit. We shall refine the formalism for the fractional powers of matrices by employing a functional calculus of Dunford, to construct matrix membrane states with a well-defined topological wrapping number. Then, the generalized (torus-circle) T-duality is verified for generic moduli of the torus on a rigorous ground. As a bonus, we shall gain some insights into the $SL(2, \mathcal{Z})$ duality by identifying the FD-string in M-theory and eliminating the unwanted degeneracy in the membrane wrapping states in a satisfying fashion.

The outline of this paper is as follows. In Sec. 2, we describe in detail the Matrix Theory on $\mathcal{C}^3/\mathcal{Z}_N^2$ orbifold, which is our starting point, in the form of quiver matrix mechanics. In Sec. 3, we (de)construct in the large N limit the toroidal world-volume geometry with generic moduli, with a $(1+2)$ -dimensional SYM defined on it, which recovers Matrix Theory compactification on a dual torus. Moreover, we are able to identify the projection of the $(1+3)$ -dimensional SYM involved in previous discussions in the literature on IIB/M(atrrix) duality. Since the geometric information of the target torus is (re)constructed from the orbifold data, this results in a precise formulation of the duality between the target torus and the world-volume torus of SYM. In Sec. 4, we suggest a matrix construction of the membrane wrapping on the compactified torus, by using *Dunford calculus*. The IIB spectroscopy including the bound FD-string is analyzed, with the unwanted degeneracy of the wrapping states eliminated as expected. Moreover the $SL(2, \mathcal{Z})$ symmetry among the wrapping states is presented in the general setting. In Sec. 5, we dwell upon

remarks, problems and a few perspectives. The relationship of this work to various approaches in the literature is discussed too.

2 Quiver Matrix Mechanics from Matrix Theory on $\mathcal{C}^3/\mathcal{Z}_N^2$

In the IIB/M(atrrix) duality, the dynamics on the M-Theory side is described by the Matrix Theory compactified on a two-torus. We (de)construct the two-torus in terms of the orbifold $\mathcal{C}^3/\mathcal{Z}_N^2$, which is defined in the following way. Parameterize \mathcal{C}^3 by three complex numbers z^a , $a = 1, 2, 3$; the actions of the discrete group \mathcal{Z}_N^2 on \mathcal{C}^3 introduces two classes of equivalence relations

$$\begin{aligned}
 \text{I:} \quad & z^1 \sim \omega_N^* z^1, z^2 \sim \omega_N z^2, z^3 \sim z^3; \\
 \text{II:} \quad & z^1 \sim \omega_N^* z^1, z^2 \sim z^2, z^3 \sim \omega_N z^3
 \end{aligned} \tag{1}$$

where $\omega_N = e^{i2\pi/N}$.

The action of $U(K)$ Matrix Theory on this orbifold reads

$$\begin{aligned}
 S = \int dt \text{Tr} \{ & \frac{1}{2R_{11}} [D_t, Y^i]^2 + \frac{R_{11}}{4} [Y^i, Y^j]^2 \\
 & + \frac{1}{R_{11}} [D_t, Z^a][D_t, Z^{a\dagger}] + \frac{R_{11}}{2} ([Z^a, Z^{a'\dagger}][Z^{a\dagger}, Z^{a'}] + [Z^a, Z^{a'}][Z^{a\dagger}, Z^{a'\dagger}]) \\
 & + R_{11} [Y^i, Z^a][Y^i, Z^{a\dagger}] \\
 & - \frac{i}{2} \Lambda^\dagger [D_t, \Lambda] + \frac{R_{11}}{2} \Lambda^\dagger \gamma_i [Y^i, \Lambda] + \frac{R_{11}}{\sqrt{2}} \Lambda^\dagger (\tilde{\gamma}_a [Z^a, \Lambda] + \tilde{\gamma}_a^\dagger [Z^{a\dagger}, \Lambda]) \}. \tag{2}
 \end{aligned}$$

In Eq. (2), the eleven-dimensional planck length is taken to be unity, R_{11} is the radius of the compactified light-cone in the infinite momentum frame (IMF), t the world-line time; $D_t = d/dt + i[A_0, \cdot]$ with A_0 the $U(K)$ gauge connection in the temporal direction; both indices i and a run from 1 to 3 and a representation of the gamma matrices is given by

$$\gamma_1 = -\tau_2 \otimes \tau_3 \otimes \tau_3 \otimes \tau_3, \quad \gamma_2 = -\tau_1 \otimes \mathbf{1} \otimes \tau_3 \otimes \tau_3,$$

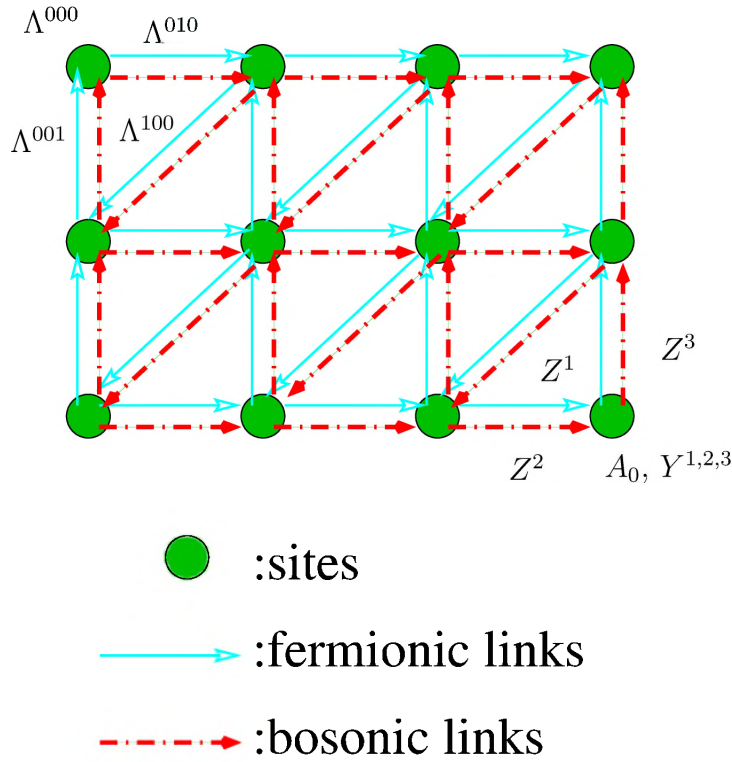


$$\begin{aligned}\gamma_3 &= -\tau_3 \otimes \mathbf{1} \otimes \tau_3 \otimes \tau_3 \quad , \quad \tilde{\gamma}_1 = -\epsilon \otimes \tau_- \otimes \mathbf{1} \otimes \tau_3, \\ \tilde{\gamma}_2 &= i\mathbf{1} \otimes \tau_3 \otimes \tau_- \otimes \tau_3 \quad , \quad \tilde{\gamma}_3 = -i\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \tau_-, \end{aligned} \quad (3)$$

in which $\tau_{1,2,3}$ are conventional Pauli matrices and $\tau_- = (\tau_1 - i\tau_2)/2$.

Y^i and Z^a are the coordinates of K D-particles; Λ their fermionic partner, which is an $SO(9)$ Majorana spinor with 16 real components. According to the tensorial decomposition in Eq. (3), the components of the fermionic coordinate are denoted as $\Lambda^{s_0 s_1 s_2 s_3}$ for $s_c = 0, 1, c = 0, 1, 2, 3$, in which Λ is real for s_0 and $\Lambda^{0i} = \Lambda^1$ for the other s_c . Because of the stringy nature of D-branes and the orbifold actions, all of these coordinates are lifted to be $KN^2 \times KN^2$ -matrices. Regarding the $N^2 \times N^2$ indices from orbifolding, their transformation properties under the gauge symmetry of each variable can be directly read off from the quiver diagram in Fig. 1, in which only six unit cells are presented (prolongable in two directions to give $N \times N$ unit cells), and Λ is labelled only by $s_{1,2,3}$; the orbifold conditions in Eqs. (1) are automatically incorporated in these transformation rules. Note that all of the variables can be interpreted to reside in the orbifolding group \mathcal{Z}_N^2 ; therefore, the terms “site” and “link” in Fig. 1 are understood in the circumstance of the discrete group \mathcal{Z}_N^2 , which here can be viewed as an approximated (or discretized) world-volume. So, sites are labelled by the elements in \mathcal{Z}_N^2 (pairs of integers (m, n) , with m, n modulo N). In the jargons of quiver theory, here from the target space point of view, site variables are adjoint matters while link variables are bi-fundamental matters.

The orbifolded Matrix Theory in Eq. (2) provides the basic machinery to generate, in the large N limit, both the geometry of the SYM world-volume and that of the torus in the target space. The next two sections are devoted to show how the geometries emerge in the continuum limit.


 Figure 1: Quiver diagram for the Matrix Theory on $\mathcal{C}^3/\mathcal{Z}_N^2$

3 Construction of Geometry of Dual Tori and SYM in Large N Limit

This section is dedicated to extract geometric information for compactified torus which is (de)-constructed with our orbifold setting.

3.1 Target Toroidal Geometry from Orbifolding

To see how a toroidal geometry in target space arises from the orbifolds $\mathcal{C}^3/\mathcal{Z}_N^2$, we introduce the following parametrization of \mathcal{C}^3 :

$$\begin{aligned}
 z^1 &= \rho_1 e^{i(\varphi^1 - \varphi^2 - \varphi^3)} / \sqrt{2}, \\
 z^2 &= \rho_2 e^{i(\varphi^1 + \varphi^2)} / \sqrt{2},
 \end{aligned}$$



$$z^3 = \rho_3 e^{i(\varphi^1 + \varphi^3)} / \sqrt{2} \quad (4)$$

where all of φ^a run from 0 to 2π . The orbifold conditions in Eqs. (1) now are expressed as

$$\begin{aligned} \text{I: } & \varphi^1 \sim \varphi^1, \varphi^2 \sim \varphi^2 + 2\pi/N, \varphi^3 \sim \varphi^3; \\ \text{II: } & \varphi^1 \sim \varphi^1, \varphi^2 \sim \varphi^2, \varphi^3 \sim \varphi^3 + 2\pi/N. \end{aligned} \quad (5)$$

Note that the angular parametrization of φ^1 is not unique, but the above choice in Eq. (4) will be convenient for our purposes.

