Renormalization of the Statistics Parameter in Three-Dimensional Electrodynamics

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We argue that the β function for the statistics parameter in (2+1)-dimensional electrodynamics is zero to all orders in perturbation theory beyond one loop. We show that there can be finite radiative corrections from massless charged scalar fields.

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Gauge theories in (2+1) dimensions have several features not seen in their (3+1)-dimensional counterparts. These center on the possible appearance of the Chern-Simons three-form in the action[1] and are connected with parity anomalies[2] and exotic charge,[2] angular momentum,[3] and statistics.[4] Original motivation for their study was their resemblance to the high-temperature limit of gauge theories in (3+1) dimensions. More recently, there has been interest in (2+1)-dimensional electrodynamics in models of certain condensed-matter phenomena, particularly high-T_c superconductors[5-7] and the quantized Hall effect.[8]

The Euclidean action of scalar electrodynamics with both Chern-Simons and Maxwell kinetic terms for the gauge fields is

$$S=\int d^3x\left[(D_\mu\phi_0)^*(D_\mu\phi_0)+m_0^2\phi_0^*\phi_0+i\frac{\alpha_0}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu^0\,\partial_\nu A_\lambda^0+\frac{1}{4e_0^2}F_{\mu\nu}^0F_{\mu\nu}^0\right], \tag{1}$$

where $D_\mu=\partial_\mu+ig_0A_\mu^0$. This is a prototype for the effective field theory of the CP^1 model discussed in Ref. 9 (see also Ref. 10). The Chern-Simons term gives the ϕ quanta fractional statistics.[4] (With the Maxwell term this is only true at large distances.) The gauge field and e_0^2 have dimension 1 and g_0 and α_0 are dimensionless. This field theory is superrenormalizable with computable, finite physical parameters. There is considerable interest in a model with only Chern-Simons kinetic terms.[4-10] It is defined by the limit $e_0^2\to\infty$ where it is renormalizable with dimensionless bare coupling $g_0|\alpha_0|^{-1/2}$ and $F_{\mu\nu}^0F_{\mu\nu}^0$ is an irrelevant operator. Since all physical parameters are not finite in this limit, renormalization is required. In Ref. 9 it was argued that the dimensionless statistics parameter g_0^2/α_0 can have an interesting renormalization flow. We show in this paper that this is not the case. When the renormalized mass $m^2\neq0$, the corresponding β function is zero to all orders in loop expansion. For the critical case $m^2=0$ (although there is no symmetry which prevents radiative mass generation, it is natural at a critical point for the scalar to be massless) we demonstrate that β is zero to two-loop order and we conjecture that it is zero to all orders. We also show that, when $m^2=0$, g_0^2/α_0 has finite renormalization in higher loops. With $\phi_0=Z^{1/2}\phi$, $A_\mu^0=Z_3^{1/2}A_\mu$, $g_0=Z^{-1}Z_3^{-1/2}Z_2g$, $\alpha_0=Z_3^{-1}Z_\alpha\alpha$, $m_0^2=m^2+\delta m^2$, and the Ward-Takahashi identity $Z=Z_2$, the action is

$$S=\int\left\{Z[\partial_\mu\phi^*\,\partial_\mu\phi+(m^2+\delta m^2)\phi^*\phi]+\frac{Z_3}{4e_0^2}F_{\mu\nu}F_{\mu\nu}+Z_\alpha i\,\frac{\alpha}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu\,\partial_\nu A_\lambda+ZgA_\mu\phi^*i\,\overleftrightarrow{\partial}_\mu\phi+Zg^2A_\mu A_\mu\phi^*\phi\right\}. \tag{2}$$

