A Tutorial on Stochastic FDTD

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***Abstract*—The Stochastic FDTD (S-FDTD) method provides a way to determine the mean and variance of the electric and magnetic fields in a model where the electrical properties (conductivity and permittivity) vary stochastically, from a single S-FDTD simulation. This method is a realistic and more efficient alternative to Monte Carlo analysis, which requires numerous FDTD simulations.**

I. Introduction

This paper is a tutorial on the Stochastic Finite Difference Time Domain (S-FDTD) method. S-FDTD can determine or at least bound the mean and variance of electromagnetic waves in a model with stochastically variable electrical properties (conductivity and permittivity) from a single S-FDTD run. Common examples of materials that vary stochastically are biological materials, soils/dirts/geophysical materials, composite materials, building materials, etc. The S-FDTD method is derived from the normal FDTD method by adding variables for the variance of the electric fields throughout the model. The variances are calculated as a function of space and time at the same time as the fields are calculated.

II. Stochastic FDTD (S-FDTD) Derivation

A. Traditional FDTD Equations

We begin with the normal FDTD equations in one dimension: [2]

$$B_y\Big|_{k+1/2}^{n+1/2} - B_y\Big|_{k+1/2}^{n-1/2} = -\frac{\Delta t}{\Delta z}\left(E_x\Big|_{k+1}^{n} - E_x\Big|_{k}^{n}\right)$$

$$E_x\Big|_k^{n+1} - \frac{\dfrac{\varepsilon_r|_k\varepsilon_o}{\Delta t} - \dfrac{\hat{\sigma}|_k}{2}}{\dfrac{\varepsilon_r|_k\varepsilon_o}{\Delta t} + \dfrac{\hat{\sigma}|_k}{2}} E_x\Big|_k^n = \frac{-\left(H_y\Big|_{k+1/2}^{n+1/2} - H_y\Big|_{k-1/2}^{n+1/2}\right)}{\left(\dfrac{\varepsilon_r|_k\varepsilon_o}{\Delta t} + \dfrac{\hat{\sigma}|_k}{2}\right)\Delta z} \qquad (1)$$

Field terms have superscripts (n) that represent the time and sub-scripts (k) that represent space along with time step Δt and the spatial step Δz. Electrical properties (permittivity ε and conductivity $\hat{\sigma}$) are assumed to vary stochastically.

B. Perturbation Theory

A stochastic function $g(\varepsilon_R,\hat{\sigma},...)$ can be expanded using a truncated Taylor series about the mean of the random variables ε_R (permittivity) and $\hat{\sigma}$ (conductivity). In this paper $\langle\bullet\rangle$ indicates the mean of the material parameter i.e. $\langle\varepsilon_R\rangle,\langle\hat{\sigma}\rangle$. These are then used in the expansion of $\langle g(\varepsilon_R,\hat{\sigma},...)\rangle$ and the variance $\sigma^2\{g(\varepsilon_R,\hat{\sigma},...)\}$ of the same stochastic function. The variance $\sigma^2\{g(\varepsilon_R,\hat{\sigma},...)\} = \langle g(\varepsilon_R,\hat{\sigma},...)^2\rangle - \langle g(\varepsilon_R,\hat{\sigma},...)\rangle^2$. These equations are expanded using the Taylor series with higher order terms discarded[3], yielding the equations also known as the delta method. [1]

The mean of a function of stochastic variables is approximated by the same function using the mean of each of the stochastic variables as the independent variables:

$$\langle g(x_1,x_2,x_3,...,x_n)\rangle \approx g(\langle x_1\rangle,\langle x_2\rangle,\langle x_3\rangle,...,\langle x_n\rangle) \qquad (2)$$

The variance is approximated:

$$\sigma^2\{g(x_1,x_2,x_3,...,x_n)\} \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}\bigg|_{\langle x_1\rangle,\langle x_2\rangle,...,\langle x_n\rangle}\left\langle\left(x_i-\langle x_i\rangle\right)\left(x_j-\langle x_j\rangle\right)\right\rangle \qquad (3)$$

