Transitions between the Quantum Hall States and Insulators Induced by Periodic Potentials

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Transitions between two quantum Hall states or between a quantum Hall state and a Mott insulator induced by periodic potentials are studied in the $1/N$ expansion. The transitions are found to be continuous in the large-N limit and are described by a critical point that depends on a real parameter θ, which is determined by the topological orders in the quantum Hall states involved in the transition. Some critical exponents and universal quantities are calculated in the large-N limit and shown to be θ dependent.

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Recent experiments clearly show that the transitions between some quantum Hall (QH) states are described by critical phenomena [1]. It is generally believed that such transitions are related to the localization of the quasiparticles or electrons [the latter case corresponds to the QH-Anderson-insulator (QH-AI) transition] in the presence of *random potentials* [2-6]. We know that a hierarchical QH state at level $l+1$ is obtained by a condensation of the quasiparticles on top of a level l hierarchical state. If the random potential destroys the condensation and localizes the quasiparticle, the level $l+1$ QH state will change to the level l QH state [5]. Thus the QH-QH transition can be viewed as a QH-AI transition for the quasiparticles. We have little theoretical understanding about the QH-AI transitions. One is not even sure, based on theory, whether the transition is continuous or not. To show the transition to be continuous we must show that the associated critical point contains one and only one relevant operator. However, some properties of the QH-AI transitions can be studied based on general principles by assuming the transitions to be continuous [3-6], or by numerical calculations [7]. Reference [5] has given detailed discussions on the QH-QH transitions.

In this paper, we will study a different but related problem: Instead of random potentials, we will turn to a *periodic potential* to drive a QH-QH or QH-*Mott-insulator* (QH-MI) transition. The periodic potential is chosen to be such that there is one electron per unit cell (which will induce a QH-MI transition) or one quasiparticle per unit cell (which will induce a QH-QH transition) [8]. The main idea is to map the problem of the QH-MI transition to the problem of the superfluid-Mott-insulator (S-MI) transition by introducing a statistical gauge field and the Chern-Simons (CS) term [5,9]. Then we will include the fluctuations of the statistical gauge field at the critical point to obtain new corrections.

It was suggested [5] that the critical exponents are the same for all transitions between QH states (at least in the presence of disorder), because it was argued that the gauge fluctuations about the mean-field solution are irrelevant. One purpose of this paper is to study the relevance of the gauge fluctuations for the transitions induced by periodic potential. We find the gauge fluctuations represent a marginal perturbation and the critical exponents depend on the coefficient in front of the CS term.

We start with the QH-MI transition of an anyon system in a magnetic field—later we will apply the results to the QH-QH transitions—with the Lagrangian

$$\mathcal{L}=i\Phi^\dagger(\partial_t-ie^*A_0-ia_0)\Phi+\frac{1}{2m}\Phi^\dagger(\partial_i-ie^*A_i-ia_i)^2\Phi-V(x)\Phi^\dagger\Phi-g|\Phi|^4+\frac{1}{4\theta}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}, \tag{1}$$

where Φ is a bosonic field, A_μ the external electromagnetic field, V a periodic potential, and θ the statistical angle of the anyons. When $V=0$, (1) has a mean-field ground state $\Phi=$const and $A_\mu+a_\mu=0$, which describes a QH state. Now let us turn to the periodic potential V with one anyon per unit cell (on average). For large V, the on-site repulsion is strong and the intersite hopping is weak, because the anyons are strongly localized at each site. We expect the system to become a Mott insulator. In particular, if we ignore the fluctuations of the gauge field a_μ around its mean-field value, the problem is identical to the problem of the S-MI transition for bosons, which has been studied in detail in Refs. [10-12]. The critical point of the S-MI transition in d spatial dimensions is in the same universality class as the $(d+1)$-dimensional X-Y model. Near the critical point, the boson theory is described by the following effective theory [10-12] (from now on we will work in the Euclidian space replacing $t\rightarrow -it$):

$$\mathcal{L}=+|(\partial_\mu-ie^*A_\mu-ia_\mu)\phi|^2+m^2\phi^\dagger\phi+g|\phi|^4, \tag{2}$$