Z_α is fixed by the requirement that the renormalized Chern-Simons term has coefficient $\alpha/4\pi$. The physical statistics parameter is $g^2/\alpha=Z_\alpha g_0^2/\alpha_0$. Propagators, vertices, and counterterms are summarized in Fig. 1. We use e_0^2 as a gauge-invariant cutoff to be taken to ∞ at the end of computations. To see that when $m^2>0$, Z_α receives no contributions beyond one loop,[11] consider the irreducible Euclidean N-photon correlation function for $N\geq4$ in the one-loop approximation $\Gamma_{\mu_1\ldots\mu_N}(p_1;\ldots;-\sum_1^{N-1}p_k)$ shown in Fig. 2. Γ is symmetric. Radiative corrections to Z_α with two or more loops are one-photon-irreducible graphs obtained either by sewing together all but two legs of Γ [see Fig. 3(a)], then finding the term linear in the external momentum and antisymmetric in vector indices of the resulting two-point function,

$$\lim_{p\to0}\epsilon_{\mu\nu\rho}\frac{\partial}{\partial p_\rho}\int dp_3dp_4\cdots\Gamma_{\mu\nu\lambda_3\ldots\lambda_N}\left(p;-p;p_3;\ldots;-\sum_3^{N-1}p_k\right)\mathcal{K}_{\lambda_3\ldots\lambda_N}(p_3;p_4;\ldots), \tag{3}$$

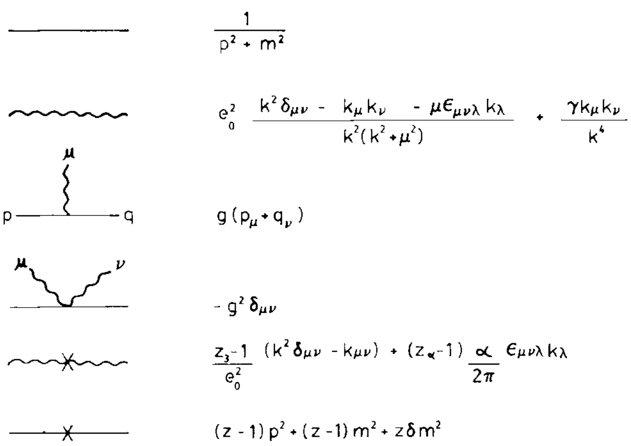

FIG. 1. Progagators and vertices for scalar electrodynamics. The topological mass is $\mu = \alpha e_0^2/2\pi$, $e_0^2 \to \infty$, and γ is the gauge-fixing parameter.

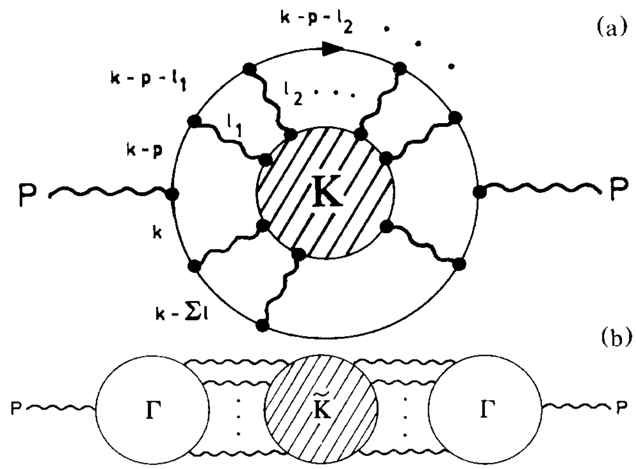

FIG. 3. How $\Pi_{\mu\nu}$ is constructed from Γ. $\tilde{\mathcal{K}}$ depends on p whereas $\mathcal{K}$ is independent of p.

or by sewing two Γ's together in all possible (one-photon irreducible) ways [Fig. 3(b)],

$$\lim_{p\to 0} \epsilon_{\mu\nu\rho} \frac{\partial}{\partial p_\rho} \int dl_1 \cdots dq_1 \cdots \Gamma_{\mu\lambda \ldots}(p;l_1;\ldots)\Gamma_{\nu\rho\ldots}(-p;q_1;\ldots)\tilde{\mathcal{K}}_{\lambda\ldots\rho\ldots}(p;l_1;\ldots;q_2;\ldots)\,. \tag{4}$$

