Massless fermions and Kaluza–Klein theory with torsion

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A pure Kaluza–Klein theory contains no massless fermion in four-dimensional theory. We investigate the effect of introducing torsion on the internal manifold and find that there are massless fermions. The hope is that given an isometry group the representation to which these fermions belong is fixed, in contrast to the situation in Yang–Mills theory. We show that this is indeed the case, but the representations do not appear to be the ones favored by current theoretical prejudice. The cases with parallelizable torsions on a group manifold as the internal manifold are analyzed in detail.

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I. INTRODUCTION

One of the stumbling blocks facing theorists trying to treat Kaluza–Klein theory[1–3] as the theory of the world is the difficulty of obtaining low mass fermions.[3,4]

Consider a Kaluza–Klein theory[5] based on $M^4 \times B$, where B is an n-dimensional compact manifold with an isometry group G. The resulting theory contains Yang–Mills fields transforming according to the gauge group G. The length scale $l_{\rm kk}$ of B is of the order of the Planck length divided by the gauge coupling constant. It is well known by now that if the $(4 + n)$-dimensional theory contains a massless fermion field, then the resulting four-dimensional theory contains an infinite spectrum of fermion fields whose masses are determined by the eigenvalues of the internal Dirac operator $i\mathscr{D}^{(\rm int)}$ appropriate to B. Since the natural mass scale is set by the huge Kaluza–Klein mass $M_{\rm kk} \equiv l_{\rm kk}^{-1}$, the observed quarks and leptons must correspond to the zero eigenvalue of $i\mathscr{D}^{(\rm int)}$.

The difficulty is that if B is a homogeneous space $B = G/H$, then $i\mathscr{D}^{(\rm int)}$ has no zero eigenvalue. The reason is as follows.[5] For G/H (with standard metric) one can show that the scalar curvature R is positive. On the other hand, the square of the Dirac operator may be evaluated[5,6] to be $-D^2 + \frac{1}{4}R$ and is therefore the sum of a non-negative operator and a positive operator.

One possible way around this difficulty involves introducing explicit gauge fields not related to the metric. If the ground state of the theory is such that these explicit gauge fields assume a topological configuration on B, then zero modes exist for the internal Dirac operator.[7,8]

In this paper we investigate an alternative possibility, that of introducing torsion on the manifold B. This means that we treat the connection $\omega^a{}_{bc}$ on B as an object unrelated to the Vielbein $e^a{}_i$. (Our convention is that of Ref. 5. See also Appendix A.) Since the internal Dirac operator, which we identify henceforth as the mass operator M, is given by

$$M \equiv i\mathscr{D}^{(\rm int)} = i\gamma^a (e^{ai}\,\partial_i - (i/4)\omega^a{}_{bc}\sigma^{bc}), \qquad (1.1)$$

one might easily imagine that with some choices of ω one can find zero modes of M.

We show below that the introduction of torsion indeed allows the existence of numerous fermion zero modes. Unfortunately, there are an equal number of left- and right-handed zero modes. This is a problem which has cropped up repeatedly in contemporary particle theory.[9] To demonstrate that this is generally the case, one would have to show that (1) the Atiyah–Singer theorem is unaffected by the introduction of torsion and (2) torsion does not change the Pontryagin number of a manifold. (A partial discussion of these points will be given in Appendices B and C.) Thus, our discussions would appear to be irrelevant for the real world unless we suppose that the fundamental interaction at the preon level is left–right symmetric and that the left–right symmetry is broken by some as-yet unknown mechanism. Nevertheless, we feel that the effect of torsion is worth investigating in some detail.

One class of manifolds with torsion consists of the parallelizable manifolds defined by Cartan and Schouten.[10] The Cartan–Schouten program is a particularly restrictive way of introducing torsion on certain manifolds so that the Riemann curvature tensor vanishes. The preceding discussion makes it suggestive that zero curvature might allow M to have zero modes. Compact Lie groups form a wide class of parallelizable manifolds and we will focus on group manifolds in this paper. Not surprisingly, we are led, after some work, to face certain equations endowed with a rather neat algebraic structure which may be of some mathematical interest in themselves.

We have solved these algebraic equations. It turns out that normally fermion zero modes form an even number of "families," but the number of zero modes escalates rapidly as the rank of the group increases. For example, for SU(5), there are four "families" of zero modes in the representation 1024.

After our work was completed, we learned that Orzalesi and collaborators[11] had launched an extensive program of studying torsion in Kaluza–Klein theory. In particular, Destri, Orzalesi, and Rossi (in Ref. 11) were the first to point out the relevance of torsion for the existence of Dirac zero modes and have studied the case of parallelized group manifolds. Their analysis, while employing a slightly different formalism, is essentially the same as ours, but they do not determine the representation in question for a general simple Lie group as explicitly as we do. Also we give a method for reducing this representation and point out the appearance of the repetitive structure in this reduction. They also studied the dynamical basis for compactification with torsion which we do not do. For an alternative application of parallelizable torsions in Kaluza–Klein theories, see Ref. 12.

In Sec. II, a brief review of the Cartan–Schouten program is given. In Sec. III, we work out the reduction of fermions in Kaluza–Klein theory. Putting together the material from these two sections, we find in Sec. IV that the search for fermion zero modes leads us to some interesting group theory problems which we solve in Secs. V and VI.

II. TORSION ON GROUP MANIFOLDS

A manifold is said to have torsion if the connection 1-form $\omega^a{}_b$ is treated as independent of the Vielbein 1-form e^a. Define the torsion two-form by

$$T^a = de^a + \omega^a{}_b e^b. \quad (2.1)$$

Without torsion, $T^a = 0$ and so $\omega^a{}_b$ is determined in terms of e^a. The Riemann curvature is given in any case by

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b. \quad (2.2)$$

We focus our attention on compact Lie groups G. The points of the manifold are associated with group elements g. At a given point, one defines the Vielbein by

$$g^{-1}\, dg = \sum_a i(\lambda^a/2) e^a. \quad (2.3)$$

We normalize the generators of the Lie algebra of G by

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = if^{abc}\,\frac{\lambda^c}{2}. \quad (2.4)$$

