Equalization of Recursive Polynomial Systems

Alberto Carini, *Member, IEEE*, V. John Mathews, *Senior Member, IEEE*, and Giovanni L. Sicuranza, *Senior Member, IEEE*

***Abstract*— This letter presents some theorems for the exact and pth-order equalization of nonlinear systems described by recursive polynomial input-output relationships. It is shown that the nonlinear equalizers derived on the basis of this theory have simple and computationally efficient structures. Furthermore, the pth-order equalizers can be shown to operate in a stable manner for a finite range of values of the input amplitude when the linear component of the nonlinear system being equalized has minimum phase properties.**

***Index Terms*— Equalization, nonlinear systems, pth-order equalization.**

I. Introduction

There are many nonlinear systems with long memory spans that can be modeled efficiently using recursive polynomial models. This letter presents a theory for the equalization of nonlinear systems characterized by the input-output relationship

$$B(q)y(n) = A(q)x(n) + \mathcal{N}[x(n), y(n)] \tag{1}$$

where q^{-1} is the delay operator, $x(n)$ and $y(n)$ are the input and output signals, $A(q), B(q)$ are polynomials in the variable q^{-1}, and $\mathcal{N}$ is a causal, discrete-time, polynomial operator with respect to the input and output signals. A *nonlinear equalizer* is a filter that, when connected in cascade before or after a nonlinear system, results in an overall system whose characteristics correspond to those of an identity system in the frequency band and in the range of input signal amplitudes of interest. When an equalizer is connected before (after) a nonlinear system, it is called a *pre- (post-) equalizer*.

We assume in this work that the input-output relationship of the nonlinear system to be equalized is completely known. Adaptive algorithms for estimation of the parameters and the equalizer online are currently under investigation. In the next section, we derive expressions for the input-output relationship of the exact equalizer for a recursive polynomial nonlinear system. In many situations, the exact equalizer does not exist or cannot operate in a stable manner. These problems are addressed in Section III, where we derive expressions for the pth-order equalizer for recursive polynomial systems. It is shown that pth-order equalizer tends to the exact equalizer as p tends to ∞ for a finite range of input amplitude values when both $A(q)$ and $B(q)$ in (1) satisfy the minimum phase property.

Manuscript received May 21, 1999. This work was supported in part by NATO under Grant CRG.950379 and Grants 60% of the University of Trieste. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Y. Hua.

A. Carini is with Telit Mobile Terminals, 34010 Sgonico, Trieste, Italy (e-mail: alberto.carini@telital.it).

V. J. Mathews is with the Department of Electrical Engineering, University of Utah, Salt Lake City, UT 84112 USA.

G. L. Sicuranza is with the DEEI, Università degli Studi di Trieste, 34127 Trieste, Italy.

Publisher Item Identifier S 1070-9908(99)09287-1.

II. Exact Equalization

Throughout this letter, we assume causal signals, i.e., all the signals are identically zero for time indices less than zero. This assumption implies that all recursive polynomial systems under consideration are initially at rest. The nonlinear system in (1) is assumed to be described by the following equations:

$$A(q) = \sum_{i=0}^{N} a_i q^{-i}, \tag{2}$$

$$B(q) = 1 + \sum_{i=1}^{N} b_i q^{-i} \tag{3}$$

and

$$\begin{aligned}
&\mathcal{N}[x(n), y(n)] \\
&\quad = \sum_{k=2}^{L} \sum_{i_1=r}^{N} \cdots \sum_{i_k=i_{k-1}}^{N} h_{i_1 i_2 \ldots i_k} x(n-i_1) \\
&\qquad \cdot \cdots \cdot x(n-i_k) \\
&\qquad - \sum_{k=2}^{L} \sum_{i_1=1}^{N} \cdots \sum_{i_k=i_{k-1}}^{N} r_{i_1 i_2 \ldots i_k} y(n-i_1) \\
&\qquad \cdot \cdots \cdot y(n-i_k) \\
&\qquad + \sum_{i=r}^{N} \sum_{j=1}^{N} c_{ij} x(n-i) y(n-j)
\end{aligned} \tag{4}$$

where the parameter r is either one or zero. In the above representation, we use only the simplest form of mixed products involving the input and output samples in the nonlinear operator $\mathcal{N}[x(n), y(n)]$. This is done for simplicity of presentation, and can be extended to more general systems in a relatively easy manner.