where m^{-1} is proportional to the correlation length ξ, which diverges at the critical point. We did not include the term $i2C\phi^\dagger(\partial_\mu-ie^*A_\mu-ia_\mu)\phi$ in (2), since it can be absorbed in the $|\phi|^2$ term by a gauge transformation, resulting in a shift $m^2 \rightarrow m^2+C^2$. The dynamical exponent z equals 1 for the S-MI transition, so the effective theory is relativistic and we have chosen the velocity of the superfluid mode to be one. With no fluctuations of the gauge field a_μ included, (2) certainly cannot describe the QH-MI transition. However, if the fluctuations of a_μ are not too strong, we can study the critical point of the QH-MI transition by doing perturbation around that of the S-MI transition. To have a systematic expansion, let us consider the large-N limit of the S-MI transition and then include the fluctuations of a_μ, resulting in the following effective theory for the QH-MI transition:

$$\mathcal{L}=\sum_{i=1}^{N}[|(\partial_\mu-ie^*A_\mu-ia_\mu)\phi_i|^2+m^2\phi_i^\dagger\phi_i] + \frac{g}{N}\left(\sum_{i=1}^{N}\phi_i^\dagger\phi_i\right)^2+\frac{iN}{4\theta}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}. \quad (3)$$

Here we use a_μ and A_μ to denote the fluctuations around the mean-field solution. When $m^2<0$, the system is in a QH phase ($\langle\phi_i\rangle\neq 0$) with a Hall conductance $\sigma_{xy} = -(\pi/\theta)e^{*2}/h$. When $m^2>0$, the system is in an insulating phase ($\langle\phi_i\rangle=0$). The critical properties of the QH-MI transitions at $m^2=0$ can be obtained perturbatively in $1/N$ expansions. (Here m^2 should be viewed as the renormalized mass.)

In the limit $\theta\rightarrow 0$, the fluctuations of a_μ can be ignored and (3) is equivalent to the spin model with $2N$ components [13]. The spin model has two fixed points in three dimensions, the unstable Guassian one and the strong coupling one. The latter has only one spin-singlet relevant operator which couples to the chemical potential and controls the S-MI transition. The properties of the strong coupling fixed point are well known in the large-N limit, so we have a good starting point to do the $1/N$ expansion. The Lagrangian (3) with $N=1$ and $m^2=0$ was first studied in Ref. [14]. Also Park studied in detail the CP^N Lagrangian with a CS term, a model closely related to ours [15]. Part of our study confirms her results except for some numerical factors. Below we will discuss physical properties at or near the critical point of the QH-MI transition.

Let $\Pi_{\mu\nu}$ be the exact polarization tensor of the ϕ field, or the exact self-energy of the gauge field a_μ. In terms of it, we have the following effective theory for a_μ:

$$\mathcal{L}_{\text{eff}}=-\tfrac{1}{2}(a_\mu+e^*A_\mu)\Pi_{\mu\nu}(a_\nu+e^*A_\nu)+\frac{iN}{4\theta}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}. \quad (4)$$

Choosing the gauge $a_0=A_0=0$ and integrating out a_i we get

$$\mathcal{L}_{\text{eff}}=-\frac{1}{2}A_i\left[\frac{1}{\Pi^{-1}+\Pi_{\text{cs}}^{-1}}\right]A_j, \quad (5)$$

where $\Pi_{\text{cs}}^{\mu\nu}=(N/2\theta)\epsilon^{\mu\nu\lambda}k_\lambda$ in momentum space. (5) allows us to calculate physical conductivity, compressibility, and orbital magnetic susceptibility from $\Pi_{\mu\nu}$. For example, let $\sigma_{\phi,ij}=-\Pi_{ij}/k_0$ be the conductivity of the boson ϕ; the physical resistivity tensor is given by

$$\rho_{xx}=\frac{1}{e^{*2}}\rho_{\phi,xx},\quad \rho_{xy}=\frac{1}{e^{*2}}\left[\rho_{\phi,xy}+\frac{2\theta}{N}\right], \quad (6)$$

which is just the composition rule discovered in Ref. [16]. Here $\rho_{\phi,ij}=(\sigma_\phi^{-1})_{ij}$.