(We have chosen the Cartan metric on the group to be just δ^{ab} so that we need not distinguish between upper and lower group indices.) Differentiating Eq. (2.3), one finds the Cartan–Maurer equation

$$de^a = \tfrac{1}{2} f^{abc} e^b e^c. \quad (2.5)$$

Without torsion, we see by referring to Eq. (2.1) that the Cartan–Maurer equation implies $\omega^{ab} = \frac{1}{2} f^{abc} e^c$. It is natural and pleasing that the connection ω^{abc} is related to the structure constant f^{abc}. We now take a "physicist's" constructive approach to parallelizable torsions. Adopt the ansatz

$$\omega^{abc} = \tfrac{1}{2} K f^{abc}; \quad (2.6)$$

this defines a one-parameter family of possible connections. A simple computation using Eq. (2.2) and Jacobi's identity gives

$$R^{ab} = \tfrac{1}{4} K (1 - \tfrac{1}{2} K) f^{abc} f^{cdg} e^d e^g. \quad (2.7)$$

By definition, a parallelizable connection leads to a vanishing curvature tensor. This yields two solutions:

$$K = 0, \quad \omega^{abc} = 0, \quad \tau = +1, \quad (2.8)$$

$$K = 2, \quad \omega^{abc} = f^{abc}, \quad \tau = -1. \quad (2.9)$$

The quantity τ specifies the corresponding torsion by

$$T^a = \tfrac{1}{2} \tau f^{abc} e^b e^c. \quad (2.10)$$

Referring to Eq. (A5) in Appendix A, we determine the Christoffel symbol to be

$$\Gamma_{ij}{}^k = e_a{}^k \partial_i e_j{}^a \quad \text{in the } \tau = +1 \text{ case,} \quad (2.8')$$

and

$$\Gamma_{ij}{}^k = -(\partial^k e_i{}^a) e_{aj} \quad \text{in the } \tau = -1 \text{ case.} \quad (2.9')$$

Note that since $\omega^a{}_b$ transforms as a connection and not as a tensor 1-form, the relations in Eqs. (2.8) and (2.9) are clearly specific to the Vielbein basis defined by the left-invariant one-form in Eq. (2.3).

The preceding is a simple realization for group manifolds of the Cartan and Schouten theory of parallelizable connections. They posed themselves the following problem. Given a Riemannian manifold with a metric g_{ij} and a Christoffel symbol $\overset{(0)}{\Gamma}{}_{ij}{}^k$ constructed from the metric, is it possible to find a tensor $S_{ij}{}^k$ such that the following three conditions hold (defining $\Gamma_{ij}{}^k \equiv \overset{(0)}{\Gamma}{}_{ij}{}^k + S_{ij}{}^k$)?

(1) The geodesics for Γ are the same as those for $\overset{(0)}{\Gamma}$.

(2) The covariant derivative of g_{ij} relative to Γ vanishes.

(3) The curvature tensor constructed from g_{ij} and $\Gamma_{ij}{}^k$ vanishes.

It is easy to see that conditions (1) and (2) imply that S_{kij} is totally antisymmetric. We understand that Cartan and Schouten prove a nontrivial theorem stating parallelizable manifolds (i.e.., those allowing parallelizable connections) besides Euclidean spaces are group manifolds and the seven-sphere S^7. To appreciate this last statement consider simply setting the connection one-form $\omega^a{}_b = 0$ on an arbitrary manifold. This certainly insures $R^a{}_b = 0$. However, in the Cartan–Schouten construction, one replaces a torsion-free connection $\overset{(0)}{\omega}{}^a{}_b$ by

$$\omega^a{}_b = \overset{(0)}{\omega}{}^a{}_b + \lambda^a{}_b, \quad (2.11)$$

where $\lambda^a{}_b$ is required to be a tensor one-form. By simply setting $\lambda^a{}_b = -\overset{(0)}{\omega}{}^a{}_b$ one defines, in a coordinate-dependent way, a $\lambda^a{}_b$ (and therefore $\omega^a{}_b$) which is not globally defined (i.e., singular somewhere) on the manifold, except for those named in the Cartan–Schouten theorem.

We return to our discussion of group manifolds. There exists a beautiful theorem that on group manifolds the Vielbein defined in Eq. (2.3), which is obviously left-translation invariant, gives a set of Killing vectors

$$\xi^a_i = e^a_i. \quad (2.12)$$

This can be easily proven in the following way.

Written out in component form, the Cartan–Maurer equation states

$$\partial_i e^a_j - \partial_j e^a_i = f^{abc} e^b_i e^c_j. \quad (2.13)$$

Multiplying by $e^{di} e^{fj} e^{ak}$ we find that the Killing vectors defined in Eq. (2.12) satisfy Lie's equation

$$\xi^{ai} \partial_i \xi^{bj} - \xi^{bi} \partial_i \xi^{aj} = -f^{abc} \xi^{cj}. \quad (2.14)$$

It is now easy to show that the metric $g^{ij} = e^{ai} e^{aj}$ admits the ξ^a_i as Killing vectors. One verifies, by using Lie's equation, that the Lie derivative of the metric along a Killing vector vanishes:

$$\mathscr{L}_{\xi^a} g^{ij} = \xi^{ak}(\partial_k g^{ij}) - (\partial_k \xi^{ai}) g^{kj} - (\partial_k \xi^{aj}) g^{ik} = 0. \quad (2.15)$$

[Incidentally, a group manifold may be thought of as the symmetric space $G \times G/G_D$, where G_D is the diagonal subgroup of $G \times G$. The isometry could be effected either by left or by right multiplication of group elements, corresponding to the two choices of writing $g^{-1}dg$ or $dg\,g^{-1}$ in Eq. (2.3). It is well known in mathematics that the $\tau = +1$ connection in the right-invariant Vielbein corresponding to $dg\,g^{-1}$ has the same form as the $\tau = -1$ connection in the left-invariant one, and vice versa. So we need to do the analysis only with the left-invariant Killing vectors.]

III. REDUCTION OF FERMION FIELD

We discuss here the reduction of Dirac fields in Kaluza–Klein theory with torsion. The discussion is for a general internal manifold B with an isometry group G.