Using the above notation, the input-output relationship of the inverse of the system in (1) can be expressed as [1]

$$A(q)w(n) = B(q)u(n) - \mathcal{N}[w(n), u(n)] \tag{5}$$

where $u(n)$ and $w(n)$ are the input and output signals, respectively, of the inverse system. If $r = 1$, (4) corresponds to a system for which an explicit expression for the output of the inverse system always exists because the system model does not depend on the input sample $x(n)$ in a nonlinear manner [1]. Only an implicit expression for the inverse of the more general system for which $r = 0$ in (4) can be derived. However, we will see in Section III that explicit expressions do exist for pth order equalizers for the general case when $r = 0$.

The output signal in (1) and (5) can also be expressed as

$$y(n) = B^{-1}(q)A(q)x(n) + B^{-1}(q)\mathcal{N}[x(n), y(n)] \quad (6)$$

and

$$w(n) = A^{-1}(q)B(q)u(n) - A^{-1}(q)\mathcal{N}[w(n), u(n)] \quad (7)$$

respectively.

In many applications, the input signal may be bandlimited, and therefore we are interested in equalization of the distortions in a given frequency band. The following two theorems characterize the structure of the post- and pre-equalizers for a specific range of frequencies.

Theorem 1: Let the input signal $x(n)$ to the system in (1) be bandlimited in a certain frequency band B and let $u(n)$ be its output. Let $\tilde{A}^{-1}(q)$ be the linear equalizer of the system $A(q)$ in the band B with zero response outside B. The system defined by

$$w(n) = \tilde{A}^{-1}(q)B(q)u(n) - \tilde{A}^{-1}(q)\mathcal{N}[w(n), u(n)] \quad (8)$$

is a post-equalizer in the band B for the system of (1).

Proof: We consider post-equalization of the nonlinear system (1) in the band B and the elimination of all other frequencies at the output. For this purpose, we first cascade the system in (1) with the linear filter $\tilde{A}^{-1}(q)B(q)$ and then equalize the resulting nonlinear system. Cascading (1) with the linear system $\tilde{A}^{-1}(q)B(q)$ eliminates all the frequencies outside the band B. The resulting system is described by the input-output relationship

$$z(n) = \tilde{A}^{-1}(q)A(q)x(n) + \tilde{A}^{-1}(q)\mathcal{N}[x(n), u(n)] \quad (9)$$

with $u(n) = y(n)$. Since $x(n)$ has frequency components only on B, the above system is equivalent to

$$z(n) = x(n) + \tilde{A}^{-1}(q)\mathcal{N}[x(n), u(n)] \quad (10)$$

whose post-inverse system is given by

$$w(n) = z(n) - \tilde{A}^{-1}(q)\mathcal{N}[w(n), u(n)]. \quad (11)$$

Since $z(n)$ is bandlimited to the band B, the output of the system in (11) is also bandlimited to the band B. Thus, the cascade of $z(n) = \tilde{A}^{-1}(q)B(q)u(n)$ and the system in (11) is the ideal equalizer for the system in (1) in the band B. It is straightforward to see that this system is identical to that described by (8). □

Theorem 2: Let the input signal $u(n)$ be bandlimited to a certain frequency band B. Let $\tilde{A}^{-1}(q)$ be the linear equalizer of the system $A(q)$ in the band B with zero response outside B. The system defined by

$$w(n) = \tilde{A}^{-1}(q)B(q)u(n) - \tilde{A}^{-1}(q)\mathcal{N}[w(n), u(n)] \quad (12)$$

is a pre-equalizer in the band B for the system of (1).

The proof is similar to that for Theorem 1. The system in (12) has the same input-output relationship as that in (8). However, such a system used as a pre-equalizer is unable to compensate for the frequency components of $y(n)$ that fall outside the band B.

The nonlinear recursive systems in (1), (7) and (12) are not stable unless the linear components $B(q)$, $A(q)$ and $\tilde{A}(q)$, respectively, have minimum phase characteristics. When $B(q)$, $A(q)$ and $\tilde{A}(q)$ represent minimum phase systems, the corresponding nonlinear systems in (1), (7), and (12) can be shown to operate in a stable manner whenever their input signal is sufficiently small. The bound on each input signal depends on the zeros of $B(q)$, $A(q)$ and $\tilde{A}(q)$, respectively, and on the coefficients of $\mathcal{N}[\cdot,\cdot]$ [2], [3]. Thus, our inverse systems will equalize the nonlinear system in (1) only in a finite range of amplitudes of the input signal to the inverse system for which they are stable.

