A note on the Oslo Algorithm

Tom Lyche
Dept. of Computer Science, University of Utah [1]

UUCS-87-018
August 6, 1987

Abstract The Oslo algorithm is a recursive method for updating the B–spline representation of a curve or tensor product surface when extra knots are added. In the present note the derivation of this method is simplified.

Keywords B–splines, knot insertion, subdivision, discrete B-splines.

1 Introduction

The Oslo algorithm was derived in Cohen, Lyche, Riesenfeld[1] using discrete B–splines and divided differences. In the spirit of de Boor and Höllig[2], where the elementary B–spline theory is simplified the derivation of both the algorithm in Cohen, Lyche, Riesenfeld[1] and the refined algorithm presented in Lyche and Mørken[3] will be simplified.

Starting with the recurrence relation for discrete B–splines, a discrete version of Marsden's identity is derived. The dual linear functionals then give the connection between discrete B–splines and knot insertion. The method for adding one knot in Böhm[4] follows as a special case.

Many proofs of the recurrence relation in Cohen, Lyche, Riesenfeld[1] exist. See Prautzsch[5] (using recurrence relations, but assuming simple knots), Lee[6] (using dual functionals on integral form), Mørken[7] (using deBoor/Fix dual linear functionals), and Barry[8] Chapter 5 (using Pólya curves and duality).

For computational aspects see Lyche, Cohen, Mørken[9], Böhm and Prautzsch[10], and Böhm and Prautzsch[11]. Detailed algorithms are presented in Lyche and Mørken[3].

2 Discrete B–splines

Let $\boldsymbol{t} = (t_i)$ be a nondecreasing sequence of real numbers, suppose $\boldsymbol{\tau} = (\tau_i)$ is a subsequence of $\boldsymbol{t}$, and let k be a positive integer. It is assumed that $\boldsymbol{\tau}$ is a finite or infinite sequence which contains at least $k+1$ elements. The *discrete B–spline of order k on $\boldsymbol{t}$*

---

[1] On leave from the Univ. of Oslo. Permanent address: Institutt for informatikk, Box 1080, Blindern, 0316 Oslo 3, Norway

with knots $\boldsymbol{\tau}$, denoted by $\alpha_{j,k}(i) = \alpha_{j,k,\boldsymbol{\tau},\mathbf{t}}(i)$, is defined recursively as follows

$$\alpha_{j,k}(i) = \omega_{j,k}(t_{i+k-1})\alpha_{j,k-1}(i) + (1 - \omega_{j+1,k}(t_{i+k-1}))\alpha_{j+1,k-1}(i), \tag{1}$$

where

$$\omega_{j,k}(x) = \begin{cases} (x - \tau_j)/(\tau_{j+k-1} - \tau_j), & \text{if } \tau_{j+k-1} > \tau_j; \\ 0, & \text{otherwise}; \end{cases}$$

and

$$\alpha_{j,1}(i) = \begin{cases} 1, & \text{if } \tau_j \leq t_i < \tau_{j+1}; \\ 0, & \text{otherwise.} \end{cases}$$

If $x = t_i = t_{i+1} = \ldots = t_{i+k-1}$ then (1) reduces to the usual recurrence relation for B–splines $B_{j,k,\boldsymbol{\tau}}$. Thus $\alpha_{j,k,\boldsymbol{\tau},\mathbf{t}}(i) = B_{j,k,\boldsymbol{\tau}}(x)$ in this case. If $\tau_\mu \leq t_i < \tau_{\mu+1}$ then (1) implies that $\alpha_{j,k}(i) = 0$ for $j \leq \mu - k$ and $j > \mu$. Moreover, if

$$\tau_\mu < t_{i+1} \leq \ldots \leq t_{i+k-1} < \tau_{\mu+1} \tag{2}$$

then it easily follows by induction in (1) that $\alpha_{j,k}(i) > 0$ for $j = \mu - k + 1, \ldots, \mu$.

The following discrete version of Marsden's identity will be useful.

Theorem 2.1 *Suppose* $\tau_\mu \leq t_i < \tau_{\mu+1}$. *Then for any positive integer* k *and any real* y

$$\psi_{i,k,\mathbf{t}}(y) = \sum_{j=\mu-k+1}^{\mu} \psi_{j,k,\boldsymbol{\tau}}(y)\alpha_{j,k,\boldsymbol{\tau},\mathbf{t}}(i), \tag{3}$$

where for any integer r *and any knot sequence* $\mathbf{u} = (u_j)$,

$$\psi_{r,k,\mathbf{u}}(y) = \begin{cases} (u_{r+1} - y)\cdots(u_{r+k-1} - y), & \text{if } k \geq 2; \\ 1, & \text{if } k = 1. \end{cases}$$