With the parametrization (4), the metric of the orbifold $\mathcal{C}^3/\mathcal{Z}_N^2$ is

$$ds^2 = \sum_{a=1}^3 (d\rho_a^2 + \rho_a^2 d\vartheta_a^2), \quad (6)$$

in which $\vartheta_1 = \varphi^1 - \varphi^2/N - \varphi^3/N$, $\vartheta_2 = \varphi^1 + \varphi^2/N$, $\vartheta_3 = \varphi^1 + \varphi^3/N$. If we suppress the variations of the radial coordinates and of φ^1 , taking ρ_a to be nonnegative constants $c_a =: Nf_a$, then the orbifold metric in Eq. (6) becomes

$$ds^2 = \sum_{\alpha, \beta=2}^3 g_{\alpha\beta} d\varphi^\alpha d\varphi^\beta, \quad (7)$$

in which

$$(g_{\alpha\beta}) = \begin{pmatrix} f_1^2 + f_2^2 & f_1^2 \\ f_1^2 & f_1^2 + f_3^2 \end{pmatrix}. \quad (8)$$

Let us take a moment here to recall the condition(s) under which Eq. (7) gives rise to a legitimate Riemannian geometry, namely the metric $g_{\alpha\beta}$ is positive definite. From linear algebra this requires that $f_1^2 + f_2^2 > 0$ and the determinant $g \equiv \det g_{\alpha\beta} > 0$. Therefore, at most one of f_a can vanish.

The metric described by Eq. (8) is just that of a flat torus. (From now on we will omit the adjective “flat”.) It is known that the geometry of a torus is specified by the complex structure modulus $\tau = \tau_1 + i\tau_2$ and by its area. To extract them, we rewrite the metric in Eq. (8) in the conformally flat form:

$$ds^2 = e^{2\omega} |d\varphi^2 + \tau d\varphi^3|^2 \quad (9)$$

where $e^{2\omega} = f_1^2 + f_2^2$. Then the modular parameter can be read off as

$$\tau = e^{-2\omega}(f_1^2 + i\sqrt{g}). \quad (10)$$

As for the area of the torus, it is the coordinate area, $(2\pi)^2$, multiplied by \sqrt{g} :

$$A_{T^2} = (2\pi)^2\sqrt{g} = (2\pi)^2\sqrt{f_1^2 f_2^2 + f_2^2 f_3^2 + f_3^2 f_1^2}. \quad (11)$$

Note that the global geometry of a two-torus can be described either by $g_{\alpha\beta}$ locally with the fixed coordinate domain (for a Euclidean worldsheet) or by the global characters of the area and the modular parameter. Thus one may use the local parametrization (f_1, f_2, f_3) or the global one (ω, τ_1, τ_2) , or even a mixed set (f_1, \sqrt{g}, ω) to describe the geometry of the target torus. For example, the area A_{T^2} in Eq. (11) can also be calculated by embedding the torus as a parallelogram spanned by two vectors $2\pi e^\omega$ and $2\pi e^\omega \tau$ in a complex plane, namely

$$A_{T^2} = (2\pi e^\omega)(2\pi e^\omega \tau_2). \quad (12)$$

In summary, the geometry of the toroidal compactification of the target space is encoded in the limit $N \rightarrow \infty$, $c_a \equiv \langle \rho_a \rangle \rightarrow \infty$ with $f_a = c_a/N$ fixed.

3.2 World Volume Toroidal Geometry from (De)Construction

In BFSS Matrix theory and, subsequently, in our quiver matrix mechanics model (2), the target space coordinates are promoted to matrices. Though this increases technical complications to certain extent, we will see that the ideas on how a toroidal geometry emerges in the continuum limit (as a large- N limit) still apply. Furthermore, besides the geometry for the compactified torus in target space, we will see another toroidal geometry, dual to the former, emerging on the world volume that is (de)constructed in the same limit. This is another incarnation of the so-called target-space/world-volume duality that we realized before in [10, 12] in orbifolded Matrix theory.

First let us try to implement the angular parametrization at the matrix level, and to see whether a discrete geometry can make sense when we assign non-vanishing vacuum expectation values (VEV) to the matrix counterpart of the variables ρ_a .

As the solution to the orbifold conditions (1), the block decomposition of the bi-fundamental bosonic matrix variables Z^a in Eq. (2) can be read off directly from the quiver diagram Fig. 1:

$$Z_{mn,m'n'}^a = z^a(m, n)(\hat{V}_a)_{mn,m'n'}, \quad (13)$$

in which

$$\begin{aligned} (\hat{V}_2)_{mn,m'n'} &= (V_N)_{m,m'}\delta_{n,n'}, \\ (\hat{V}_3)_{mn,m'n'} &= \delta_{m,m'}(V_N)_{n,n'} \end{aligned} \quad (14)$$

and $\hat{V}_1 := \hat{V}_2^\dagger \hat{V}_3^\dagger$. Here the clock and shift matrices U_N, V_N of rank N are defined by

$$U_N^N = \mathbf{1}_N, V_N^N = \mathbf{1}_N, V_N U_N = \omega_N U_N V_N, \quad (15)$$

with $\mathbf{1}_N$ the unit matrix of rank- N . The block decomposition of other variables can be read off in the same way; for example,

$$Y_{mn,m'n'}^i = y^i(m, n)\delta_{mm'}\delta_{nn'}. \quad (16)$$

At a fixed site (m, n) , z^a (as well as y^i) is a K -by- K matrix. To (de)construct the toroidal geometry, after orbifolding we need to assign nonzero vacuum expectation value (VEV) to each $z^a(m, n)$; namely, we make the following decomposition

$$z^a = \langle z^a \rangle + \tilde{z}^a, \quad (17)$$

in which $\langle z^a \rangle$ are the VEV and \tilde{z}^a the fluctuations. We take

$$\langle z^a \rangle \equiv \frac{f_a}{\sqrt{2}\delta\sigma} \mathbf{1}_K \quad (18)$$

where

$$\delta\sigma := 2\pi/N \quad (19)$$

f_a will be understood as the same quantities that we have introduced in last subsection, while $\delta\sigma$ as the lattice constant for world volume coordinates later.

The most direct way to look for an interpretation in terms of discrete geometry is to rewrite the following term in Eq. (2)

$$S_{YZ} = - \int dt \text{Tr} \{ ||Z^a, Y^i||^2 \}, \quad (20)$$

as the discretized kinetic term of Y^i . Here we have absorbed R_{11} into a redefinition of the world-line time $t' = R_{11}t$ and suppressed the superscript prime.

We will introduce the discretized derivatives by using the shift matrix. In this paper, the clock matrix is represented by $U_N = \text{diag}(\omega_N, \omega_N^2, \dots, \omega_N^N)$ and the shift matrix by

$$V_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (21)$$

Note that the representation of V_N in Eq. (21) is the hermitian conjugate of the representation used in ref. [10]. Now let f be a diagonal matrix in the site indices: $f_{mn, m'n'} = f(m, n)\delta_{mm'}\delta_{nn'}$. The action of the shift operator, S_a , by a unit along the a -th direction is given by

$$S_a f = \hat{V}_a f \hat{V}_a^\dagger; \quad (22)$$

indeed we have explicitly

$$\begin{aligned} S_1 f(m, n) &= f(m-1, n-1), \\ S_2 f(m, n) &= f(m+1, n), \\ S_3 f(m, n) &= f(m, n+1). \end{aligned} \quad (23)$$

Subsequently we define the discrete partial derivatives by

$$\hat{\partial}_a f := (S_a f - f)/\delta\sigma. \quad (24)$$

(So $\delta\sigma$ serves as a (coordinate) lattice constant.) Because of the relation

$$\hat{\partial}_1 = -S_2^{-1}S_3^{-1}\hat{\partial}_2 - S_3^{-1}\hat{\partial}_3, \quad (25)$$

$\hat{\partial}_a$ are not algebraically independent.