III. pTH-ORDER EQUALIZATION

The ideal equalizer of the previous section may not always be realizable. For example, we saw in the previous section that the equalizer does not have an explicit input-output relationship when $r = 0$ in (4). Furthermore, because of its recursive structure, the equalizer may also be unstable. In this section, we derive the input-output relationships for the pth-order equalizers for the nonlinear system (4). A *pth-order equalizer* of a nonlinear system is a filter that, when connected in cascade before or after a nonlinear system, results in an overall system whose Volterra series expansion, in the frequency band and in the range of input signal amplitudes of interest, correspond to those of a parallel connection of an identity system and a nonlinear component having Volterra kernels of order lower than or equal to p identically equal to zero. The pth-order inverse of a nonlinear system was first defined by Schetzen [5], [6]. In the original definition of a pth order inverse, the inverse system was constrained to be of order p or lower. This constraint was relaxed by Sarti and Pupolin [4] by allowing the inverse system to possess nonzero Volterra kernels of order greater than p. This work uses the relaxed definition of pth-order inverses since such a definition allows the derivation of simpler and more efficient structures for the pth-order equalizers. The following theorem states the main result of this section.

Theorem 3: Let $A(q)$ and $B(q)$ be minimum phase polynomials. The sequence of systems defined by

$$w_1(n) = A^{-1}(q)B(q)u(n), \quad (13)$$
$$w_p(n) = w_1(n) - A^{-1}(q)\mathcal{N}[w_{p-1}(n), u(n)] \quad (14)$$

defines pth-order post-inverses of the system of (1) according to the definition in [4]. Moreover, the sequence of systems converges to the system in (7) when the input signal is bounded by some finite constant $\tau > 0$.

Proof: We prove the theorem using induction. Let us process the output of the system of (1) with the system defined by (13). Since $u(n) = y(n)$, we obtain the following relationship for $w_1(n)$:

$$\begin{aligned} w_1(n) &= x(n) + A^{-1}(q)\mathcal{N}[x(n), y(n)] \\ &= x(n) + T_1(n). \end{aligned} \quad (15)$$

We can easily prove that $T_1(n)$ is a Volterra operator of order greater than one in $x(n)$ by substituting $y(n)$ with its Volterra series expansion. This proves that $w_1(n)$ is the output of a first-order inverse. Let us suppose that $w_p(n)$ is the output of a pth order inverse according to the definition in [4], i.e.,

$$w_p(n) = x(n) + T_p(n) \tag{16}$$

where $T_p(n)$ is a Volterra operator of order greater than p. We now show that the system defined by

$$w_{p+1}(n) = A^{-1}(q)B(q)u(n) - A^{-1}(q)\mathcal{N}[w_p(n), u(n)] \tag{17}$$

is a $(p+1)$th-order inverse of (1). Substituting $y(n)$ for $u(n)$ and (16) for $w_p(n)$ in (17) gives

$$\begin{aligned} w_{p+1}(n) &= x(n) + A^{-1}(q)\mathcal{N}[x(n), y(n)] \\ &\quad - A^{-1}(q)\mathcal{N}[x(n) + T_p(n), y(n)] \\ &= x(n) + A^{-1}(q)\mathcal{N}[x(n), y(n)] \\ &\quad - A^{-1}(q)\mathcal{N}[x(n), y(n)] + T_{p+1}(n) \end{aligned} \tag{18}$$

where T_{p+1} is an operator of order greater than $p+1$. Thus, the sequence of systems in (13)–(14) defines pth-order inverses of (1). If $A(q)$ and $B(q)$ satisfy the minimum phase property, the sequence of systems (13)–(14) will converge to (5) at least in an amplitude interval around zero. In order to prove convergence, it is required to show that $||T_{p+1}(n)||_\infty$ tends to zero when p tends to ∞ and the input signal $x(n)$ is bounded by some finite constant $\tau > 0$, i.e., when $||x(n)||_\infty < \tau$.

Let $||x(n)||_\infty$ be sufficiently smaller than one and be such that the output $y(n)$ of the system (1) is bounded by one, i.e., $||y(n)||_\infty < 1$. Using the results in [2], [3], we can find a bound k_2 on $x(n)$ such that the bound on $y(n)$ is satisfied whenever $B(q)$ satisfies the minimum phase property. Proceeding in a similar manner, we can show that

$$||T_1(n)||_\infty < k_1 ||x(n)||_\infty^2 \tag{19}$$

for some positive constant k_1. It follows immediately that if $||x(n)||_\infty < \min((1/k_1), k_2)$, $||T_1(n)||_\infty < ||x(n)||_\infty$. In the following, we prove by induction that for sufficiently small $||x(n)||_\infty$, there exists a constant $k < 1$ such that $||T_{p+1}(n)||_\infty < k^p ||x(n)||_\infty$. To see this, let us assume that $||T_p(n)||_\infty < ||x(n)||_\infty$ and observe that