Start with the Dirac Lagrangian in $(4+n)$-dimensional theory:

$$\mathscr{L} = \bar{\Psi} i \hat{\gamma}^\mu (\partial_{\hat{\mu}} - (i/4)\sigma^{\hat{\alpha}\hat{\beta}}\, \hat{\omega}_{\hat{\alpha}\hat{\beta}\hat{\mu}})\, \hat{\Psi} \equiv \bar{\Psi} i \hat{\gamma}^\mu \mathscr{D}_{\hat{\mu}}\, \hat{\Psi}. \quad (3.1)$$

Our notation is the same as in Ref. 5. Briefly, Greek indices, μ, ν, ..., α, β... refer to the "external" four-dimensional space while Latin indices $i, j,...a,b,...$ refer to the "internal" space. The "hat" notation is used when we have to refer to the entire $(4+n)$-dimensional space. We also find it convenient occasionally to refer to the "internal" coordinates x^i collectively as y and to the "external" coordinates as x.

Fortunately, if we are interested only in dimension-4 terms in the four-dimensional theory, we do not have to compute every component of $\hat{\omega}_{\hat{\alpha}\hat{\beta}\hat{\mu}}$. By a dimensional argument we can see that we can effectively set the connection in the Vielbein basis to be

$$\begin{aligned} &\hat{\omega}_{abc} \to \omega^{(\text{int})}_{abc}, \\ &\hat{\omega}_{\alpha\beta\gamma} \to \omega^{(\text{ext})}_{\alpha\beta\gamma}, \\ &\hat{\omega}_{ab\alpha} \to e_{ai}(e^j_b\, \partial_j \xi^i_r - \xi^j_r \partial_j\, e^i_b) A^r_\mu e^\mu_\alpha. \end{aligned} \quad (3.2)$$

Here ξ^r_j denotes the Killing vector corresponding to the generator of G labeled by the index r and A^r_μ denotes the corresponding gauge potential. We have chosen the Cartan metric to be flat $\propto \delta^{rs}$.

The Dirac field $\hat{\Psi}$ transforms as a spinor under the local group SO$(4+n)$ [or SO$(3+n,1)$]. Recall that the theory of orthogonal groups is such that the spinor index carried by $\hat{\Psi}$ factorizes into a spinor index for SO(4) and a spinor index for SO(n). The gamma matrices factorize accordingly:

$$\gamma^\alpha = \gamma^\alpha \times \gamma^5, \quad (3.3a)$$

$$\gamma^a = 1 \times \gamma^a, \quad (3.3b)$$

$$\gamma_5 = \gamma_5 \times \gamma_5. \quad (3.3c)$$

Note that the presence of γ_5 in Eq. (3.3a) but not in (3.3b) is necessary in order for $\{\gamma^\alpha, \gamma^a\} = 0$. In Eq. (3.3c) the notation is such that the three γ_5 denote the γ_5 matrix for the Clifford algebra corresponding to SO$(4+n)$, SO(4), and SO(n), respectively.

Combining Eqs. (3.2) and (3.3) we find that

$$\gamma^{\hat{\mu}} \mathscr{D}_{\hat{\mu}} \to \gamma^\mu \mathscr{D}^{(\text{ext})}_\mu + \gamma^i \mathscr{D}^{(\text{int})}_i - \gamma^a e^\mu_a A^r_\mu T_r. \quad (3.4)$$

Here $\mathscr{D}^{(\text{ext})}_\mu$ and $\mathscr{D}^{(\text{int})}_\mu$ are the covariant derivatives (not including the Yang–Mills potential) constructed out of $\omega^{(\text{ext})}$ and $\omega^{(\text{int})}$, the "external" and "internal" connections, respectively. The eigenvalues of the operator $M \equiv i\gamma^i \mathscr{D}^{(\text{int})}_i$ determine the fermion mass spectrum in the resulting four-dimensional theory and so M may be identified as the mass operator.

The operators

$$iT_r \equiv \xi^j_r\, \partial_j + (i/4) e_{ai} (e^j_b \partial_j \xi^i_r - \xi^j_r\, \partial_j\, e^i_b) \quad (3.5)$$

are very interesting. We see that if Eq. (3.4) is to describe correctly the coupling of the Yang–Mills potential to fermion fields, we must have

$$[T_r, T_s] = i f_{rst}\, T_t, \quad (3.6)$$

$$[T_r, M] = 0. \quad (3.7)$$

The operators T_r have been discussed previously by Wetterich[13] and by Tanaka.[14] The derivation given by Tanaka is different from the direct approach followed here and offers additional insight into the origin of T_r. We continue this discussion paying special attention to the case with torsion.

Since we want to add torsion only in the internal manifold, among the components of $\hat{\omega}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}$ in Eq. (3.2) none other than $\hat{\omega}_{abc}$ should be changed, and the change of $\hat{\omega}_{abc}$ is simply to add torsion in $\omega^{(\text{int})}_{abc}$. The operators T_r given by Eq. (3.5) are unchanged by addition of torsion only in the internal manifold. However, if we express T_r in terms of the covariant derivative in the internal space, the expression differs in the cases with and without torsion. Recall that in the case without internal torsion, we have

$$iT_r \equiv \xi^i_r (\partial_i - (i/4)\omega_{abi}\sigma^{ab}) + (i/4) e^j_a e^k_b\, \xi^r_{j;k} \sigma^{ab}, \quad (3.8)$$

where $\xi^r_{j;k}$ is the usual covariant derivative of ξ^r_j. But, if there is internal torsion, using Eqs. (A4) and (A6) of Appendix A we can rewrite T_r as

$$iT_r = \xi^i_r\, \partial_i + (i/4)\sigma^{ab} \times [e^i_a e^j_b (\nabla_j\, \xi_{ri} - T_{ijk}\, \xi^k_r) - \omega_{abk}\, \xi^k_r], \quad (3.9)$$

where

$$\nabla_j\, \xi^i_r = \partial_j\, \xi^i_r + \Gamma^i_{kj} \xi^k_r. \quad (3.10)$$

According to Eqs. (3.6) and (3.7) the eigenmodes of M corresponding to a given eigenvalue furnish a representation of the group G. One defines[14] spinor harmonics $U(y)$ by the equation

$$(T_r)_{NN'}\, U^{(\sigma\pm)K}_{N'}(y) = (t^{(\sigma)}_r)^{KK'}\, U^{(\sigma\pm)K'}_N(y). \quad (3.11)$$