Since we will take the large N limit eventually, we need to regularize the trace that appeared in Eqs. (2) and (20) by

$$Tr\{.\} \rightarrow \sum_{m,n} \delta\sigma^2 \kappa \{tr\{.\}\}, \quad (26)$$

in which κ is a regularization constant to be specified later, and tr the trace on the subspace supporting the gauge group $U(K)$. Now with a little algebra, we can rewrite S_{YZ} as

$$S_{YZ} = - \int dt \sum \delta\sigma^2 \kappa tr \{ |\delta\sigma z^a \hat{\partial}_a y^i + [\tilde{z}^a, y^i]|^2 \}. \quad (27)$$

Below, we assume scalings that all of the variables including the fluctuations in Eq. (17) are of $\mathcal{O}(1)$ in the large N limit except for $\langle z^a \rangle$ which behaves like $\mathcal{O}(N)$, provided the constants f_a are independent of N . This is the common circumstance for deconstruction in the present literature. Separating the fluctuation field \tilde{z}^a into hermitian and anti-hermitian part, $\Re\tilde{z}^a$ and $i\Im\tilde{z}^a$ respectively, Eq. (27) can be further written as

$$S_{YZ} = - \int dt \sum \delta\sigma^2 \kappa tr \{ |\frac{f_a}{\sqrt{2}}(\hat{\partial}_a y^i + \frac{i}{f_a}[\sqrt{2}\Im\tilde{z}^a, y^i]) + [\Re\tilde{z}^a, y^i] + \delta\sigma\tilde{z}^a\hat{\partial}_a y^i|^2 \}. \quad (28)$$

To reveal the discrete geometry on the quiver diagram, let us switch off the fluctuations in Eq. (28), resulting in

$$S_{Y\langle Z \rangle} = - \int dt \sum \delta\sigma^2 \kappa \frac{1}{2} tr \{ \tilde{g}^{22}(\hat{\partial}_2 y^i)^2 + \tilde{g}^{33}(S_2\hat{\partial}_3 y^i)^2 + 2\tilde{g}^{23}(\hat{\partial}_2 y^i)(S_2\hat{\partial}_3 y^i) \}. \quad (29)$$

Here the (contravariant) metric is defined by

$$(\tilde{g}^{\alpha\beta}) = \begin{pmatrix} f_1^2 + f_2^2 & f_1^2 \\ f_1^2 & f_1^2 + f_3^2 \end{pmatrix}, \quad (30)$$

with $\alpha, \beta = 2, 3$. It is amusing to notice that comparing with Eq. (8), we have

$$\tilde{g}^{\alpha\beta} = g_{\alpha\beta}. \quad (31)$$

As a corollary of either Eq. (30) or Eq. (31), we see that $\tilde{g}^{\alpha\beta}$ is independent of N ! As a simplest application of the metric (30) in discrete geometry, we can assign an area to the elementary parallelogram, or *plaquette* in the jargon of lattice gauge theory, spanned by two edges labelled by Z^2 and Z^3 in the quiver diagram in Fig. 1:

$$\delta A_N = \frac{\delta\sigma^2}{\sqrt{g}}. \quad (32)$$

where $\delta\sigma^2$ is the coordinate area. Because of translation invariance, by counting the total number of the plaques we get a total area for the quiver diagram:

$$A_N = N^2 \cdot \delta A_N = \frac{(2\pi)^2}{\sqrt{g}}, \quad (33)$$

which is also independent of N .

In the continuum limit, i.e. in the large N limit with f_a fixed, we have

$$(\mathcal{Z}_N^2, \delta\sigma^2) \rightarrow (\tilde{T}^2, d\sigma^2 d\sigma^3); \quad (34)$$

namely the quiver diagram (de)constructs a continuum torus \tilde{T}^2 , with continuous coordinates σ^α ($\alpha = 2, 3$) running from 0 to 2π . So the metric in Eq. (30) on the discrete quiver diagram survives the large N limit, and becomes the metric on the torus \tilde{T}^2 .

Moreover, in the large N limit,

$$\hat{\partial}_2 \rightarrow \partial/\partial\sigma^2, \hat{\partial}_3 \rightarrow \partial/\partial\sigma^3, \quad (35)$$

and because of the linear relation in Eq. (25)

$$\hat{\partial}_1 \rightarrow -\partial/\partial\sigma^2 - \partial/\partial\sigma^3. \quad (36)$$

Now it is easy to work out the large N limit of $S_{Y\langle Z \rangle}$:

$$S_{Y\langle Z \rangle} = - \int dt d^2\sigma \kappa \text{tr} \left\{ \frac{1}{2} \tilde{g}^{\alpha\beta} \partial_\alpha y^i \partial_\beta y^i \right\}, \quad (37)$$

where, as usual, $d^2\sigma = d\sigma^2 d\sigma^3$. The same positive-definiteness condition analyzed below Eq. (8) should be imposed to ensure the positive-definiteness of $\tilde{g}^{\alpha\beta}$. It is a constraint on VEV in Eq. (18), which gives rise to a normal Riemannian geometry to the toroidal membrane, \tilde{T}^2 , (de)constructed with our quiver diagram.

To see that the toroidal geometry \tilde{T}^2 is dual to that of the compactified target torus T^2 , we choose the regularization constant κ in Eq. (26) to be $\kappa = 1/\sqrt{\tilde{g}} = \sqrt{\bar{g}}$, where \tilde{g} is the determinant of $\tilde{g}_{\alpha\beta}$, the inverse of $\tilde{g}^{\alpha\beta}$ in Eq. (30), such that $d^2\sigma \cdot \kappa$ becomes the invariant measure on the dual torus. Here we introduce the definition that two tori $(T^2, (\varphi^2, \varphi^3), g_{\alpha\beta})$ and $(\tilde{T}^2, (\sigma^2, \sigma^3), \tilde{g}_{\alpha\beta})$ are dual to each other, if and if all of the affine parameters have the same domain from 0 to 2π and Eq. (31) is satisfied. Consequently, the area of the dual torus is

$$A_{\tilde{T}^2} = \int d^2\sigma \kappa = \frac{(2\pi)^2}{\sqrt{\tilde{g}}}, \quad (38)$$

which coincides with A_N in Eq. (33). Just as the case for A_{T^2} in Eq. (12), the result for $A_{\tilde{T}^2}$ in Eq. (38) can also be obtained by rewriting the dual metric as

$$d\tilde{s}^2 = e^{2\Omega} |d\sigma^2 + \tilde{\tau} d\sigma^3|^2, \quad (39)$$

in which $e^{2\Omega} = (f_1^2 + f_3^2)/g$, and the dual modular parameter is identified to be

$$\tilde{\tau} = \tilde{\tau}_1 + i\tilde{\tau}_2 = \frac{-f_1^2 + i\sqrt{g}}{e^{2\Omega}g} = -\frac{1}{\tau}! \quad (40)$$

Thus, Eq. (37) can be written as

$$S_{Y\langle Z \rangle} = - \int dt d^2\sigma \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} \text{tr} \left\{ \frac{1}{2} \partial_\alpha y^i \partial_\beta y^i \right\}, \quad (41)$$

Without any additional pain we can safely claim that in Eq. (2),

$$S_{Y^4} := \int dt Tr \left\{ \frac{1}{4} [Y^i, Y^j]^2 \right\} \xrightarrow{N \rightarrow \infty} S_{Y^4} = \int dt d^2 \sigma \sqrt{\tilde{g}} tr \left\{ \frac{1}{4} [y^i, y^j]^2 \right\}. \quad (42)$$

To summarize, the above terms are of the usual form of the action integral for fields on the world volume of a torus, with $\tilde{g}^{\alpha\beta}$ as contravariant metric. Later we will see that in the continuum limit, all other terms in our quiver model contain the same metric. This feature indeed identifies $\tilde{g}^{\alpha\beta}$ as the metric on the world volume (de)constructed by our orbifolds. In last subsection, we drew the quiver diagram in Fig. 1 as a square lattice; however, no notion of length was introduced at that stage. It was only after assigning non-zero VEV as in Eq. (18), the quiver diagram becomes a lattice with meaningful lattice constant, and in the large N limit becomes a continuum torus, \tilde{T}^2 , with a flat metric. The relation (31) implies that the toroidal geometry of \tilde{T}^2 is *dual to* that of the compactified torus, T^2 , in target space as we discussed in last subsection. In this way, our (de)construction procedure (orbifolding, assigning non-zero VEV and taking the continuum limit) exhibits the so-called target-space/world-volume duality. In the literature, including our previous paper [10], this duality was shown only for regular tori; here we have shown the validity of this duality when the compactified target torus is of a generic (oblique) shape.

3.3 1 + 2-Dimensional Super Yang-Mills and the Detour into Four Dimensions

We devote this subsection to a complete discussion of the continuum limit of our orbifolded quiver matrix mechanics. On the one hand, we will show that all terms in the continuum action contain one and same metric $\tilde{g}^{\alpha\beta}$, justifying the emergence of the world volume geometry in (de)construction through orbifolding. On the other hand, we will show that in this continuum limit, the quiver matrix mechanics

approaches to 1 + 2-dimensional SYM on \tilde{T}^2 . Previously to argue for the S -duality and rotational invariance in Matrix Theory compactified on a torus, a connection between 1 + 2 and 1 + 3 dimensional SYM was proposed in refs. [6, 7]. In this subsection we will see that indeed this detour into four dimensions is something very natural in the present approach.

In [10] we have studied the case with $f_1 = 0$, leading to a regular torus. To consider torus of more general shape, here we study another simplified case, corresponding to a triangular lattice, with

$$f_1 = f_2 = f_3 = L. \quad (43)$$

In accordance with the parametrization (4) and the VEV in Eqs. (18) and (43), we parameterize fluctuations in Eq. (17) by

$$\begin{aligned} \tilde{z}^1 &= (\phi^1 + iL(\phi'_1 - A_2 - A_3))/\sqrt{2}, \\ \tilde{z}^2 &= (\phi^2 + iL(\phi'_1 + A_2))/\sqrt{2}, \\ \tilde{z}^3 &= (\phi^3 + iL(\phi'_1 + A_3))/\sqrt{2}. \end{aligned} \quad (44)$$

All the new variables here, with the site indices (m, n) omitted, are K -by- K matrices. In the following we will discuss the dynamics of these fluctuations.

3.3.1 Discrete Geometry and Equilateral (Triangular) Lattice

With the symmetric VEV (43), the quiver diagram in Fig. 1 becomes a equilateral triangular lattice, shown in Fig. 2, in which re-label the fermionic coordinates

$$\begin{aligned} \Lambda^{s_0 s_1 s_2 s_3} &= \lambda^{s_0 s_1 s_2 s_3} \hat{V}_{s_1+2s_2+3s_3}, \quad \text{for } s_1 + s_2 + s_3 = 0, 1, \\ \Lambda^{s_0 s_1 s_2 s_3} &= \lambda^{s_0 s_1 s_2 s_3} \hat{V}_{(s_1+1)+2(s_2+1)+3(s_3+1)}^\dagger \quad \text{for } s_1 + s_2 + s_3 = 2, 3 \end{aligned} \quad (45)$$

with $s_c + 1$ (with $c = 1, 2, 3$) defined modulo 2.