A. Mean Approximation

Taking the mean of both sides of the equations found in (**Error! Reference source not found.**) and solving for the future term yields:

$$\left\langle B_y\Big|_{k+1/2}^{n+1/2}\right\rangle = \left\langle B_y\Big|_{k+1/2}^{n-1/2}\right\rangle - \frac{\Delta t}{\Delta z}\left[\left\langle E_x\Big|_{k+1}^{n}\right\rangle - \left\langle E_x\Big|_{k}^{n}\right\rangle\right] \qquad (4)$$

Ampere's equation is not as easily separated. Using the Delta method approximation yields:

$$
\left\langle E_x\Big|_k^{n+1}\right\rangle = \frac{\dfrac{\varepsilon_r\big|_k \varepsilon_o}{\Delta t} - \dfrac{\hat{\sigma}\big|_k}{2}}{\dfrac{\varepsilon_r\big|_k \varepsilon_o}{\Delta t} + \dfrac{\hat{\sigma}\big|_k}{2}} \left\langle E_x\Big|_k^{n}\right\rangle - \frac{\left(\left\langle H_y\Big|_{k+1/2}^{n+1/2}\right\rangle - \left\langle H_y\Big|_{k-1/2}^{n+1/2}\right\rangle\right)}{\left(\dfrac{\varepsilon_r\big|_k \varepsilon_o}{\Delta t} + \dfrac{\hat{\sigma}\big|_k}{2}\right)\Delta z} \tag{5}
$$

B. Variance Approximation

We next need to determine the variance of each of the two FDTD equations, using the Delta method to approximate the variance of any function of stochastic variables that are not easily separated into their separate variance or covariance terms. For example, terms like $\sigma^2\{aX+bY\}$ can be expanded to $a^2\sigma^2\{X\}+\ b^2\sigma^2\{Y\}$ assuming a and b are constant, but terms like this next are not easily separated. For example, the Delta method provides an approximation for $\sigma^2\{abXY\}$ that allows this to be separated into terms involving $\sigma^2\{X\}$, $\sigma^2\{Y\}$, and their covariance.

1) Faraday's Equation

Taking the variance of Faraday's Law yields:

$$
\sigma^2\left\{B_y\Big|_{k+1/2}^{n+1/2} - B_y\Big|_{k+1/2}^{n-1/2}\right\} = \frac{\Delta t^2}{\Delta z^2}\sigma^2\left\{E_x\Big|_{k+1}^{n} - E_x\Big|_{k}^{n}\right\}
$$

In expanding the previous equation correlation coefficients between the B and E field terms were approximated (over-estimated) to be equal to one. The standard deviation, indicated by $\sigma\{\bullet\}$, is:

$$
\sigma\left\{B_y\Big|_{k+1/2}^{n+1/2}\right\} = \sigma\left\{B_y\Big|_{k+1/2}^{n-1/2}\right\} + \frac{\Delta t}{\Delta z}\left(\sigma\left\{E_x\Big|_{k+1}^{n}\right\} - \sigma\left\{E_x\Big|_{k}^{n}\right\}\right) \tag{6}
$$

1) Ampere's Equation

Following the same procedure for Ampere's equation we start with:

$$
\sigma^2\left\{E_x\Big|_k^{n+1} - \frac{\dfrac{\varepsilon_r\big|_k \varepsilon_o}{\Delta t} - \dfrac{\hat{\sigma}\big|_k}{2}}{\dfrac{\varepsilon_r\big|_k \varepsilon_o}{\Delta t} + \dfrac{\hat{\sigma}\big|_k}{2}} E_x\Big|_k^{n}\right\}
= \sigma^2\left\{\frac{-\left(H_y\Big|_{k+1/2}^{n+1/2} - H_y\Big|_{k-1/2}^{n+1/2}\right)}{\left(\dfrac{\varepsilon_r\big|_k \varepsilon_o}{\Delta t} + \dfrac{\hat{\sigma}\big|_k}{2}\right)\Delta z}\right\}
$$