Here N denotes a spinor index (of the internal space), σ specifies the representation of the group G, and K labels the components of the representation σ. We write the representation matrix in the representation σ as $t^{(\sigma)}_r$. Furthermore, since T_r commutes with γ_5 of the internal space, the harmonics can be chosen to be chiral eigenstates

$$\gamma_5 U^{(\pm)} = \pm\, U^{(\pm)}. \quad (3.12)$$

On the other hand, M anticommutes with γ_5 and so takes $U^{(+)}$ into $U^{(-)}$:

$$MU^{(\sigma\pm)} = m^{(\sigma)}U^{(\sigma\mp)}. \tag{3.13}$$

We can now expand the field Ψ as a sum of the spinor harmonics over the internal manifold:

$$\Psi^{(+)} = \sum_{\sigma,K} \{ \psi^{(\sigma+)K}(x)U^{(\sigma+)K}(y) + \psi^{(\sigma-)K}(x)U^{(\sigma-)K}(y)\}. \tag{3.14}$$

The coefficients $\psi^{(\sigma\pm)}(x)$ in this expansion are the fermion fields of the four-dimensional theory. The expansion in Eq. (3.14) is for a right-handed field

$$\gamma_5 \Psi^{(+)} = + \Psi^{(+)}. \tag{3.15}$$

The correlation of chirality in Eq. (3.14) is dictated by Eq. (3.3c).

IV. FERMIONS ON GROUP MANIFOLDS

Here we specialize the discussion of the preceding sections to group manifolds. In this case, the operators T simplify to the form

$$T^a = -i(\xi^{aj}\partial_j - (i/4)\sigma^{bc}f^{abc})$$
$$\equiv X^a + Y^a. \tag{4.1}$$

(Since the number of group labels is now equal to the dimension of the manifold, we identify the indices $r, s, ...$ as $a, b ...$, keeping in mind that T^a is independent of the presence of torsion.)

We find it useful to define the operators

$$X^a \equiv -i\xi^{aj}\partial_j \quad \text{and} \quad Y^a \equiv -\tfrac{1}{4}\sigma^{bc}f^{abc},$$

as indicated in Eq. (4.1). The presence of Y^a in this equation reminds us that under a Killing displacement one has to turn the spinor indices on a Dirac field. We note the algebraic structure

$$[X^a,X^b] = if^{abc}X^c, \tag{4.2}$$

$$[X^a,Y^b] = 0, \tag{4.3}$$

$$[Y^a,Y^b] = if^{abc}Y^c, \tag{4.4}$$

which follows from Lie's equation and from Jacobi's identity. This insures the correct commutation relation for T^a.

Clearly, if one were to reduce scalar fields in Kaluza–Klein theory, X^a would play the role of T^a. Scalar harmonics are defined by an equation analogous to Eq. (3.11):

$$X^aV^{(\sigma)K}(y) = (t^{(\sigma)a})^{KK'}V^{(\sigma)K'}(y). \tag{4.5}$$

[For $G = SO(3)$ the $V(y)$'s are just the standard rotation functions.] Similarly, we can represent the algebra as realized by the Y^a's [Eq. (4.4)]:

$$(Y^a)_{NN'}W_{N'}^{(\sigma)K} = (t^{(\sigma)a})^{KK'}W_N^{(\sigma)K'}. \tag{4.6}$$

We learn from Eqs. (4.1)–(4.4) that we have here a problem analogous to the addition of angular momentum in quantum mechanics. We can decompose

$$U_N^{(\sigma)K}(y) = \sum_{\substack{\sigma_1\sigma_2\\K_1K_2}} C(\sigma K;\sigma_1K_1,\sigma_2K_2) \times V^{(\sigma_1)K_1}(y)W_N^{(\sigma_2)K_2}. \tag{4.7}$$

Here C denotes generalized Clebsch–Gordon coefficients. The dependences of U on the spinor index and on the coordinates of the internal manifold separate.

The mass operator

$$M = i\gamma^c(e^{ci}\partial_i - (i/4)\omega_{abc}\sigma^{ab}) \tag{4.8}$$

depends on torsion. Referring to Eqs. (2.8), (2.9), and (2.12), we see that for group manifolds M has the elegant algebraic form

$$M = -\gamma^a(X^a + \tfrac{1}{2}KY^a), \tag{4.9}$$

where K is a real parameter as defined in Eq. (2.6). For the case of no torsion, $K = 1$. For $\tau = +1$, $K = 0$, and for $\tau = -1$, $K = 2$. The fact that M commutes with T^a follows from the algebraic equations, Eqs. (4.1)–(4.4), and from the fact that γ^a transforms in the adjoint representation:

$$[T^a,\gamma^b] = [Y^a,\gamma^b] = if^{abc}\gamma^c. \tag{4.10}$$

We know from a general theorem that for $K = 1$, the torsion-free case, M has no zero mode. As K varies over the real line, zero modes may appear. In particular, consider the two parallelizable cases. The $\tau = +1$ case is easier and will be discussed first.

For $\tau = +1$, $M = i\gamma^a\xi^{aj}\partial_j$, and so it follows immediately that there is one class of zero modes for which $U(y)$ is independent of y. To put it more formally, we set σ_1 in Eq. (4.7) to be the trivial representation so that we simply have to solve Eq. (4.6). This group theoretic problem is treated in the next section. It can be shown that such zero modes are the only ones, because

$$M^2U = (X^a + Y^a)^2U - (Y^a)^2U = 0, \tag{4.11}$$

and because of the property of the Casimir invariant.

V. A GROUP THEORY PROBLEM

We now address the group theoretic problem encountered in the last section. We will phrase the problem in somewhat more general terms in order to clarify the problem.

Given a Lie algebra G, let $A(G) = SO(N)$, where $N =$ the number of generators of G. [With the choice that the Cartan metric can be just the Kronecker delta, as in Sec. II, the elements of $SO(N)$ are automorphisms of G.] Let the generators of $SO(N)$ be represented by the matrices σ_{bc}, $b,c = 1...N$, in some representation r. Define a set of matrices Y_a by

$$Y_a = -\tfrac{1}{4}f_{abc}\sigma_{bc}. \tag{5.1}$$

Then, one can verify, using the Jacobi identity, that

$$[Y_a,Y_b] = if_{abc}\,Y_c. \tag{5.2}$$

Thus, Y_a furnishes a representation of G. The definition of Y_a [Eq. (5.1)] defines a map from the set of representation of $A(G)$ into the set of representations of G. We will refer to this map as a projection and write $P(r)$ as the representation of G corresponding to a representation r of $A(G)$. In the last section we are specifically interested in the projection of the spinor representation s of $A(G) = SO(N)$ [or, strictly speaking, of the spin (N) covering of $SO(N)$]. We want to determine $P(s)$.