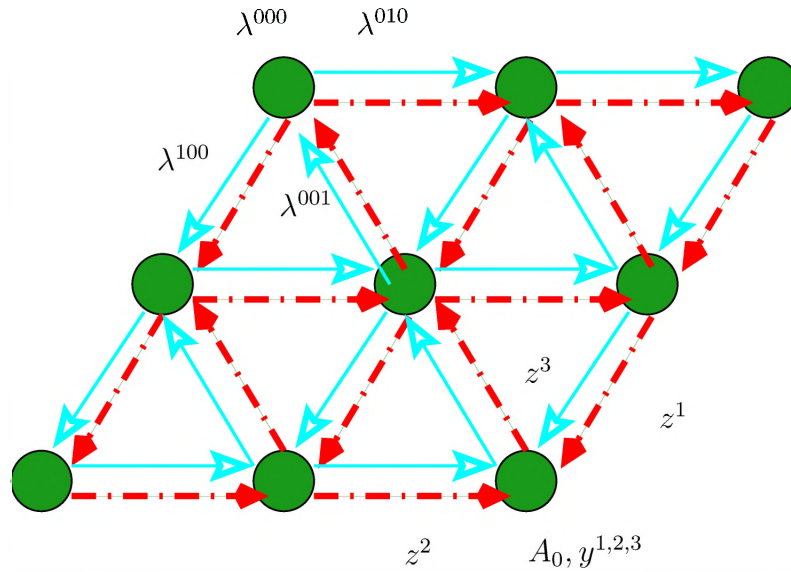


Figure 2: Two-dimensional Equilateral Lattice from (De)Construction

In fact, with the VEV (43), the metric in Eq. (30) becomes

$$(\tilde{g}^{\alpha\beta}) = L^2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (46)$$

whose inverse is

$$(\tilde{g}_{\alpha\beta}) = \frac{1}{3L^2} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (47)$$

From either the definition of the discrete partial derivatives in Eq. (24) or the large N limit of the measure on \mathcal{Z}_N^2 in Eq. (34), we learn that the role of the coordinate lattice constant in both two directions on \mathcal{Z}_N^2 is played by $\delta\sigma$, so that the total coordinate length of each cycles in either discrete or continuum cases is just 2π . Now we compute the *proper length* of the unit vectors e_a , denoted as $\|e_a\|$, in three directions in Fig. 2 labelled by z^a , with $e_1 = \delta\sigma \cdot (-1, -1)^T$, $e_2 = \delta\sigma \cdot (1, 0)^T$, $e_3 = \delta\sigma \cdot (0, 1)^T$. The result is

$$\|e_a\|^2 = (e_a, e_a) = \tilde{g}_{\alpha\beta} e_a^\alpha e_a^\beta = \frac{2\delta\sigma^2}{3L^2}, \quad (48)$$

for all $a = 1, 2, 3$ (no summation on a). Therefore the lattice is equilateral, with the area $A_{\tilde{T}^2} = (2\pi)^2/\sqrt{3}L^2$ from Eq. (38).

A linear transformation

$$\begin{pmatrix} \sigma^2 \\ \sigma^3 \end{pmatrix} = \frac{L}{\sqrt{2}} \begin{pmatrix} \sqrt{3} & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}, \quad \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \frac{1}{\sqrt{6}L} \begin{pmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} \sigma^2 \\ \sigma^3 \end{pmatrix}, \quad (49)$$

transforms the dual metric into the standard form

$$d\tilde{s}^2 = (dw^1)^2 + (dw^2)^2, \quad (50)$$

with the measure $d^2\sigma/\sqrt{3}L^2 = d^2w$. We can calculate the area $A_{\tilde{T}^2}$ in the w -frame. In fact, \tilde{T}^2 in σ -frame spanned by two basis vectors $E_2 := 2\pi \cdot (1, 0)^T$, $E_3 := 2\pi \cdot (0, 1)^T$; by the second formula in Eq. (49), in w -frame,

$$E_2 = \frac{\sqrt{2}}{\sqrt{3}L} \cdot 2\pi \cdot (1, 0)^T, \quad E_3 = \frac{\sqrt{2}}{\sqrt{3}L} \cdot 2\pi \cdot \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T. \quad (51)$$

The value of $A_{\tilde{T}^2}$ follows hence because of Eq. (50).

3.3.2 Four Dimensions

Now it is the time to revisit S_{YZ} in Eq. (28). Recall the term $\delta\sigma\tilde{z}^a\hat{\partial}_ay^i$ is of order $\mathcal{O}(1/N)$, it is not difficult to deduce the continuum limit of S_{YZ} ;

$$S_{YZ} = - \int dt d^2\sigma \frac{1}{\sqrt{3}L^2} \frac{1}{2} \text{tr} \{ L^2 (D_\alpha y^i + i[\phi'_1, y^i])^2 - [\phi^a, y^i]^2 \} \quad (52)$$

where the index a runs from 1 to 3, and $D_\alpha = \partial_\alpha + i[A_\alpha, \cdot]$ for $\alpha = 2, 3$. Recall the fact that

$$D_1 + D_2 + D_3 = 0; \quad (53)$$

we get

$$S_{YZ} = - \int dt d^2\sigma \frac{1}{\sqrt{3}L^2} \frac{1}{2} \text{tr} \{ \tilde{g}^{\alpha\beta} D_\alpha y^i D_\beta y^i - 3L^2 [\phi'_1, y^i]^2 - [\phi^a, y^i]^2 \}. \quad (54)$$

Eq. (54) is of a standard form in SYM after a rescaling of ϕ'_1 . More amusing is the assertion that the continuum action Eq. (52) can be obtained from dimension reduction from SYM in one more dimension! Our key observation here is that the role of ϕ'_1 in Eqs. (44) and (52) is very similar to a gauge connection, whose direction can be parameterized virtually by a coordinate σ^1 . This implies that S_{YZ} in Eq. (52) can be regarded as a 1 + 3-dimensional theory subject to the dimensional reduction constraint

$$\partial/\partial\sigma^1 \equiv 0. \quad (55)$$

This motivates us to introduce a three-dimensional Euclidean space (x^1, x^2, x^3) , in the sense of a covering space, such that

$$\begin{aligned} \xi \frac{\partial}{\partial x^1} &= \eta \frac{\partial}{\partial \sigma^1} - \frac{\partial}{\partial \sigma^2} - \frac{\partial}{\partial \sigma^3}, \\ \xi \frac{\partial}{\partial x^2} &= \eta \frac{\partial}{\partial \sigma^1} + \frac{\partial}{\partial \sigma^2}, \\ \xi \frac{\partial}{\partial x^3} &= \eta \frac{\partial}{\partial \sigma^1} + \frac{\partial}{\partial \sigma^3} \end{aligned} \quad (56)$$

with two constants ξ , η , and $\phi'_1 = \eta A_1$ with A_1 the gauge connection in the σ^1 -direction. Since covariant derivatives transform in the same way as ordinary derivatives, from Eq. (56) we can solve the coordinate transformation

$$\begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \end{pmatrix} = \xi^{-1} \begin{pmatrix} \eta & \eta & \eta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}. \quad (57)$$

Now we suppose the terms $[\phi^a, y^i]^2$ in Eq. (52) are intact in dimensional reduction, and the metric in the x -frame is the standard Euclidean metric. Then we have to modify the regularization of the trace in Eq. (26) and interpret the measure in Eq. (52) in the following way

$$Tr \rightarrow \int \int \frac{d^2\sigma}{\sqrt{3}L^2} \kappa' = \int \int d\sigma^2 d\sigma^3 \int d\sigma^1 \sqrt{G^\sigma} = \int \int \int d^3x, \quad (58)$$

where we have written the multiple integral explicitly, and κ' is the proper length of the one-dimensional space swept by σ^1 and G^σ is the determinant of the covariant metric in the three-dimensional σ -frame. Keeping Eq. (58) in mind and naming the covariant derivatives in x -frame as ∇_i for $i = 1, 2, 3$, the first terms in Eq. (52) must be of the standard form $\nabla_j y^i \nabla_j y^i$, from which ξ is fixed to be $1/L$.

Consequently, we can write the covariant metric in σ -frame as

$$(G_{\sigma}^{aa'}) = L^2 \begin{pmatrix} 3\eta^2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad (59)$$

whose restriction to the lower-right 2-by-2 block is identical to $\tilde{g}^{\alpha\beta}$ in Eq. (46); similarly for the contravariant metric

$$G_{aa'}^{\sigma} = \frac{1}{3L^2} \begin{pmatrix} \frac{1}{\eta^2} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \quad (60)$$

Note that η is a free parameter at this stage because only ϕ'_1 is “visible” after dimensional reduction. To make Eqs. (56) and (57) nonsingular, we only require η nonvanishing. In fact, η controls the scale of the basis vector in σ^1 -direction, whose magnitude is not relevant in the present context. For later convenience we rescale $\sigma^1 \rightarrow \sigma^1/\sqrt{3}L\eta$ such that $G_{11}^{\sigma} \rightarrow 1$ and that the domain of σ^1 changes to $(0, \kappa')$; now η is entirely absorbed.