When $m^2>0$, Γ is analytic at $p_i^{\mu_i}=0$. $\tilde{\mathcal{K}}$ depends on p where as $\mathcal{K}$ does not. The Ward-Takahashi identity is

$$0 = p_i^{\nu_i}\Gamma_{\mu_1\ldots\nu_1\ldots\mu_N}\left(p_1;\ldots;p_i;\ldots;-\sum_k^{N-1} p_k\right).$$

Taking a derivative by p_i, setting $p_i=0$, and using analyticity yields

$$0 = \Gamma_{\mu_1\ldots\nu_i\ldots\mu_N}\left(p_1;\ldots;0;\ldots;-\sum_1^{N-1} p_k\right),$$

This immediately implies that (4) vanishes. It further implies

$$\lim_{p\to 0}\frac{\partial}{\partial p_\rho}\Gamma_{\mu\nu\ldots}(p;-p;\ldots) = \lim_{p\to 0}\left(\frac{\partial}{\partial p_\rho}\Gamma_{\mu\nu\ldots}(p;0;\ldots) - \frac{\partial}{\partial p_\rho}\Gamma_{\mu\nu\ldots}(0;-p;\ldots)\right) = 0\,,$$

and (3) also vanishes. (This generalizes to other massive matter such as spinor and vector fields. It is also valid at finite temperature. There the Euclidean momentum vectors have one component the discrete Matsubara frequency $p_i^0 = 2\pi k_B T n_i$ with T the temperature, n_i an integer, and the other two components continuous variables. For $N>4$, Euclidean N-photon functions have unique extensions to functions of complex momenta p_i^0. With massive matter, these functions are analytic at zero frequency and momentum and the Ward-Takahashi identity also holds. The above reasoning shows that no two-photon vertex with one power of derivatives is generated by radiative corrections beyond one loop at finite temperature.) Thus, given the known *finite* one-loop corrections from both scalars and spinors,[2] their contributions to the β function for the

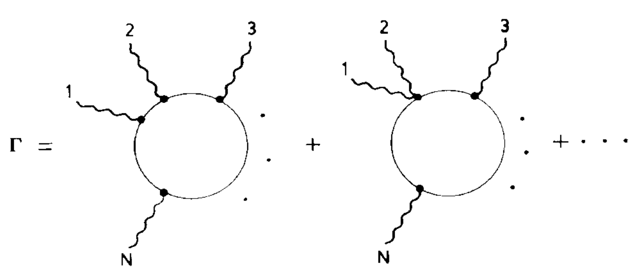

FIG. 2. The N-photon correlation function at one-loop order is the symmetrized sum with all combinations of both three- and four-point vertices.

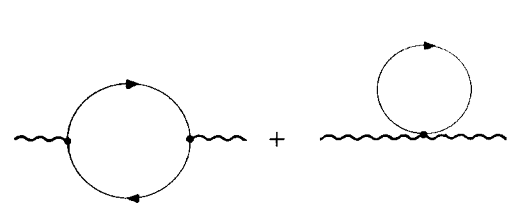

FIG. 4. One-loop photon self-energy.

statistics parameter is zero.

This leaves the possibility of a nonzero β function from massless charged fields where the N-point functions Γ cannot be assumed analytic at zero external momenta. The zero mass limit which defines a massless field theory may not commute with the zero momentum limit which determines the renormalization of a coupling constant. However, since the β function is independent of momentum, it will be zero in the massless limit as it is in massive theory. This will be verified by explicit calculation to two-loop order in the following. We conjecture that it holds to all orders. This argument does not exclude finite renormalizations and we shall find explicit nonzero finite corrections at two-loop order from massless scalars.