[Our knowledge of the mathematical literature is rather limited. However, as far as we can determine by cursory discussions with a couple of mathematicians, our treatment is not in the standard mathematical literature. Partial results have been given for the special cases SU(2), SU(3), and SU(5) by Destri *et al.*[11]]

One can easily prove a series of fairly obvious theorems.

Theorem 1: The projection of a reducible representation is reducible. The projection of an irreducible representation may or may not be reducible.

Theorem 2: $P(r_1 \times r_2) = P(r_1) \times P(r_2)$.

This theorem allows us to determine the projection of any representation of SO(N) once we know $P(s)$.

Theorem 3: The projection of the vector representation of SO(N) is the adjoint representation of g.

For example, for $G = \mathrm{SU}(3)$, $A(G) = \mathrm{SO}(8)$, and $P(\underset{\sim}{8}) = \underset{\sim}{8}$. Theorem 2 allows us to find, for instance, $P(28) = 10 + \overline{10} + \underset{\sim}{8}$. The reducibility of $P(28)$ illustrates Theorem 1.

Clearly, the dimension of $P(r)$ is the same as the dimension of r. This provides one clue to determining $P(s)$: the dimension of $P(s)$ is equal to an integral power of 2. Incidentally, this proves rather indirectly the nonobvious theorem that any Lie algebra has a representation with dimensions equal to an integral power of 2.

We can exploit the fact that we know the explicit form of $\sigma_{ab} = (i/2)\,[\gamma_a, \gamma_b]$ for the spinor representation to evaluate the quadratic Casimir invariant for $P(s)$:

$$Y_a Y_a = (i^2/16) f_{abc} f_{ade} \gamma_b \gamma_c \gamma_d \gamma_e = \tfrac{1}{8} f_{abc} f_{abc}. \tag{5.3}$$

We used Jacobi's identity and properties of the gamma matrices. The fact that $Y_a Y_a$ comes out to be proportional to the unit matrix proves another theorem.

Theorem 4: $P(s)$ is either irreducible or a single irreducible representation repeated. (The number of repetitions is a power of 2.)

These considerations allow us to determine $P(s)$. After all, for a given G, there are not many representations of G with dimensions equal to $2^{[N/2]}/2^{k+1}$ with k a non-negative integer. (Here $[N/2]$ is the smallest integer not less than $N/2$.) It turns out that we have to express our solution using the Dynkin language. Our notation is the standard one as may be found in Refs. 15 and 16, for instance.

Recall that a representation is characterized by its highest weight Λ. Label the representation by Λ. The roots of the algebra are denoted by α_i. Let δ be half of the sum of the positive roots:

$$\delta = \frac{1}{2} \sum_{\substack{\text{positive}\\\text{roots}}} \alpha_i. \tag{5.4}$$

There exists a theorem that[16]

$$2(\delta, \alpha_j) = (\alpha_j, \alpha_j). \tag{5.5}$$

The scalar product between two vectors α and β is given by

$$(\alpha, \beta) = \sum_{i,j} \alpha_i G_{ij} \beta_j, \tag{5.6}$$

where the Dynkin metric G_{ij} is listed in tables.[15] The α_i are the components of α in the Dynkin basis. The number of components is equal to the rank of G. Also, recall the famous Weyl formula for the dimension of a representation Λ:

$$\dim(\Lambda) = \prod_{\substack{\text{positive}\\\text{roots}}} \frac{(\Lambda + \delta, \alpha_i)}{(\delta, \alpha_i)}. \tag{5.7}$$

The quadratic Casimir invariant of the representation Λ is given by

$$C(\Lambda) = (\Lambda, \Lambda + 2\delta). \tag{5.8}$$

In this language, we can write Eq. (5.3) as

$$C(\Lambda(P(s))) = \tfrac{1}{8} \dim(\mathrm{adj}) C(\mathrm{adj}). \tag{5.3$'$}$$

Here "adj" refers to the adjoint representation. We have to find a representation Λ which satisfies Eq. (5.3′).

We assert the following.

Theorem 5: The highest weight of $P(s)$ is δ.

From Weyl's formula [Eq. (5.7)] we see immediately that

$$\dim(\delta) = 2^{(\text{number of positive roots})}, \tag{5.9}$$

which indeed is a power of 2.

To check Eq. (5.3′) we evaluate

$$C(\delta) = 3(\delta, \delta) \tag{5.10}$$

using Eq. (5.8). Now, according to the theorem in Eq. (5.5), δ has the elegant form

$$\delta = (1,1,1,\ldots,1) \tag{5.11}$$

in the Dynkin basis, and so

$$C(\delta) = 3 \sum_{i,j} G_{ij} \tag{5.12}$$

is just the sum of all the entries in the Dynkin metric. Unfortunately, the Dynkin metric G_{ij} differs from Lie algebra to Lie algebra and so we have to evaluate $C(\delta)$ separately for the different cases in Cartan's classification. Furthermore, Λ_{adj} has different forms for different algebras and the evaluation of $C(\mathrm{adj})$ also has to proceed case by case.

Before we go to the general evaluation there are some simple cases for which the preceding formalism is not necessary. For $G = \mathrm{SU}(2)$, $A(G) = \mathrm{SO}(3)$, we can use the explicit form $\sigma_{ab} = -\epsilon_{abc}\tau_c$ for the spinor representation to evaluate $Y_a = (1/2)\tau_a$ so that $P(s) = \underset{\sim}{2}$. The projection in this case obviously just expresses the local isomorphism between SO(3) and SU(2). For $G = \mathrm{SU}(3)$, $A(G) = \mathrm{SO}(8)$. After some thoughts, one finds $P(s) = \underset{\sim}{8}$. According to Theorem 3, P (vector) $= \underset{\sim}{8}$. This is consistent with the famous automorphism of SO(8) in which the two $\underset{\sim}{8}$'s of spinor and the $\underset{\sim}{8}$ of vector can be transformed into each other. In fact, we can exploit this automorphism to determine $P(s)$ in the first place. Let σ^s_{ab} and σ^v_{ab} be the generators of SO(8) in the spinor and vector representations, respectively. There exists a similarity transformation

$$\sigma^s_{ab} = \mathscr{S} \sigma^v_{ab} \mathscr{S}^{-1}. \tag{5.13}$$

Multiplying by f_{abc} and summing over a, b we obtain an explicit construction of $P(s)$.