From Eq. (56), one can easily infer that viewed in x -frame, σ^2 runs along the $(-1, 2, -1)$ -direction, σ^3 in the $(-1, -1, 2)$ -direction, while σ^1 in the $(1, 1, 1)$ -direction. The geometric significance of the dimensional reduction condition $\partial/\partial\sigma^1 \equiv 0$ is nothing but the requirement the directional derivative in the $(1, 1, 1)$ -direction in x -frame should vanish, i.e. to restrict the theory to the sector invariant under translations in $(1, 1, 1)$ -direction. (Of course, on (σ^2, σ^3) -plane, we need to impose periodic boundary conditions to make it into the torus \tilde{T}^2 .)

As a final remark in this sub-subsection, we see that κ' provides the room for the electric-magnetic duality argument in [7] to convince $O(8)$ rotational symmetry in IIB string theory. We know that the Matrix Theory compactified on a three-dimensional torus is equivalent to 1 + 3-dimensional SYM. In fact, there exist two different limits from this 1 + 3-dimensional SYM to IIB string theory, in which κ' are treated differently. The first is in the sense of Sethi and Susskind in [7], that the κ' is tuned to be proportional to the overall size of \tilde{T}^2 . Then in the second one, the σ^1 -direction is taken as a KK circle, equivalently to decompactify the dual circle in the target space, and the original 1 + 2-dimensional theory is in the KK limit; in this case, there appears an additional wave function normalization such that effectively $\kappa' = 1$ (the dimensional reduction condition (55) is equivalent to the prescription to keep only the zero-modes along the KK-circle).

3.3.3 SYM in the Continuum Limit

With the above preparations, now it is straightforward to derive the continuum limit of the quiver matrix mechanics (2) and show that the full action is none other than a 1 + 2-dimensional SYM with 16 supercharges. We will bring the power of the KK dimensional reduction of 1 + 3-dimensional SYM into full play, with κ' taken to be unity.

I. Scalar

Collect Eqs. (42), (54) and the kinetic term of Y^i (the first term) in Eq. (2).

$$S_Y = \int dt \frac{d^2\sigma}{\sqrt{3}L^2} \text{tr} \left\{ \frac{1}{2} ((D_t y^i)^2 - \tilde{g}^{\alpha\beta} D_\alpha y^i D_\beta y^i) + \frac{1}{2} [\phi^A, y^i]^2 + \frac{1}{4} [y^i, y^j]^2 \right\} \quad (61)$$

where the index A runs from 1 to 4 and $\phi^4 := \sqrt{3}L\phi'_1$; or in a mixed fashion, with three-dimensional measure and four-dimensional Lagrangian:

$$S_Y = \int dt \frac{d^2\sigma}{\sqrt{3}L^2} \text{tr} \left\{ \frac{1}{2} ((D_t y^i)^2 - (\nabla_j y^i)^2) + \frac{1}{2} [\phi^a, y^i]^2 + \frac{1}{4} [y^i, y^j]^2 \right\} \quad (62)$$

II. Yang-Mills

For the bosonic bi-fundamental variables Z^a , we rewrite the relevant terms in Eq. (2)

$$S_Z = \int dt \text{Tr} \{ |[D_t, Z^a]|^2 - \frac{1}{2} (|[Z^a, Z^{a\dagger}]|^2 + |[Z^a, Z^{a'}]|^2) \}. \quad (63)$$

Introduce $[Z^a, Z^{a\dagger}] =: \hat{V}_a^\dagger P_{a'a} \hat{V}_a$ such that $P_{a'a}^\dagger = P_{aa'}$ and

$$P_{a'a} = S_{a'} z^a S_a z^{a'\dagger} - z^{a'\dagger} z^a = \delta\sigma \hat{\partial}_{a'} z^a S_a z^{a'\dagger} + \delta\sigma z^a \hat{\partial}_a z^{a'\dagger} + [z^a, z^{a'\dagger}]. \quad (64)$$

Then $[Z^a, Z^{a'}] =: Q_{aa'} \hat{V}_a \hat{V}_{a'}$ such that $Q_{a'a} = -Q_{aa'}$ and

$$Q_{aa'} = z^a S_a z^{a'} - z^{a'} S_{a'} z^a = \delta\sigma z^a \hat{\partial}_a z^{a'} - \delta\sigma z^{a'} \hat{\partial}_{a'} z^a + [z^a, z^{a'}]. \quad (65)$$

Also $[D_t, Z^a] =: S_{0a} \hat{V}_a$ with

$$S_{0a} = \dot{z}^a - i\delta\sigma z^a \hat{\partial}_a A_0 + i[A_0, z^a]. \quad (66)$$

The action S_Z in Eq. (63) is recast into

$$S_Z = \int dt \sum \delta\sigma^2 \kappa \text{tr} \{ |S_{0a}|^2 - \frac{1}{2} (|P_{a'a}|^2 + |Q_{a'a}|^2) \}. \quad (67)$$

Now we separate the VEV and the fluctuation as in Eq. (17), impose the VEV (43) which, at large N , is the moduli condition in the language of quantum field theory, and substitute the parametrization of fluctuation in Eq. (44); remember the remark below Eq. (56) and definition of ∇_i above Eq. (59), with subscripts a and a' in the sense of the directions in the quiver diagram replaced by j and j' in three-dimensional Euclidean space or three-torus, what follows in the continuum limit is

$$P_{j'j} = \frac{1}{2} ((\nabla_j \phi^{j'} + \nabla_{j'} \phi^j) + i(F_{j'j} - i[\phi^j, \phi^{j'}])), \quad (68)$$

$$Q_{jj'} = \frac{1}{2} ((\nabla_j \phi^{j'} - \nabla_{j'} \phi^j) + i(-F_{j'j} - i[\phi^j, \phi^{j'}])), \quad (69)$$

$$S_{0j} = \frac{1}{\sqrt{2}} (D_t \phi^j + iF_{0j}), \quad (70)$$

in which the gauge field strength $F_{jj'}$, F_{0j} are defined conventionally and each operator in Eqs. (68), (69) and (70) is sorted in the form $F = \Re F + i\Im F$ so that $|F|^2 = (\Re F)^2 + (\Im F)^2$. Then similar to Eq. (62), Eq. (67) can be put into

$$S_Z = \int dt \frac{d^2\sigma}{\sqrt{3}L^2} \text{tr} \left\{ \frac{1}{2} F_{0j}^2 - \frac{1}{4} F_{jj'}^2 + \frac{1}{2} ((D_t \phi^a)^2 - (\nabla_j \phi^a)^2) + \frac{1}{4} [\phi^a, \phi^{a'}]^2 \right\} \quad (71)$$

where we recover the a, a' indices for scalar fields. Eq. (71) should be understood with the help of the dimensional reduction condition (55); however we are not bothered with deducing a similar expression like Eq. (61), since the goal here is just to check the 1 + 2-dimensional SYM in the continuum limit. It is easy to see that Eq. (62) plus Eq. (71) are simply a dimensional reduction of the 4-dimensional Yang-Mills theory subjected to Eq. (55).

III. Fermions

Once again, we try to deduce the continuum limit for fermions from

$$S_F = \int dt \text{Tr} \left\{ -\frac{i}{2} \Lambda^\dagger [D_t, \Lambda] + \frac{1}{2} \Lambda^\dagger \gamma_i [Y^i, \Lambda] + \frac{1}{\sqrt{2}} \Lambda^\dagger (\tilde{\gamma}_a [Z^a, \Lambda] + \tilde{\gamma}_a^\dagger [Z^{a\dagger}, \Lambda]) \right\} \quad (72)$$

in the four-dimensional point of view.

This time, we do it in the fastest way. From our experience with S_Y and S_Z , we know that the effects of \hat{V}_a in Eqs. (45) are washed out in the large N limit except the terms containing the VEV of Z^a . We only emphasize that we can take the VEV's to be nonnegative in Eq. (18) because their phases can be absorbed into the redefinition of the fermionic coordinates. Accordingly, we can write down the continuum limit of S_F directly

$$S_F = \int dt \frac{d^2\sigma}{\sqrt{3}L^2} \text{tr} \left\{ -i\lambda^\dagger D_t \lambda + \lambda^\dagger \gamma^{2j+3} [\nabla_j, \lambda] + \lambda^\dagger \gamma_i [y^i, \lambda] + \lambda^\dagger \gamma_{2a+2} [\phi^a, \lambda] \right\} \quad (73)$$

where $\gamma_{2a+2} = \tilde{\gamma}_a + \tilde{\gamma}_a^\dagger$, $\gamma_{2j+3} = i(\tilde{\gamma}_j - \tilde{\gamma}_j^\dagger)$.

In summary, we have shown that the continuum limit of the quiver matrix mechanics in Eq. (2) is a $d = 1+2$ SYM: Eqs. (62), (71) plus Eq. (73) constitute precisely the action of the $d = 1 + 3$ SYM with 16 supercharges dimensionally reduced by



Eq. (55). This outcome in the continuum limit justifies our creed to approximate a compactification in Matrix Theory via a sequence of orbifolds, demonstrating that we are on the right track for IIB/M duality.

4 Wrapping Matrix Membrane and $SL(2, \mathcal{Z})$ Duality

It is generally believed that in M-theory framework, IIB string theory can be described by M-theory compactified on a torus. In [9], Schwarz suggested that in such a theory there exist solitonic states, describing M2-branes wrapping on the target torus, which correspond to the doubly charged (q_1, q_2) -strings (or bound FD strings) in IIB theory; and it is very tempting to identify the $SL(2, \mathcal{Z})$ duality in IIB string theory with the geometric $SL(2, \mathcal{Z})$ invariance for a torus. He already noted a serious problem in this identification, i.e. the degeneracy of wrapping membrane states would be generally greater than that of (q_1, q_2) strings, unless there is a way to identify the degenerate wrapping membrane states which he assumed is true. However, the explicit description of these solitonic wrapping membranes and the details of how the elimination of their degeneracy happens are still in demand in the literature.