In general, $\Pi_{\mu\nu}$ takes the following form:

$$\Pi_{\mu\nu}(p_\mu)=(\delta_{\mu\nu}-p_\mu p_\nu/p^2)p^{-1}f(p^2,m^2,\Lambda) + \epsilon_{\mu\nu\lambda}p_\lambda h(p^2,m^2,\Lambda), \quad (7)$$

where Λ is the high-energy cutoff. Because $\Pi_{\mu\nu}$ is the correlation of the conserved current which has no anomalous dimensions, we expect that as $\Lambda\rightarrow\infty$, Π is finite near the critical point. (For this to be true, m should be regarded as the renormalized mass or the inverse of the physical correlation length.) Thus in the limit $p,m\ll\Lambda$, one has $f(p^2,m^2,\Lambda)=\tilde{f}(p^2/m^2)$, $h(p^2,m^2,\Lambda)=\tilde{h}(p^2/m^2)$, where $\tilde{f}$ and $\tilde{h}$ are finite for finite p as $m^2\rightarrow 0$ [17].

Here we would like to address an important issue in this paper. Namely, whether the inclusion of the gauge field fluctuations for finite θ is a relevant, marginal, or an irrelevant perturbation with respect to the $\theta=0$ case which describes the S-MI transition. The above argued finiteness of h as both $\Lambda\rightarrow\infty$ and $m^2\rightarrow 0$ implies that the CS term does not receive infinite renormalization so that the a_μ fluctuations represent an exactly marginal perturbation. We have confirmed this through an explicit calculation to the next-to-leading order in $1/N$ expansion [18]. Furthermore, one can show from the Ward identity that with finite mass $m^2>0$, the CS term does not receive any radiative corrections, i.e., $h(0,m^2)=0$. Of course, all of these agree with the famous nonrenormalization theorem for the CS term in the literature [14,19]. Thus the S-MI critical point survives the gauge field fluctuations, but the exponents and universal quantities are expected to be modified by the fluctuations and acquire a θ dependence.

In the leading order of the $1/N$ expansion, $(1/N)^{-1}$ (and to all orders in θ), only the graph in Fig. 1(a) contributes to f and h and gives [11,14], at the critical point $m^2=0$,

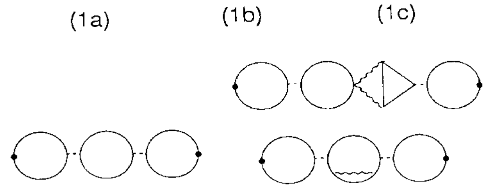

FIG. 1. (a) The graph for the self-energy of the gauge field a_μ. (b),(c) The graphs for the correlation $\langle\phi_i^\dagger\phi_i\rangle$. (d),(e) The graphs for the correlation $\langle(\phi_i^\dagger\phi_i)(\phi_i^\dagger\phi_i)\rangle$.

$$-f=\sigma_{\phi,xx}=N\tfrac{1}{16}+O(N^0),\quad -h=\sigma_{\phi,xy}=0+O(N^0)\,. \tag{8}$$

Here σ_ϕ can be viewed as the critical conductance of the bosons.

Next we calculate the scaling exponents at the QH-MT transition. The dynamical exponent is $z=1$. This is because the model (3) has the Lorentz symmetry and there are no spin-singlet relevant operators that break the Lorentz symmetry [20]. To obtain the scaling dimension of the operator ϕ_i at the first order of the $1/N$ expansion $(1/N)$, we only need to calculate the graphs in Figs. 1(b) and 1(c) for the propagator of ϕ_i. The dotted line represents the $|\phi|^4$ interaction, and the wavy line the fully dressed a_μ propagator, which has the form

$$G_{a;\mu\nu}=F(\delta_{\mu\nu}-p_\mu p_\nu/p^2)p^{-1}+H\epsilon_{\mu\nu\lambda}p_\lambda p^{-2}\,,$$
$$F=\frac{\sigma_{\phi,xx}}{(\sigma_{\phi,xx})^2+(\sigma_{\phi,xy}-N/2\theta)^2}\,, \tag{9}$$
$$H=\frac{\sigma_{\phi,xy}-N/2\theta}{(\sigma_{\phi,xx})^2+(\sigma_{\phi,xy}-N/2\theta)^2}\,,$$

in the Landau gauge $(\partial_\mu a_\mu=0)$ and at the critical point. Note F and H are of order $1/N$. We find $\langle\phi^\dagger_{-p}\phi_p\rangle \propto p^{-2}[1+(4F/3\pi^3)\ln p+\alpha\ln p]$. The F term comes from the gauge interactions in Fig. 1(c) and the α term comes from the ϕ^4 interactions in Fig. 1(b) that were calculated before [13]. Thus the scaling dimension of ϕ is [21]

$$[\phi]=\frac{1}{2}+\frac{4}{3\pi^2}\frac{1}{2N}+\frac{2F}{3\pi^2}+O(N^{-2})\,. \tag{10}$$