The general evaluation below uses Tables 7 and 8 in Ref. (15):

$$A_n(\mathrm{SU}_{n+1}):\quad C(\delta) = \tfrac{1}{4} n(n+1)(n+2),\quad C(\mathrm{adj}) = 2n+2,\quad \dim(\mathrm{adj}) = n(n+2);$$

$B_n(\mathrm{SO}_{2n+1})$: $C(\delta) = \frac{1}{4} n(2n-1)(2n+1)$,
$C(\mathrm{adj}) = 2(2n-1)$,
$\dim(\mathrm{adj}) = n(2n+1)$;
$C_n(\mathrm{Sp}_{2n})$: $C(\delta) = \frac{1}{4} n(n+1)(2n+1)$,
$C(\mathrm{adj}) = 2n+2$,
$\dim(\mathrm{adj}) = n(2n+1)$;
$D_n(\mathrm{SO}_{2n})$: $C(\delta) = \frac{1}{2} n(n-1)(2n-1)$,
$C(\mathrm{adj}) = 4n-4$,
$\dim(\mathrm{adj}) = n(2n-1)$;
G_2: $C(\delta) = 14$, $\dim(\delta) = 64$,
$C(\mathrm{adj}) = 8$, $\dim(\mathrm{adj}) = 14$;
F_4: $C(\delta) = 117$,
$\dim(\delta) = 16\,777\,216$,
$C(\mathrm{adj}) = 52$, $\dim(\mathrm{adj}) = 52$;
E_6: $C(\delta) = 234$,
$C(\mathrm{adj}) = 24$, $\dim(\mathrm{adj}) = 78$;
E_7: $C(\delta) = 1197/2$,
$C(\mathrm{adj}) = 36$, $\dim(\mathrm{adj}) = 133$;
E_8: $C(\delta) = 1860$,
$C(\mathrm{adj}) = 60$, $\dim(\mathrm{adj}) = 248$.

It is amusing to see that for all simple compact Lie groups the irreducible representation δ with the highest weight $(1,1,...,1)$ in the Dynkin basis satisfies the relation $C(\delta) = \frac{1}{8} N(\mathrm{adj}) C(\mathrm{adj})$.

The repetition number, mentioned in Theorem 4, of the irreducible representation δ in $P(s)$ is simply the quotient of the dimension of s, the spinor on the group manifold, over that of δ, namely $2^{[N/2]}/\dim(\delta)$. Thus, here see the natural emergence of something like family structure, with the number of families restricted to be a power of 2. For example, for SU(5), the zero modes form four families of the representation $\underline{1024}$.

Thus, except for the lowest ranked groups, we obtain an exceedingly large number of zero modes. For instance, for F_4, there are more than 16 million fermion zero modes! The reason is clearly that the number of zero modes increases exponentially in the number of generators in the group. We find it extremely unlikely that these zero modes could correspond to quarks and leptons. It is perhaps conceivable that at some preon level the gauge group is small, SU(2) say. One could also imagine a Kaluza–Klein theory with the internal manifold $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(1)$. But the fermion representation appears to be incorrect.

We recognize that Eq. (5.3′) is a necessary but not sufficient condition. Thus, strictly speaking, we have only found a candidate solution and we have not proved that our solution is the solution. However, it seems highly unlikely that another representation exists with first the right dimension (a power of 2), and second, the right Casimir invariant [Eq. (5.3′)]. In particular, for those groups in which tables exist,[17] one can easily verify that our solution is unique. We have not bothered to try to complete the proof because unfortunately these zero modes appear to be irrelevant for phenomenology.

VI. SEARCH FOR ZERO MODES

We now return to the other parallelizable case in which $\tau = -1$. The mass operator simplifies to

$$M = i\gamma^a T^a. \tag{6.1}$$

The equation for zero modes $MU = 0$ can then be written as

$$(\gamma^a \otimes t^a) U = 0, \tag{6.2}$$

with γ^a and t^a acting on the spinor and group indices, respectively. (We suppress the index σ.) Alternatively, regard U as a (rectangular) matrix with a spinor index and a group index and write the rather strange matrix equation

$$\gamma^a U \tilde{t}^a = 0 \tag{6.2′}$$

($\tilde{t}$ is the transpose of t.)

A direct approach would involve using the harmonic expansion in Eq. (4.7). The equation $MU = 0$ then gives an equation involving $t^{(\sigma_1)}$, $t^{(\sigma_2)}$, and Clebsch–Gordon coefficients.

Let us apply M to Eq. (6.2) again

$$(\gamma^a \gamma^b \otimes t^a t^b) U = 0 = (1 \otimes t^2 + \tfrac{1}{2}\sigma^{ab} f^{abc} \otimes t^c). \tag{6.3}$$

We recognize the appearance of the operator Y_a so that Eq. (6.3) may be written as

$$(2Y_a \otimes t_a) U = t^2 U. \tag{6.4}$$

Here t^2 is the second Casimir invariant of the representation σ which U transforms as. The discussion of the preceding section on the property of Y_a tells us that for a given group the representation σ_2 appearing in the Clebsch–Gordon decomposition of $U^{(\sigma)}$ in Eq. (4.7) is determined [to be the one with the highest weight $\delta = (1,1,1,...,1)$]. For a given σ, the representation σ must appear in the direct product $\sigma \otimes \sigma_2^*$. Alternatively, one can regard Eq. (6.4) as an eigenvalue problem determining the representation σ to which the zero modes, if any, belong. [Of course, one must still insure that Eq. (6.2) is satisfied.]