Since our quiver matrix mechanics (de)constructs M-theory compactified on a torus, with a toroidal geometry for the dynamical membrane (see Subsections 3.1 and 3.2), it provides a natural platform for dealing with the above mentioned problems involving wrapping membranes on target torus in IIB/M(atrrix) theory. In a previous work [12] we have resolved these problems for the case when the compactified target torus is a regular one. In this section we generalize the discussions for an oblique torus with a generic complex modular parameter τ .

4.1 Wrapping a Membrane on Orbifold

How to define the states with a definite wrapping number for wrapping matrix membranes?

For a continuous membrane of toroidal topology, we use a pair of real coordinates (q, p) , with the equivalence $q \sim q + 2\pi$, $p \sim p + 2\pi$. The continuous wrapping map from the membrane to the target T^2 satisfies the periodic boundary conditions

$$\begin{aligned}\varphi^2(q + 2\pi, p) &= \varphi^2(q, p) + 2\pi m^2, & \varphi^2(q, p + 2\pi) &= \varphi^2(q, p) + 2\pi n^2, \\ \varphi^3(q + 2\pi, p) &= \varphi^3(q, p) + 2\pi m^3, & \varphi^3(q, p + 2\pi) &= \varphi^3(q, p) + 2\pi n^3,\end{aligned}\quad (74)$$

for four arbitrary integers m^2, n^2, m^3, n^3 . The solution, up to homotopy and large diffeomorphisms, is of the form

$$\begin{pmatrix} \varphi^2(q, p) \\ \varphi^3(q, p) \end{pmatrix} = W(\underline{m}, \underline{n}) \begin{pmatrix} q \\ p \end{pmatrix}\quad (75)$$

with the wrapping map matrix

$$W(\underline{m}, \underline{n}) := (\underline{m}, \underline{n}), \underline{m} := (m^2, m^3)^T, \underline{n} := (n^2, n^3)^T.\quad (76)$$

Subsequently the pull-back geometry from T^2 to the membrane is

$$ds^2 = e^{2\omega^*} |dq + \tau^* dp|^2.\quad (77)$$

here the pull-back conformal factor is $e^{\omega^*} = e^\omega |m^2 + m^3 \tau|$, with the pull-back modular parameter to be

$$\tau^* = \frac{n^3 \tau + n^2}{m^3 \tau + m^2}.\quad (78)$$

(τ is the modular parameter of T^2 .) So the induced measure is

$$dqdp \cdot e^{2\omega^*} \tau_2^* = dqdp \cdot e^{2\omega} \cdot \Im((n^3 \tau + n^2)(m^3 \bar{\tau} + m^2)) = dqdp \cdot w \cdot \sqrt{g},\quad (79)$$

or simply $d^2\varphi = wdqdp$, with $w = \det W$ the *wrapping number*.¹

¹Some authors gave alike construction in different circumstances; for example, in [13], Bars considered the connections between discrete area preserving diffeomorphisms, reduced Yang-Mills and strings.

The wrapping map (75) is fully characterized by the induced structure (78) and (79). We know that, in the form of gauge-fixed metric (9), a torus possesses an $SL(2, \mathcal{Z})$ symmetry containing all large diffeomorphisms:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} \varphi^2 \\ \varphi^3 \end{pmatrix} \rightarrow C \cdot \begin{pmatrix} \varphi^2 \\ \varphi^3 \end{pmatrix}, \quad (80)$$

where C is an $SL(2, \mathcal{Z})$ matrix

$$C = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \quad (81)$$

Because $\det C = 1$, none of $d^2\varphi$, \sqrt{g} and the wrapping number w is changed under this $SL(2, \mathcal{Z})$.

4.2 Matrix States of Wrapping Membrane and Fractional Powers

The investigation in the previous subsection is carried out solely for the continuous torus. By (de)constructing a torus with a sequence of orbifolds, the wrapping map in Eqs. (74) change to be the following form

$$\begin{aligned} z^1(q + 2\pi, p) &= e^{-i2\pi(m^2+m^3)/N} z^1(q, p) \quad , \quad z^1(q, p + 2\pi) = e^{-i2\pi(n^2+n^3)/N} z^1(q, p), \\ z^2(q + 2\pi, p) &= e^{i2\pi m^2/N} z^2(q, p) \quad , \quad z^2(q, p + 2\pi) = e^{i2\pi n^2/N} z^2(q, p), \\ z^3(q + 2\pi, p) &= e^{i2\pi m^3/N} z^3(q, p) \quad , \quad z^3(q, p + 2\pi) = e^{i2\pi n^3/N} z^3(q, p). \end{aligned} \quad (82)$$

Below, introducing $m^1 = -m^2 - m^3$, $n^1 = -n^2 - n^3$. solution to Eqs. (82) can be simply written as

$$z^a(q, p) = e^{i(m^a q + n^a p)/N} f^a(q, p) \quad (83)$$

where $f^a(q, p)$ are periodic functions in q and p .

The membrane in Matrix Theory is “quantized” by the prescription first to cut off all the components with the frequency higher than certain K in the Fourier

series of any world-volume function and then to substitute the two algebraic basis functions e^{iq} , e^{ip} with the clock and shift matrices U_K and V_K . Keeping track of the tensorial structure in the orbifolding, we are motivated to promote Eq. (83) to the Matrix Ansatz:

$$Z^a(U_K, V_K) = U_K^{m^a/N} V_K^{n^a/N} F^a(U_K, V_K) \otimes \hat{V}_a \quad (84)$$

with F^a polynomials in U_K and V_K .²

To make sense of Eq. (84), we define the fractional powers of clock and shift matrices U_K , V_K with “nice” properties, a key technicality in this work. For mathematical rigor, we apply the *Dunford functional calculus* [14] to matrices with finite rank K by defining

$$U_K^{a/c} = \frac{1}{2\pi i} \oint_{\Gamma} \zeta^{a/c} (\zeta - U_K)^{-1} d\zeta, \quad (85)$$

$$V_K^{b/d} = \frac{1}{2\pi i} \oint_{\Gamma'} \zeta^{b/d} (\zeta - V_K)^{-1} d\zeta \quad (86)$$

where a, b, c and d are arbitrary integers. From Eq. (15), it is easy to show that the spectrum of U_K contains all of the K -th roots of unity, ω_K^j for $j = 0, 1, \dots, K-1$. To single out this spectrum, the contour Γ by definition consists K disjoint small circles, each encircling an eigenvalue ω_K^j , say $|\zeta - \omega_K^j| = \epsilon$ for some small $\epsilon > 0$. A cut on ζ -plane, running from the origin to the infinity, is drawn to sort out an analytic branch of the function $\zeta^{a/c}$. The cut can not have any intersections with the contour Γ , say passing between two neighboring circles. The same rules on the contour and cut should be applied for the definition of Γ' as well. Note that, as Eq. (15), the definition in Eqs. (85) and (86) is independent of specific matrix representation for U_K and V_K .

²We are not bothered with commensurability of N with respect to m^a and n^a , because eventually N is taken to infinity while keeping m^a and n^a finite. See ref. [13] also.

The following properties are generic results from the Dunford calculus [15].

$$U_K^{a/c} \cdot U_K^{a'/c'} = U_K^{a/c+a'/c'} \quad , \quad V_K^{b/d} \cdot V_K^{b'/d'} = V_K^{b/d+b'/d'} \quad , \quad (87)$$

$$U_K^{a/c\dagger} = (U_K^\dagger)^{a/c} \quad , \quad V_K^{b/d\dagger} = (V_K^\dagger)^{b/d} \quad , \quad (88)$$

$$(U_K^{a/c})^{a'/c'} = U_K^{aa'/cc'} \quad , \quad (V_K^{b/d})^{b'/d'} = V_K^{bb'/dd'} \quad . \quad (89)$$

For example, Eq. (87) follows by manipulating Cauchy's integral formula and adopting the resolvent identity

$$(\zeta' - \zeta)^{-1}[(\zeta - U_K)^{-1} - (\zeta' - U_K)^{-1}] = (\zeta - U_K)^{-1}(\zeta' - U_K)^{-1} \quad (90)$$

(and a similar formula for V_K). A direct consequence of Eqs. (88) and (89) is

$$U_K^{a/c\dagger} = U_K^{-a/c} \quad , \quad V_K^{b/d\dagger} = V_K^{-b/d} \quad , \quad (91)$$

namely $U_K^{a/c}$ and $V_K^{b/d}$ are unitary.

Because of the first two formulas in Eq. (15), one has

$$(\zeta - U_K)^{-1} = (\zeta^K - 1)^{-1} \sum_{j=1}^K \zeta^{K-j} U_K^{j-1} \quad , \quad (\xi - V_K)^{-1} = (\xi^K - 1)^{-1} \sum_{j=1}^K \xi^{K-j} V_K^{j-1} \quad ; \quad (92)$$

and therefore,

$$U_K^{a/c} = \frac{1}{2\pi i} \sum_{j=1}^K \oint_{\Gamma} \frac{\zeta^{a/c+K-j}}{\zeta^K - 1} U_K^{j-1} d\zeta \quad , \quad V_K^{b/d} = \frac{1}{2\pi i} \sum_{j=1}^K \oint_{\Gamma'} \frac{\xi^{b/d+K-j}}{\xi^K - 1} V_K^{j-1} d\xi \quad . \quad (93)$$

With the help of these equations, we have the following theorem:

Theorem 1

$$V_K^{b/d} \cdot U_K^{a/c} = \omega_K^{ab/cd} U_K^{a/c} \cdot V_K^{b/d} \quad . \quad (94)$$

The proof of this theorem is presented in Appendix A.