Compound terms are separated with the Delta approximation in (3). After a great deal of manipulation which is detailed in [2] we arrive at the following equation :

$$
\begin{aligned}
\sigma\left\{E_x\Big|_k^{n+1}\right\} \approx\ & \\
& \frac{2\varepsilon_o\left\langle \varepsilon_r\big|_k\right\rangle - \Delta t\left\langle \hat{\sigma}\big|_k\right\rangle}{2\varepsilon_o\left\langle \varepsilon_r\big|_k\right\rangle + \Delta t\left\langle \hat{\sigma}\big|_k\right\rangle}\sigma\left\{E_x^n(k)\right\} \\
& + \frac{2\Delta t}{\Delta x\left(2\varepsilon_o\left\langle \varepsilon_r\big|_k\right\rangle + \Delta t\left\langle \hat{\sigma}\big|_k\right\rangle\right)}\left(\sigma\left\{H_y\Big|_{k+1/2}^{n+1/2}\right\} - \sigma\left\{H_y\Big|_{k-1/2}^{n+1/2}\right\}\right) \\
& + \frac{4\Delta t\varepsilon_o\left(\left\langle \hat{\sigma}\big|_k\right\rangle \rho_{\varepsilon_r,E}\sigma\left\{\varepsilon_r\big|_k\right\} - \left\langle \varepsilon_r\big|_k\right\rangle \rho_{\underline{\sigma},E}\sigma\left\{\hat{\sigma}\big|_k\right\}\right)}{\left(2\varepsilon_o\left\langle \varepsilon_r\big|_k\right\rangle + \Delta t\left\langle \hat{\sigma}\big|_k\right\rangle\right)^2}\left\langle E_x\Big|_k^{n}\right\rangle \\
& - \frac{2\Delta t}{\Delta x\left(2\varepsilon_o\left\langle \varepsilon_r\big|_k\right\rangle + \Delta t\left\langle \hat{\sigma}\big|_k\right\rangle\right)} \bullet \\
& \quad \left(\frac{\left(2\varepsilon_o\sigma\left\{\varepsilon_r\big|_k\right\}\rho_{\varepsilon_r,H_y^{n+1/2}} + \Delta t\sigma\left\{\hat{\sigma}\big|_k\right\}\rho_{\hat{\sigma},H_y^{n+1/2}}\right)}{\left(2\varepsilon_o\left\langle \varepsilon_r\big|_k\right\rangle + \Delta t\left\langle \hat{\sigma}\big|_k\right\rangle\right)} \bullet \left(\left\langle H_y\Big|_{k-1/2}^{n+1/2}\right\rangle - \left\langle H_y\Big|_{k+1/2}^{n+1/2}\right\rangle\right)\right)
\end{aligned} \tag{7}
$$

With (6) and (7) an effective variance wave can be generated within the model space. Calculating this wave as it moves in space and time gives us an approximation to the variance of the E-fields propagating through the model space, the boundary conditions are the same as those used for the FDTD simulation. The process starts with finding the $E_x|_k^{n+1}$ then the $H_y|_{k+1/2}^{n}$ field terms followed by the $\sigma\left\{E_x|_k^{n+1}\right\}$ then finally $\sigma\left\{H_y|_{k+1/2}^{n}\right\}$ and repeating the process until reaching steady state.

References

[1] R. L. B. George Casella Statistical Inference, 2nd ed. Pacific Grove: DUXBURY, Thompson Learning, 2002.

[2] S. M. Smith, "Stochastic FDTD," May, 2011 ed. Salt Lake City, Utah: University of Utah, 2011.

[3] A. Alighanbari and C. D. Sarris, "An Unconditionally Stable Laguerre-Based S-MRTD Time-Domain Scheme," Antennas and Wireless Propagation Letters, IEEE, vol. 5, pp. 69-72, 2006.