For the simplest case SU(2) it is quite easy to prove that there is no nontrivial solution. For SU(2), $Y_a = \tau_a/2$. We "square" Eq. (6.4)

$$(\tau_a \otimes t_a)(\tau_b \otimes t_b) U = (t^2)^2 U = (1 \otimes t^2 - \tau_a \otimes t_a) U. \tag{6.5}$$

Using Eq. (6.4) again we find $t^2 = t^2(1 - t^2)$ which only has the trivial solution $t^2 = 0$. For groups larger than SU(2), a similar, but not so simple, analysis[18] can be made to show that the only solution to Eq. (6.2) is that with $t^2 = 0$. Therefore, in the $\tau = -1$ case, fermion zero modes are singlets.

VII. CONCLUSION

We conclude that with the introduction of torsion, Kaluza–Klein theories can have massless fermions, but not chiral fermions. Given a gauge group the fermion representation is determined. For parallelizable torsion, the representation is enormous for any but the smallest gauge groups. This is evidently related to the high dimension of group manifolds. As one possibility, we may reduce the dimension of the manifold by looking at a coset homogeneous space G/H instead of G itself. For instance, for SO(10), well known to be a leading candidate for a group relevant to the real world, we might look at $S_9 = \mathrm{SO}(10)/\mathrm{SO}(9)$. But unfortunately, S_9 is not parallelizable in the Cartan–Schouten sense. Modulo this difficulty, theories with spheres as internal manifold look quite promising to us. On S_9, the spinor is 16 dimensional. Thus, the fact that in SO(10) grand unification fermions belong to the $\underline{16}$ may be explained in the Kaluza–Klein context. The geometry of the internal manifold may be reflected in the fermion spectrum.

In general, one can introduce an arbitrary amount of torsion, not necessarily just so as to make the Riemann curvature tensor vanish. By varying the parameter K so that the scalar curvature becomes negative, one may obtain, conceivably, massless fermions belonging to representations favored by current theoretical prejudice. But one would then be hard put to justify choosing that particular value of K.

These and other questions discussed here should be investigated further.

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APPENDIX A: CONNECTION AND TORSION

We briefly recall some elementary facts about differential geometry with torsion. Our notation is essentially that of Ref. 5 to which the reader unfamiliar with the subject may wish to turn.

One defines orthonormal basis vectors $\mathbf{e}_a$ and coordinate basis vectors $\mathbf{e}_i$ on the manifold. The Vielbein is defined by expanding

$$\mathbf{e}_a = e_a^i \mathbf{e}_i. \tag{A1}$$

The connection and the Christoffel symbol are defined by infinitesimal transport of the basis vectors:

$$\nabla_i \mathbf{e}_a = -\omega_a{}^b{}_i \mathbf{e}_b, \tag{A2}$$

$$\nabla_i \mathbf{e}_j = \Gamma^k_{ji}\, \mathbf{e}_k. \tag{A3}$$

Combining Eqs. (A1)–(A3) one finds

$$\partial_i e_a^k + \Gamma^k_{ji} e^j_a + \omega_a{}^b{}_i e^k_b = 0, \tag{A4}$$

which can be rewritten as a relation between Γ and ω,

$$\Gamma^k_{ji} = -(\partial_i e^k_a + \omega_a{}^b{}_i e^k_b) e^a_j. \tag{A5}$$

Recalling the torsion one-form is defined by $T^a = de^a + \omega^a{}_b e^b$ we see that

$$\Gamma^k_{ij} - \Gamma^k_{ji} = T^k_{ji}. \tag{A6}$$

The Christoffel symbol Γ^k_{ij} is symmetric in its two lower indices in the absence of torsion.

APPENDIX B: DIRAC OPERATOR WITH TORSION

In this appendix we discuss properties of the Dirac operator with torsion to see more closely how the usual positivity argument for the absence of zero modes breaks down in this case. We will consider only the internal manifold.

The internal Dirac operator is

$$i\not{\mathscr{D}}\psi = i\gamma^j \mathscr{D}_i\, \psi = i\gamma^a e^i_a (\mathscr{D}_i - (i/4)\sigma^{bc}\omega_{bci})\, \psi. \tag{B1}$$

Here ω_{bci} is a generic connection with torsion. Using Eq. (A5) in Appendix A we can prove that

$$\mathscr{D}_i(\gamma^j\mathscr{D}_j)\,\psi = \gamma^j\mathscr{D}_i\mathscr{D}_j\,\psi,$$

$$\text{i.e., } [\gamma^j,\mathscr{D}_i] = 0. \tag{B2}$$

Here we note that the proper definitions for the action of $\mathscr{D}_i$ on both sides are not actually identical:

$$\mathscr{D}_i(\gamma^j\mathscr{D}_j\,\psi) = (\partial_i - (i/4)\sigma^{ab}\omega_{abi})(\gamma^c e^j_c \mathscr{D}_j\,\psi), \tag{B3}$$

$$\mathscr{D}_i(\mathscr{D}_j\,\psi) = (\partial_i \delta^k_j - \Gamma^k_{ji} - (i/4)\sigma^{ab}\omega_{abi}\,\delta^k_j)(\mathscr{D}_k\,\psi) \tag{B3'}$$

because of the difference in transformation property of $(\gamma^j \times \mathscr{D}_j\psi)$ and $\mathscr{D}_j\,\psi$. Here Γ^k_{ji} is the connection with torsion in the coordinate basis.

From Eq. (B3') it follows that

$$[\mathscr{D}_i,\mathscr{D}_j] = -(i/4)\,\sigma^{ab}R_{abij} - T^k_{ij}\,\mathscr{D}_k, \tag{B4}$$

where $T^k_{ij} = \Gamma^k_{ji} - \Gamma^k_{ij}$. Therefore by using Eqs. (B2) and (B4) it is easy to obtain

$$(i\not{\mathscr{D}})^2\psi = \{ -g^{ij}\mathscr{D}_i\mathscr{D}_j + \tfrac{1}{8}\sigma^{ij}\sigma^{ab}R_{abij} - (i/2)\sigma^{ij}T^k_{ij}\mathscr{D}_k \}\,\psi. \tag{B5}$$

The first term on the right side is still a non-negative operator even if $\mathscr{D}_i$ has torsion in it,

$$\int d^n y\,\sqrt{g}\,\psi^\dagger g^{ij}\mathscr{D}_i\mathscr{D}_j\,\psi$$

$$= -\int d^n y\sqrt{g}\{\, g^{ij}(\mathscr{D}_i\,\psi)^\dagger(\mathscr{D}_j\,\psi) + (\mathscr{D}_i\, g^{ij})(\mathscr{D}_j\,\psi) + \tfrac{1}{2} g^{ij}T^k_{ji}(\mathscr{D}_k\,\psi)\}, \tag{B6}$$

since the second and third terms in this equation vanish identically. However, both the second and third terms in Eq. (B5) are not necessarily positive definite. Therefore, the addition of torsion in the Dirac operator may lead to the appearance of zero modes.