The correlation function of $\sum\phi_i^\dagger\phi_i$ up to order $(1/N)^0$ is given by Figs. 1(d) and 1(e). Other graphs containing the gauge field at this order are ultraviolet finite in dimensional regularization and have no contribution to the anomalous dimension of $\phi_i^\dagger\phi_i$. Figure 1(d) gives $\langle(\phi_i^\dagger\phi_i)_{-p}(\phi_i^\dagger\phi_i)_p\rangle\propto p+\text{const}$. Thus the leading term for the scaling dimension of $\phi_i^\dagger\phi_i$ is $[\phi_i^\dagger\phi_i]=2$. At the order of $1/N$, we have [21]

$$[\phi_i^\dagger\phi_i]=2-\frac{32}{3\pi^2}\frac{1}{2N}+\frac{4}{3\pi^2}F-\frac{N}{4\pi^2}(F^2-H^2)+O(N^{-2})\,. \tag{11}$$

The second term comes from the ϕ^4 interaction [13] and the third and fourth terms from Fig. 1(e). $\Delta\mathcal{L}=\int d^2x\,dt\sum|\phi_i|^2$ is the only relevant spin-singlet operator that controls the transition, at least in the larger-N limit. So the QH-MI transition is continuous, with the exponent ν [defined by $\xi\sim(\mu-\mu_c)^{-\nu}$, μ_c the critical chemical potential],

$$\nu=\frac{1}{3-[\phi_i^\dagger\phi_i]}=1-\frac{32}{3\pi^2}\frac{1}{2N}+\frac{4}{3\pi^2}F-\frac{N}{4\pi^2}(F^2-H^2)+O(N^{-2})\,. \tag{12}$$

The value of ν is determined by θ as one can see from (8) and (9).

Alternatively, one may introduce N statistical gauge fields $a_{I\mu}$ that couple to each ϕ_I to obtain a new large-N theory:

$$\mathcal{L}=\sum_{i=1}^{N}[|(\partial_\mu-ie^*A_\mu-ia_{I\mu})\phi_i|^2+m^2\phi_i^\dagger\phi_i]+\frac{g}{N}\left(\sum_{i=1}^{N}\phi_i^\dagger\phi_i\right)^2+\sum_I\frac{i}{4\theta}a_{I\mu}\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda}\,. \tag{13}$$

In this large-N theory, the result in Ref. [14] implies that the conductance σ_ϕ is θ dependent at the leading order $(1/N)^{-1}$:

$$-f=\sigma_{\phi,xx}=\frac{N}{16}+O(N^0)+O(\theta^2)\,,$$
$$-h=\sigma_{\phi,xy}=\frac{N\theta}{8}\left(\frac{1}{4}-\frac{1}{\pi^2}\right)+O(N^0)+O(\theta^3)\,. \tag{14}$$

Note in Ref. [5] that σ_ϕ is assumed to be independent of θ; this together with (6) forms the basis of their law of correspondence. In (14) our σ_ϕ does receive θ-dependent corrections due to gauge field fluctuations. Thus the law of correspondence does not apply to QH-QH and QH-MI transitions induced by periodic potentials.

Now let us consider the transition between two (Abelian) QH states. In the generalized hierarchical construction [22], a generic Abelian QH state is characterized by a symmetric integer matrix K_{IJ} and a charge vector t_I, with the effective Lagrangian

$$\frac{i}{4\pi}K_{IJ}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda}+i\frac{e}{2\pi}A_\mu t_I\partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda}\,. \tag{15}$$

The filling fraction is $\nu_K=\sum t_I(K^{-1})_{IJ}t_J$. There are $\dim K$ kinds of different quasiparticles, each respectively carries a unit charge of a_I. A composite quasiparticle that contains l_I quasiparticles of the Ith kind has a statistics θ_l and a charge e^*:

$$\theta_l=\pi\sum l_I(K^{-1})_{IJ}l_J,\quad e^*=\sum l_I(K^{-1})_{IJ}t_Je\,. \tag{16}$$