This theorem is the central result of this subsection. It is a very nice property of the fractional powers we have defined. Comparing Eq. (94) with the commutation relation $V_K U_K = \omega_K U_K V_K$, we see that the complex factor $\omega_K^{ab/cd}$ in the former

is just a fractional power of ω_K in the latter. This property is highly non-trivial, because here we are dealing with the commutation relations of the fractional power of two *noncommuting* operators.³

After these preparations, now we can proceed with our Matrix Ansatz (84) for the wrapping states of a matrix membrane, which is a proper, noncommutative generalization of the ordinary wrapping map (83). Our discussion below on membrane physics will heavily rely on the commutation relation (94).

The next step to define those matrix states is to specify the matrix functions $F^a(U_K, V_K)$ in Eq. (84). In this work, we will restrict ourselves to the center-of-mass motion, suppressing the oscillation modes. Therefore, F^a are just complex numbers. As for the values for m^a, n^a , we take Schwarz' Ansatz

$$n^2 = q_1, \quad n^3 = q_2 \quad (95)$$

whose reason will be explored in full length in Subsection 4.4.1. Because of Eq. (91), motion in the radial directions and in the unorbifolded angular direction are suppressed, exactly the same as from Eq. (6) to Eq. (7). In accordance with subsection 3.1, if we further require

$$|F^a| = \langle z^a \rangle \quad (96)$$

with $\langle z^a \rangle$ are the VEV given in Eq.(18), then the Matrix Ansatz (84) describes a matrix membrane wrapping on T^2 , as dictated in Eq. (95). In other words, contrary to Eq. (17), the factor $F^a U_K^{m^a/N} V^{n^a/N}$ in Eq. (84) provides a polar decomposition of the bi-fundamental variable Z^a . Here $|F^a|$ may be interpreted as the distance from the membrane to the orbifold singularity (as center-of-mass degrees of freedom), while the unitary matrix $U_K^{m^a/N} V^{n^a/N}$ describes how the constituent D0-branes of the membrane are wrapped in the orbifolded angular direction of Z^a .

³See [16] for the fractional powers of operators along a line other than the Dunford calculus.

4.3 Dynamics of Wrapping Membranes

After defining a class of wrapping membrane states with Eqs. (84), (95) and (96), we discuss their physics in this subsection. Since the wrapping states involves pulling the geometry of T^2 back to the membrane in subsection 4.1, the membrane probes the geometry constructed in subsection. 3.1 via the wrapping states.

4.3.1 Classical Motions

As part of the BFSS conjecture [4], membranes in Matrix Theory are considered to be a composite of D0 branes. Therefore, the action (2) of our quiver matrix mechanics, as an orbifolded Matrix Theory, legitimately describes the dynamics of the matrix membrane degrees of freedom for the states given by Eq. (84). Generically the center-of-mass degrees of freedom F^a are time dependent. The classical motion for $F^a(t)$ in Eq. (84) is determined by the equation of motion (EOM) derived from the action (2):

$$\ddot{Z}^a + \frac{R_{11}^2}{2}([Z^b, [Z^{b\dagger}, Z^a]] + [Z^{b\dagger}, [Z^b, Z^a]]) = 0, \quad (97)$$

in which we have recovered R_{11} explicitly. For convenience, we write $Z^a(U_K, V_K)$ shortly as Z^a hereafter without confusion. Eq. (97) is satisfied if

$$F^a(t) = \langle z^a \rangle e^{-i\omega_a t} \quad (98)$$

with

$$\omega_a^2 = \left(\frac{R_{11}N}{2\pi}\right)^2 \left(1 - \cos \frac{2\pi w}{KN^2}\right) \sum_{b \neq a} |f_b|^2. \quad (99)$$

Moreover, the solution (98) also solves an additional constraint

$$[\dot{Z}^a, Z^{a\dagger}] + [\dot{Z}^{a\dagger}, Z^a] = 0, \quad (100)$$

that descends from the gauge fixing of the membrane world-volume diffeomorphism (see for example [17]).

4.3.2 Wrapping Spectrum

First let us calculate the energy density on a wrapping membrane:

$$\mathcal{H} = \frac{R_{11}}{2} \{ |[Z^a, Z^{b\dagger}]|^2 + |[Z^a, Z^b]|^2 \}. \quad (101)$$

Taking into account Eqs. (84) and (98), one has

$$\mathcal{H} = R_{11} \left(\frac{N}{2\pi} \right)^4 (f_1^2 f_2^2 + f_1^2 f_3^2 + f_2^2 f_3^2) \left(1 - \cos \frac{2\pi w}{KN^2} \right) \mathbf{1}_{KN^2}. \quad (102)$$

Instead of Eq. (26) in the context of the SYM limit, for wrapping membrane states the trace of $\mathbf{1}_{KN^2}$ is regularized to be $(2\pi)^2 K$. Recalling the toroidal metric (8), in either the large N or the large K limit, the wrapping energy approaches to

$$P_w = Tr \mathcal{H} = \frac{M_w^2}{2P^+} \quad (103)$$

where

$$M_w = T_{M2} w A_{T^2} \quad (104)$$

with $T_{M2} = 1/(2\pi)^2$ the dimensionless membrane tension, A_{T^2} the area of T^2 given by Eq. (11), and the light-cone momentum

$$P^+ = K/R_{11}. \quad (105)$$

Eqs. (103), (104) and (105) match perfectly with the M-theory picture. The light-cone energy P_w is of the nonrelativistic form in Eq. (103), with light-cone mass P^+ ; the transverse (wrapping) mass M_w is factorized exactly into the correct membrane tension, wrapping number and the area of the torus in target space. As a finite energy state, the light-cone energy scales like $\mathcal{O}(1/K)$, as predicted by BFSS [4].

4.3.3 Stability of Configuration

From Eq. (99), the configuration in Eq. (98) is static if and only if there is no wrapping (provided $w \ll N$.) This is not a surprise that a wrapped static membrane

cannot stay stably on an orbifold because, the closer the membrane to the orbifold singularity, the less the tension energy costs in Eq. (104). (In other words, the only stable static wrapping configuration is the one in which all D-particles stay right at the origin.)

Consequently, a wrapping membrane away from the origin has to rotate to achieve a stationary state. The rotation with the angular velocity ω_a contributes a nonzero kinetic energy:

$$K_w = \frac{1}{R_{11}} Tr |\dot{Z}^a|^2 = P_w. \quad (106)$$

Another observation is

$$|\omega_a| = R_{11} \frac{N}{\pi} \sin\left(\frac{\pi w}{KN^2}\right) \left[\sum_{b \neq a} f_b^2/2\right]^{1/2}. \quad (107)$$

The interpretation of Eq. (107) is simple: due to the fuzziness introduced by finite N , the wrapping membrane generally rotates also in the un-orbifolded direction φ^1 .

4.3.4 Center-of-Mass Momenta

Eq. (84) is not the most general solution to the equations of motion (97) and (100). At least one can add a term linear in time:

$$Z^a = \frac{f_a}{\sqrt{2}2\pi} (N e^{-i\omega_a t} U_K^{m^a/N} V_K^{n^a/N} + \frac{iR_{11}}{K} k^a t) \otimes \hat{V}_a, \quad (108)$$

where real coefficients k^a are to be determined. Now the total (light-cone) energy of the configuration (108) at finite N and K is

$$H_{mem} = \frac{1}{2P^+} \frac{tr}{K} \left\{ f_a^2 |k^a - \frac{\sin \pi w / KN^2}{\pi / KN^2} [\sum_{b \neq a} f_b^2 / 2]^{1/2} e^{-i\omega_a t} U_K^{m^a/N} V_K^{n^a/N} |^2 \right\} + P_w. \quad (109)$$

To evaluate Eq. (109), we need to know $tr\{U^{m^a/N} V^{n^a/N}\}/K$. From Eq. (93) and the definition of the contours Γ, Γ' and the branch cuts, we get

$$\begin{aligned} \frac{tr}{K} \{U^{m^a/N} V^{n^a/N}\} &= \frac{1}{K^2} \left(\sum_{j=1}^K e^{i2\pi j m^a / KN} \right) \left(\sum_{j'=1}^K e^{i2\pi j' n^a / KN} \right) \\ &= \frac{e^{i2\pi m^a / KN} (e^{i2\pi m^a / N} - 1)}{K (e^{i2\pi m^a / KN} - 1)} \cdot \frac{e^{i2\pi n^a / KN} (e^{i2\pi n^a / N} - 1)}{K (e^{i2\pi n^a / KN} - 1)}. \end{aligned} \quad (110)$$



So in the large- K and large- N limit (continuous membrane and continuous torus limits), with $R_{11} = K$,

$$H_{mem} = \frac{1}{2}(f_a^2 p^{a2} + M_w^2) \quad (111)$$

where $p^a = k^a - w[\sum_{b \neq a} f_b^2/2]^{1/2}$. From the point of view of T^2 , we require $p^1 = -p^2 - p^3$; therefore k^a can not be independent. And we also require that the canonical momenta $p_\alpha = g_{\alpha\beta} p^\beta$ are quantized to take integer values l_2, l_3 ; accordingly,

$$H_{mem} = \frac{1}{2}(g^{\alpha\beta} l_\alpha l_\beta + M_w^2). \quad (112)$$

4.4 IIB/M(atrrix) Duality from Wrapping Membranes

Now we are in a position to verify the IIB/M(atrrix) duality by studying the spectrum and symmetry of matrix membrane states.

4.4.1 Elimination of Unwanted Degeneracy

Generically different wrapping map matrices may have the same wrapping number. Since the energy of wrapping states is proportional to w , we seem to encounter a possible enormous degeneracy for a given w . Even the $SL(2, \mathcal{Z})$ equivalence in Eq. (80) is not able to resolve all the degeneracy. On the other hand, if one wants to make the correspondence between wrapping membrane states and the doubly-charged (q_1, q_2) string states, in accordance with IIB string theory, the wrapping membranes with a given wrapping number w should be non-degenerate [9]. To eliminate the degeneracy of wrapping membranes, Schwarz incorporates the Kaluza-Klein direction in to picture for the (q_1, q_2) strings [9].