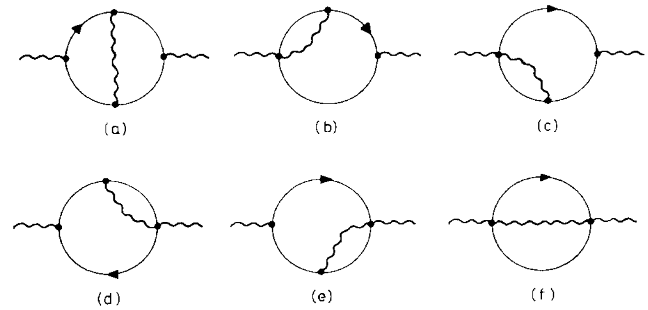

FIG. 5. Two-loop photon self-energy.

The one-loop correction to the photon two-point function from charged scalars (Fig. 4) is

$$\Pi_{\mu\nu}^{(1)}(p)=(p^2\delta_{\mu\nu}-p_\mu p_\nu)\frac{g^2}{8\pi}\left\{-\frac{2\sqrt{m^2}}{p^2}+\frac{p^2+4m^2}{p^2\sqrt{p^2}}\arcsin\left[\frac{p^2}{p^2+4m^2}\right]^{1/2}\right\}. \tag{5}$$

Although the loop integral is potentially divergent and requires a cutoff at intermediate stages of calculation, the result is finite. Its antisymmetric part vanishes and when $m^2>0$ its symmetric part has a local limit $(p^2\delta_{\mu\nu}-p_\mu p_\nu)\times g^2/24\pi\sqrt{m^2}$ which contributes to Z_3. Figure 5 depicts two-loop self-energy diagrams. The corresponding integrals are finite for $e_0^2<\infty$ and by power counting are logarithmically divergent when $e_0^2\to\infty$. The contributions of Figs. 5(b)–5(e) to $(1/2p^2)\epsilon_{\mu\nu\lambda}\Pi_{\mu\nu}^{(2)}p_\lambda$ cancel immediately upon combining the integrals. Figure 5(a) can be reduced to the Feynman parameter integral

$$\frac{g^4}{8\pi^2\alpha}\int_0^1 dx_1\cdots dx_4\,\theta(1-x_1-\cdots-x_4)\frac{1}{[(1-x_1-x_2)(x_1+\cdots+x_4)-(x_3+x_4)^2]^{5/2}}$$
$$\times\ln\frac{\mu^2(1-x_1-\cdots-x_4)(1-x_1-x_2)+m^2(x_1+\cdots+x_4)(1-x_1-x_2)+p^2\Omega}{m^2(x_1+\cdots+x_4)(1-x_1-x_2)+p^2\Omega} \tag{6}$$

where $\mu=\alpha e_0^2/2\pi$ and

$$\Omega=[(1-x_1-x_2)(x_2+x_3)-(x_3+x_4)^2]-\frac{[(1-x_1-x_2)(x_2+x_3)-(x_3+x_4)x_3]^2}{(1-x_2-x_3)(x_1+\cdots+x_4)-(x_3+x_4)^2}. \tag{7}$$

Figure 5(f) reduces to

$$-\frac{g^4}{8\pi^2\alpha}\int_0^1 dx_1dx_2\,\theta(1-x_1-x_2)\frac{x_1x_2}{[(1-x_1-x_2)(x_1+x_2)-x_1x_2]^{5/2}}$$
$$\times\ln\frac{\mu^2(1-x_1-x_2)(1-x_1)+m^2(1-x_1)(x_1+x_2)+p^2\tilde\Omega}{m^2(1-x_1)(x_1+x_2)+p^2\tilde\Omega}, \tag{8}$$

$$\tilde\Omega=x_2(1-x_1-x_2)-\frac{x_2^2(1-x_1-x_2)^2}{(1-x_1-x_2)(x_1+x_2)+x_1x_2}. \tag{9}$$