Incidentally, if there is no torsion, then the second term in Eq. (B5) collapses to

$$\tfrac{1}{8}\,\sigma^{ij}\sigma^{ab}R_{abij} = \tfrac{1}{8}\sigma^{ab}\sigma^{cd}R_{abcd} = \tfrac{1}{4}\,R, \tag{B7}$$

because of

$$R_{abcd} = R_{cdab}, \quad R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{B8}$$

However, when there is torsion, the two properties of R_{abij} in Eq. (B8) are no longer true, and so neither is Eq. (B7).

APPENDIX C: INDEX THEOREM AND PONTRYAGIN NUMBER IN THE PRESENCE OF TORSION

The fact that we obtain equal number of left- and right-handed zero modes is perhaps not surprising. A heuristic argument[19] goes roughly as follows.

Torsion can be switched on continuously. One can write the ω_t, connection one-form, as

$$(\omega_t)^a{}_b = \overset{(0)}{\omega}{}^a{}_b + t\lambda^a{}_b \tag{C1}$$

so that as t goes from 0 to 1 the connection goes from $\overset{(0)}{\omega}$ to $\omega = \overset{(0)}{\omega} + \lambda$. The difference between the number of left- and right-handed zero modes $(n_L - n_R)$ is zero for $t = 0$ and so by continuity it should not jump to an integer value as t varies.

This argument is basically correct. However, there are a few technical gaps which we need to fill in to turn it into a proof. Here, in fact, we need more than continuity since $(n_L + n_R)$ does jump discontinuously as t varies. The crucial point is that unlike $(n_L + n_R)$, the quantity $(n_L - n_R)$ is related by the Atiyah–Singer index theorem to a topological quantity, which is constant against continuous deformation of connections.

Therefore, to turn the above argument into an explicit proof we have to show that (1) the index theorem and (2) the Pontryagin numbers are not affected by the presence of torsion.

It should be straightforward to check (1). Here we only check (2).

Recall that in the index theorem[20]

$$n_L - n_R = -\int_M \left[1 - \tfrac{1}{24} P_1 + \tfrac{1}{5760}(7P_1^2 - 4P_2) + \cdots\right], \tag{C2}$$

the right-hand side involves only Pontryagin numbers ($P_k \propto \operatorname{tr} R^{2k}$). We want to show that Pontryagin numbers are not changed by torsion. Let R and $\overset{(0)}{R}$ be the curvature two-form constructed out of the connection one-forms ω and $\overset{(0)}{\omega}$, respectively. We now prove that

$$\int \operatorname{tr} R^{2k} = \int \operatorname{tr} \overset{(0)}{R}{}^{2k} \tag{C3}$$

(where the integrals are over a compact $4k$-dimensional manifold) if $\omega - \overset{(0)}{\omega} = \lambda$ is a tensor.

It suffices to show that

$$\operatorname{tr} R^{2k} - \operatorname{tr} \overset{(0)}{R}{}^{2k} = dX, \tag{C4}$$

where X transforms covariantly. We emphasize that we know $\operatorname{tr} R^{2k}$ is locally exact. Indeed, in a previous work we have derived the representation [Eq. (3.15) of Ref. 21]

$$\operatorname{tr} R^{2k} = d\left[2k \int_0^1 dt\, \operatorname{Str}\{\omega,(t\,d\omega + t^2\omega^2)^{2k-1}\}\right]. \tag{C5}$$

The properties of the symmetric trace Str may be found in Appendix B of Ref. 20. The point is that the quantity in the square bracket does not transform covariantly and so $\operatorname{tr} R^{2k}$ is not globally exact and its integral over a compact manifold does not necessarily vanish. The claim in Eq. (C4) is that $\operatorname{tr} R^{2k} - \operatorname{tr} \overset{(0)}{R}{}^{2k}$ is globally exact.

For k small, one can verify Eq. (C4) by explicit computation using Eq. (C5). For arbitrary k this approach becomes unwieldy. Instead, define ω_t as in Eq. (C1) and define

$$R_t \equiv d\omega_t + \omega_t^2 = \overset{(0)}{R} + t\overset{(0)}{D}\lambda + t^2\lambda^2. \tag{C6}$$

Here $\overset{(0)}{D}$ is the covariant derivative with the connection $\overset{(0)}{\omega}$. Then we find

$$\begin{aligned}\frac{d}{dt}\operatorname{tr}(R_t)^{2k} &= 2k\operatorname{tr}\left(\overset{(0)}{D}\lambda + 2t\lambda^2\right)R_t^{2k-1}\\ &= 2k\operatorname{tr}(D_t\lambda)R_t^{2k-1}\\ &= 2k\operatorname{Str}(D_t\lambda,R_t^{2k-1})\\ &= 2k\,d\,[\operatorname{Str}(\lambda,R_t^{2k-1})].\end{aligned} \tag{C7}$$

Here D_t is the covariant derivative with the connection ω_t. The last step in Eq. (C7) follows from a property of Str [see Eq. (B13) of Ref. 20] and from the Bianchi identity $D_t R_t = 0$. The square bracket in Eq. (C7) transforms covariantly provided that $\lambda^a{}_b$ is a tensor one-form and so Eq. (C4) follows.

Again, one may be tempted to argue that Eq. (C7) follows merely from continuity. However, one needs the additional input that λ transforms covariantly.

Equation (C4) can be easily generalized to the cases in which the Pontryagin densities are of the form $\operatorname{tr} R^{2k_1} \cdot \operatorname{tr} R^{2k_2} \cdots \operatorname{tr} R^{2k_l}$. So the generic Pontryagin members are also unchanged by addition of torsion.

Incidentally, the discussion here amounts to an indirect proof that the Pontryagin densities for group manifolds vanish. Of course, one can compute them directly by using Eq. (2.7).

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