According to the generalized T-duality that we briefed in Sec. 1, (q_1, q_2) -string winding l times on the IIB theory circle is dual to KK mode (lq_1, lq_2) on T^2 in M-theory, with q_1, q_2 coprime. Schwarz' *ansatz* says that one cycle of the membrane, say, in p -direction, *must* posit in the KK direction and winds only once, namely

$\underline{n} = (q_1, q_2)^T =: \underline{q}$. Accordingly,

$$w = m^2 q_2 - m^3 q_1. \quad (113)$$

We claim the following theorems:

Theorem 2 *For any pair of coprime integers (q_1, q_2) , there exists a pair of integers (m^2, m^3) such that Eq.(113) is satisfied with $w = 1$.*

This theorem is an elementary result in Number Theory; it has an (obviously) equivalent presentation:

Theorem 3 *For any integer w and any pair of coprime integers (q_1, q_2) , there exists a pair of integers (m^2, m^3) such that Eq.(113) is satisfied.*

The following theorem asserts the uniqueness of the considerations:

Theorem 4 *If there exists another pair (m'^2, m'^3) satisfying Eq. (113), then there exists an $SL(2, \mathcal{Z})$ transformation on the membrane coordinates (q, p) that relates these two wrapping maps.*

We present a proof to Theorem 4 here. Because both (m^2, m^3) and (m'^2, m'^3) satisfy Eq. (113), $(m'^2 - m^2)q_2 = (m'^3 - m^3)q_1$. Subsequently, there exists an integer b such that $m'^2 = m^2 + bq_1$, $m'^3 = m^3 + bq_2$. Then, as usual $\underline{m}' := (m'^2, m'^3)^T$,

$$W(\underline{m}', \underline{q}) = W(\underline{m}, \underline{q}) \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}. \quad (114)$$

Q.E.D.

Using these theorems, the original characterization of a wrapping map is traded into three parts, (q_1, q_2) -charge, wrapping number w and an $SL(2, \mathcal{Z})$ family labelled by b . Thus the wrapping states with given w is non-degenerate up to the geometric $SL(2, \mathcal{Z})$ symmetry on the membrane.

4.4.2 IIB/M Duality

To see how the generalized T-duality [9] works, on the one hand w is the wrapping number of a membrane over T^2 in M-theory; on the other hand the wrapping mass (104) can be reinterpreted as a KK momentum as suggested by IIB/M duality, namely

$$M_w = w/R_B, R_B = 1/\sqrt{g} \quad (115)$$

where the newly constructed IIB circle S_B has radius R_B . S_B becomes decompactified when the size of T^2 , measured with \sqrt{g} , shrinks to zero; hence, the wrapping contribution to the spectrum (112) becomes a continuous kinetic energy. To complete the generalized T-duality, besides Eq. (95), we add the following known constraint on the center-of-mass momenta of the membrane, coming from the argument of the stability of the FD-string bound states, that

$$l_2 = lq_1, l_3 = lq_2. \quad (116)$$

The counterpart of the “nine-eleven flip” in IIA/M duality is the identification of the modular parameter τ of T^2 and the IIB coupling $\chi + ie^{-\phi}$, where χ is the Ramond-Ramond scalar and ϕ the dilaton in IIB string theory. In accordance with this identification, the metric of T^2 can be expressed in IIB terms as

$$(g_{\alpha\beta}) = e^{2\omega} \begin{pmatrix} 1 & \chi \\ \chi & \chi^2 + e^{-2\phi} \end{pmatrix}. \quad (117)$$

Therefore, the parametrization f_a is just the moduli of an RR-scalar, dilaton and an overall radion [18].

Since both the determinant g and the wrapping number are invariant under $SL(2, \mathcal{Z})$ transformation of T^2 , so is the wrapping mass (104). Moreover, (q_1, q_2) transform covariantly under $SL(2, \mathcal{Z})$, so the kinetic energy (112) is also invariant, as well as the total energy H_{mem} .



5 Conclusions and Discussions

The logic underlying this paper can be summarized as follows. First the geometry setting is the orbifold $\mathcal{C}^3/\mathcal{Z}_N^2$, in which a discretized torus is “embedded” even at finite N . On the one hand, the collective motion of D-particles in the angular directions on this orbifold may develop a (regularized) wrapping membrane, which are described by fractional powers of the clock and shift matrices. Our quiver matrix mechanics governs the dynamics of the wrapping configurations. The wrapping modes develop a Kaluza-Klein tower, giving rise to the generalized T-duality and IIB/M(atrrix) duality. On the other hand, by deconstruction of D-particle states, the dual torus emerges in target space and 1 + 2-dimensional SYM emerges in the large N limit. This deconstruction procedure also reveals a (hidden) underlying 1 + 3-dimensional SYM, which plays a vital role in the literature of IIB/M(atrrix) Theory duality.

Our analysis of IIB/M duality concentrates mainly on the spectroscopy, leaving for the future research of the deduction of an effective theory from SYM as well as the relation between Yang-Mills coupling and IIB coupling. A spectroscopic discussion of wrapping membrane is also given in [19], where $T^3 \times A_{N-1}$ is taken to be the base space. We only note that to get the correct chirality of IIB fermions from the Matrix Theory is highly nontrivial. The deconstruction technique has been widely employed in string community, for examples see [20, 21]; more comprehensive discussion on compactification in various dimensions, especially on M5-brane, can be found in [8, 22] (see also [23] and [24] for intersectional M5-brane). As we noted before, the main technical difficulty that we have overcome is to construct the matrix membrane states wrapping on the orbifold. Our formalism of fractional powers of clock and shift matrices has a natural connection with fractional membrane, of which early intuition can be traced to [25].

We have applied our approach to both IIA/M and IIB/M dualities. While IIA/M

duality is now a somewhat common exercise⁴, the success of our quiver matrix mechanics approach in demonstrating IIB/M duality shows its powerfulness in dealing with non-perturbative aspects of string theory, in contrast to the inaptitude of compactified Matrix Theory to incorporate the wrapping matrix membrane states.

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A Proof of Theorem 1

We have to show the following lemma first.

Lemma 1 *For arbitrary integer j ,*

$$(\omega_K^j U_K)^{a/c} = \omega_K^{ja/c} U_K^{a/c}; \quad (118)$$

a similar statement holds for $V_K^{b/d}$ too.

In fact,

$$\begin{aligned} \text{R.H.S. of Eq. (118)} &= \frac{1}{2\pi i} \sum_{j'=1}^K \oint_{\Gamma} \frac{(\omega_K^j \zeta)^{a/c} \zeta^{K-j'}}{\zeta^K - 1} U_K^{j'-1} d\zeta \\ &= \frac{1}{2\pi i} \sum_{j'=1}^K \oint_{\Gamma} \frac{(\omega_K^j \zeta)^{a/c+K-j'}}{(\omega_K^j \zeta)^K - 1} (\omega_K^j U_K)^{j'-1} d(\omega_K^j \zeta). \end{aligned} \quad (119)$$

In the last line, we used the fact that $(\omega_K^j)^K = 1$. Changing the variable from ζ to $\omega_K^j \zeta$ just shifts cyclically the circles constituting the symmetric contour Γ , without

⁴For the latest discussions, see [26, 27].

the integral unchanged. Recall the definition in Eq. (93), and the L.H.S. of Eq. (118) follows.

Then the proof of Eq. (94) becomes straightforward. By substituting Eq. (93),

$$\text{L.H.S. of Eq. (94)} = \frac{1}{2\pi i} \sum_{j=1}^K \oint_{\Gamma'} \frac{\xi^{b/d+K-j}}{\xi^K - 1} \frac{1}{2\pi i} \sum_{j'=1}^K \oint_{\Gamma} \frac{\zeta^{a/c+K-j'}}{\zeta^K - 1} V_K^{j-1} U_K^{j'-1} d\zeta d\xi. \quad (120)$$

Because of the commutation relation in Eq. (15), $V_K^{j-1} U_K^{j'-1} = \omega_K^{(j-1)(j'-1)} U_K^{j'-1} V_K^{j-1}$.

Accordingly, the R.H.S of Eq. (120) is

$$\frac{1}{2\pi i} \sum_{j=1}^K \oint_{\Gamma'} \frac{\xi^{b/d+K-j}}{\xi^K - 1} \left[\frac{1}{2\pi i} \sum_{j'=1}^K \oint_{\Gamma} \frac{\zeta^{a/c+K-j'}}{\zeta^K - 1} (\omega_K^{j-1} U_K)^{j'-1} d\zeta \right] V_K^{j-1} d\xi. \quad (121)$$

By definition (93), [...] in Eq. (121) is just $(\omega_K^{j-1} U_K)^{a/c}$. Due to Lemma 1, $(\omega_K^{j-1} U_K)^{a/c} = \omega_K^{(j-1)a/c} U_K^{a/c}$. Then, the R.H.S of Eq. (121) is

$$U_K^{a/c} \frac{1}{2\pi i} \sum_{j=1}^K \oint_{\Gamma'} \frac{\xi^{b/d+K-j}}{\xi^K - 1} (\omega_K^{a/c} V_K)^{j-1} d\xi. \quad (122)$$

Again by definition in Eq. (93), Eq. (122) gives rise to $U_K^{a/c} (\omega_K^{a/c} V_K)^{b/d}$. Finally, using Lemma 1 again, the R.H.S. of Eq. (94) follows. *Q.E.D.*

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