If we keep $m^2>0$ and put $p^2=0$, these two contributions cancel for all values of $\mu^2>0$, consistent with the no-renormalization argument above. On the other hand, if we first put $m^2=0$ and then μ^2/p^2 large (since $e_0^2\to\infty$), the divergent leading terms proportional to $\ln(\mu^2/p^2)$ cancel. Therefore, as expected, there is no infinite contribution to Z_α. After some standard manipulations,[12] the finite parts are

$$\frac{g^4}{8\pi^2\alpha}\int_0^1 d\rho\,d\lambda\,dx\,dy\,\theta(1-y-\rho)\frac{\lambda^{1/2}\rho}{[1-\lambda+\lambda\rho(1-\rho)]^{5/2}}$$
$$\times\ln\frac{\rho(1-\rho)(1-\lambda)(1-\lambda\rho)}{[(1-\lambda\rho)(\rho x+y)-\lambda y^2][1-\lambda+\lambda\rho(1-\rho)]-[\rho(1-\lambda\rho)x+(1-\lambda)y]^2}\approx-2.74\frac{g^4}{8\pi^2\alpha}, \tag{10}$$

where the approximate result is obtained numerically.

To couple spinors we add $\int\psi_0^\dagger[\gamma_\mu(i\partial_\mu-i\tilde g_0A_\mu^0)+M_0]\psi_0$ to (1). Generalization of the arguments above shows that there are no contributions to Z_α beyond one loop at either zero or finite temperature when $M\neq 0$. At one loop, the correction is known[2] at $T=0$ and can be obtained at $T\neq 0$ as

$$\lim_{p\to 0}\frac{1}{2p^2}\epsilon_{\mu\nu\lambda}\Pi^{(1)}_{\mu\nu}p_\lambda=\frac{\tilde{g}^2}{8\pi}\tanh\frac{M}{k_BT}\,.$$

When $M=0$ the one-loop correction vanishes but there could be finite corrections from higher loops.

Equation (5) indicates a radiatively generated Maxwell term at one-loop (and higher) order. We determine Z_3 so that the renormalized Maxwell term has coefficient $1/4e^2$. Even when the bare action has only a Chern-Simons term, a Maxwell term is generated in the effective action. In that case it is possible to resume perturbation theory using the renormalized Maxwell term in the Feynman rules which are then superrenormalizable. The present analysis also applies to the singular case where $1/e^2=0$ which remains marginally renormalizable. In summary, for scalars when $m=0$, $Z_\alpha=1+2.74g^4/2\pi\alpha^2+\dots$, $Z_3=e_0^2/e^2+\dots$; when $m^2\neq 0$, $Z_\alpha=1$, $Z_3=e_0^2/e^2-e_0^2g^2/24\pi\sqrt{m^2}+\dots$. For spinors when $M=0$, $\delta Z_\alpha=$(finite corrections from $\geq$ two loops), Z_3 is affected at two and higher loops; when $M^2\neq 0$, $\delta Z_\alpha=-(\tilde{g}^2/2\alpha)\tanh(M/k_BT)$, $\delta Z_3=-e_0^2\tilde{g}^2/12\pi\sqrt{M^2}+\dots$ from one- and higher-loop vacuum polarization. In the massive theories we have Z_α exactly. In both cases there are no infinite corrections to α_0 and the corresponding β function vanishes. However, there is finite renormalization from massless particles in higher loops and from massive spinors at one loop. There can be an infinite renormalization from charged massive vectors at one loop.[13] The latter [as well as the CP^1 model to which (1) is approximate] is a nonrenormalizable theory and the perturbative expansion is unreliable. Also, our results for renormalizable theories are perturbative and would be improved by demonstrating Borel summability of the perturbation series. Finally, we speculate on hysteresis where we vary parameters, i.e., T, m^2, e^2 of scalar electrodynamics. The statistics parameter is unaffected if we avoid critical points. If we go to a critical point the statistics parameter gets finite computable corrections. We then leave the critical point with renormalized fractional statistics.

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