$$\begin{aligned} ||T_{p+1}(n)||_\infty &= ||A^{-1}(q)\mathcal{N}[x(n), u(n)] \\ &\quad - A^{-1}(q)\mathcal{N}[x(n) + T_p(n), u(n)]||_\infty \\ &= ||\mathcal{G}[x(n), T_p(n)]||_\infty \end{aligned} \tag{20}$$

where $\mathcal{G}[x(n), T_p(n)]$ is a finite order polynomial with no constant or linear term and with each term containing at least a factor $T_p(n)$. Since both inputs to the operator $\mathcal{G}$ are bounded by values smaller than one, and since the operator contains only a finite number of terms, it follows that there exist two positive constants k_3 and k_4 such that

$$||T_{p+1}(n)||_\infty < k_3 ||x(n)||_\infty ||T_p(n)||_\infty + k_4 ||T_p(n)||_\infty^2. \tag{21}$$

Let k be a positive constant less than one and let $||x(n)||_\infty < \tau = \min((1/k_1), k_2, (k/(k_3 + k_4)), 1)$. It follows easily from (21) that for every p, $||T_{p+1}(n)||_\infty < k ||T_p(n)||_\infty$ which means that

$$||T_{p+1}(n)||_\infty < k^p ||x(n)||_\infty. \tag{22}$$

Thus, $||T_{p+1}(n)||_\infty$ tends to zero when $||x(n)||_\infty < \tau$. $\square$

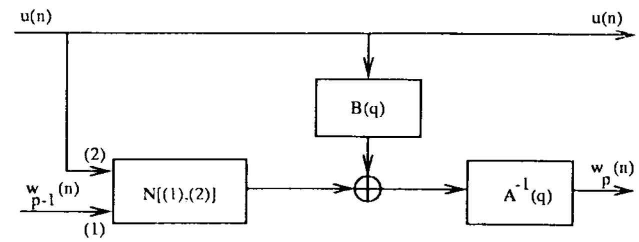

Fig. 1. ith order cell.

It can be proved in a similar manner that the sequence of systems defined by (13) and (14) defines pth-order pre-inverses of the system of (1).

The sequence of systems defined by (13) and (14) corresponds to a systolic cascade of cells. The block diagram of a pre-equalizer cell is shown in Fig. 1. Unlike the ideal equalizer which is not realizable for the system in (4) with $r = 0$, each cell of the pth-order equalizer is always realizable. Furthermore, the modularity of the systolic structure of the pth-order equalizer leads to efficient implementation of such systems using VLSI circuits. When the input signal is bounded and $A(q)$ satisfies the minimum phase property, the output of the system defined by (13) and (14) is bounded for any finite choice of p [2], [3].

IV. Conclusions

This letter presented a theory for the exact and the pth-order equalization of nonlinear systems with known recursive polynomial input-output relationships. The pth-order equalizers can be implemented by cascading modular and stable components. The theory of pth order equalizers in Section III can be extended to the case of bandlimited input signals using ideas similar to those described in Theorems 1 and 2. The resulting structures are also modular and can be implemented using a cascade of systolic cells. Finally, the extension of the theory to the exact and pth order linearizers for recursive polynomial systems is also possible and will be described elsewhere.

References

[1] A. Carini, G. L. Sicuranza, and V. J. Mathews, "On the inversion of certain nonlinear systems," *IEEE Signal Processing Lett.*, vol. 4, pp. 334–336, Dec. 1997.

[2] J. Lee and V. J. Mathews, "A stability condition for certain bilinear systems," *IEEE Trans. Signal Processing*, vol. 42, pp. 1871–1873, July 1994.

[3] E. Mumolo and A. Carini, "A stability condition for adaptive recursive second-order polynomial filters," *Signal Process.*, vol. 54, pp. 85–90, Oct. 1996.

[4] A. Sarti and S. Pupolin, "Recursive techniques for the synthesis of a pth-order inverse of a Volterra system," *Eur. Trans. Telecommun.*, vol. 3, pp. 315–322, July/Aug. 1992.

[5] M. Schetzen, "Theory of pth-order inverses of nonlinear systems," *IEEE Trans. Circuits Syst.*, vol. CAS-23, pp. 285–291, May 1976.

[6] ——, *The Volterra and Wiener Theories of Nonlinear Systems* New York: Wiley, 1980.