Let Φ_l be the field that describes the above composite quasiparticle. The low-energy dynamics of the quasiparticle is described by the effective Lagrangian [22]

$$\frac{1}{2}\left|\left(\partial_\mu+i\sum l_Ia_{I\mu}+ia_\mu\right)\Phi_l\right|^2+\frac{1}{2}m^2|\Phi_l|^2+g|\Phi_l|^4+\frac{i}{4\pi q}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} \tag{17}$$

with q an even integer. The CS term of the new gauge field a_μ binds q units of flux to the quasiparticle and does not change its statistics. It allows the quasiparticles to condense into different Laughlin states ($\langle\Phi\rangle=$const), each giving rise to a new QH state described by (15) and (17) and labeled by $(\dim K+1)$-dimensional matrix $\tilde{K}$ and vector $\tilde{t}$:

$$\tilde{K}=\begin{pmatrix}K_{IJ} & l_I\\ l_J & q\end{pmatrix},\quad \tilde{t}=\begin{pmatrix}t_I\\ 0\end{pmatrix}. \tag{18}$$

For more details, see Ref. [22]. One may view the above description of the QH-QH transitions as a generalization of those in Ref. [5].

Concerning only the transition of the quasiparticles, all the low-lying excitations are described by Φ_l and we can integrate out a_I. This leads to the effective theory

$$\frac{1}{2}|(\partial_\mu+ie^*A_\mu+ia_\mu)\Phi_l|^2+\frac{1}{2}m^2|\Phi_l|^2+g|\Phi_l|^4+\frac{i}{4(\pi q+\theta_l)}a_\mu\partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}-\frac{i\nu_K e^2}{4\pi}A_\mu\partial_\nu A_\lambda \epsilon^{\mu\nu\lambda}. \qquad (19)$$

Equation (19) is identical to (3) with $\theta=\theta_l+q\pi$, $N=1$, and an additional CS term for the electromagnetic field, which comes from the "vacuum," the QH state of filling fraction ν_K. Equation (19) describes a system of quasiparticles with charge e^* and statistics θ and an inert system of the parent QH state of filling fraction ν_K. This confirms the picture that, with the parent QH state being inert, the QH-QH transition is equivalent to the QH-MI transition for the qusiparticles. The total conductivity of the system is the conductivity of the quasiparticles calculated previously plus the conductivity of the underlying QH state [5]:

$$\sigma_{xx}=\frac{\sigma_{\phi,xx}}{[(2\theta/N)\sigma_{\phi,xx}]^2+[1-(2\theta/N)\sigma_{\phi,xy}]^2}\frac{e^{*2}}{\hbar},$$
$$\sigma_{xy}=\frac{\sigma_{\phi,xy}-(2\theta/N)(\sigma_{\phi,xx}^2+\sigma_{\phi,xy}^2)}{[(2\theta/N)\sigma_{\phi,xx}]^2+[1-(2\theta/N)\sigma_{\phi,xy}]^2}\frac{e^{*2}}{\hbar}+\nu_K\frac{e^2}{h}. \qquad (20)$$

Before the transition $\sigma_\phi=0$. After the transition $\sigma_{\phi,xx}=\infty$. At the transition σ_ϕ is given by (8) or (14). The conductivity and other linear responses at the transition point are determined by the statistics θ and the charge e^* of the quasiparticles. θ and e^* can be calculated from (16). The 2π shift in θ can be determined from the requirement $\nu_K-(\pi/\theta)e^{*2}/e^2=\nu_{\tilde K}$, where $\nu_{\tilde K}$ is the filling fraction of the $\tilde K$ QH state. These discussions apply also to the transitions between the multilayer and spin unpolarized QH states.

Let us consider hierarchical states with filling fraction $\nu_K=k/(pk+1)$ (p is even), including the $1,\frac{1}{3},\frac{2}{5},\ldots$ states. The K matrix and the charge vector t were given in Ref. [22]. For $e^*=1/(kp+1)$ quasiparticle, $\theta_l/\pi=1-p/(pk+1)$. Thus $\theta/\pi=1-q-p/(pk+1)$ (q is a nonzero even integer). The condensation of such quasiparticle leads to a new QH state with filling fraction $\nu_{\tilde K}=\nu_K+(pk+1)^{-1}[(q-1)(pk+1)+p]^{-1}$. This series includes $\frac{2}{3},\frac{2}{5},\frac{2}{7},\ldots$ QH states. In particular, for the transition between $\nu_K=k/(2k+1)$ and $\nu_{\tilde K}=(k+1)/[2(k+1)+1]$ QH states ($q=p=2$), we have $e^*=1/(2k+1)$ and $\theta/\pi=-(2k+3)/(2k+1)$.

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