Proof. Denoting the right hand side of (3) by S_k, using (1), and the fact that $\alpha_{\mu-k+1,k-1}(i) = \alpha_{\mu+1,k-1}(i) = 0$, one finds after rearranging terms

$$S_k = \sum_{j=\mu-k+2}^{\mu} c_j \alpha_{j,k-1,\boldsymbol{\tau},\mathbf{t}}(i),$$

where

$$c_j = \psi_{j,k,\boldsymbol{\tau}}(y)\omega_{j,k}(t_{i+k-1}) + \psi_{j-1,k,\boldsymbol{\tau}}(y)(1 - \omega_{j,k}(t_{i+k-1})) = (t_{i+k-1} - y)\psi_{j,k-1,\boldsymbol{\tau}}(y).$$

Thus, $S_k = (t_{i+k-1} - y)S_{k-1}$. Since $S_1 = 1$ equation (3) follows by induction. ■

The proof just given is almost identical to de Boor's simple proof of Marsden's identity for B–splines. Cf. de Boor and Höllig[2]. Barry informed the author that a more general form of (3) can be found in Barry[8] equation (5.6).

Suppose

$$f = \sum_j c_j B_{j,k,\tau}$$

is a linear combination of B-splines on the knot sequence τ. Since τ is a subsequence of t , f can also be written as a linear combination of B-splines on the knot sequence t ,

$$f = \sum_i d_i B_{i,k,t}$$

for some coefficients d_i. It was shown in Cohen, Lyche, Riesenfeld[1] (using a different proof) that the d's and c's are related as follows.

Theorem 2.2 *If* $t_{i+k} > t_i$ *then*

$$d_i = \sum_{j=\mu-k+1}^{\mu} c_j \alpha_{j,k,\tau,t}(i), \tag{4}$$

where the integer μ *is such that* $\tau_\mu \leq t_i < \tau_{\mu+1}$.

Proof. Note first that $B_{i,k,t} \equiv 0$ if $t_{i+k} = t_i$. Thus d_i is not uniquely defined in this case. For any $y \in [t_i, t_{i+k})$, the coefficient d_i can be expressed in terms of f as follows

$$d_i = \sum_{r=0}^{k-1} (-1)^{k-1-r} {\psi_{i,k,t}}^{(k-1-r)}(y) f^{(r)}(y)/(k-1)!. \tag{5}$$

See de Boor and Höllig[2] for a simple proof of this formula. Similarly, for any $y \in [\tau_j, \tau_{j+k})$,

$$c_j = \sum_{r=0}^{k-1} (-1)^{k-1-r} {\psi_{j,k,\tau}}^{(k-1-r)}(y) f^{(r)}(y)/(k-1)!. \tag{6}$$

Choose $y \in (t_i, \tau_{\mu+1})$. Then $y \in [\tau_j, \tau_{j+k})$ for $j = \mu-k+1, \ldots, \mu$, and also $y \in [t_i, t_{i+k})$. Taking the $(k-1-r)$th derivative in (3) with respect to y , multiplying each side by $(-1)^{k-1-r} f^{(r)}(y)/(k-1)!$, summing over r, using (5) on the left hand side, exchanging the order of summation on the right hand side, and finally using (6), one obtains (4). ∎

For simplicity assume from now on that $t_{i+k} > t_i$ and $\tau_{j+k} > \tau_j$ for all integers i and j. Following Lyche and Mørken[3] let us take a closer look at the polynomial $\psi_{i,k,t}$ which is used to define the $\alpha_{j,k,\tau,t}(i)$ for fixed i. Suppose

$$\psi_i = \psi_{i,k,t}(y) = \prod_{j=1}^{h} (z_j - y)^{r_j},$$

where $z_1 < z_2 < \cdots < z_h$. Define

$$\rho_j = \begin{cases} \min\{r_j, s_j\}, & j = 1, h; \\ s_j, & \text{otherwise:} \end{cases}$$

where $s_j = \#\{\tau_i : \tau_i = z_j\}$. Since $\boldsymbol{\tau} \subset \boldsymbol{t}$ it follows that $\rho_j \leq r_j$ for all j. Factor ψ_i into a product of two polynomials $\psi_i = \psi_{\boldsymbol{\omega}}\psi_{\boldsymbol{\xi}}$ where

$$\psi_{\boldsymbol{\omega}}(y) = \prod_{j=1}^{h}(z_j - y)^{\rho_j}, \quad \psi_{\boldsymbol{\xi}}(y) = \prod_{j=1}^{h}(z_j - y)^{r_j-\rho_j}.$$

The $\nu = \sum_j(r_j - \rho_j)$ roots, say $\xi_1, \ldots, \xi_\nu$ of $\psi_{\boldsymbol{\xi}}$ are the elements of $t_{i+1}, \ldots, t_{i+k-1}$ which do not correspond to a τ - knot. As an example, if (2) holds then $\psi_{\boldsymbol{\xi}} = \psi_{i,k,\boldsymbol{t}}$ while $\psi_{\boldsymbol{\omega}} = 1$.

A more precise version of (4) can now be given.

Theorem 2.3 *Let for fixed i the nondecreasing sequences $\boldsymbol{\tau}'$ and $\boldsymbol{t}'$ be obtained from $\boldsymbol{\tau}$ and $\boldsymbol{t}$ by removing the $k-1-\nu$ roots of $\psi_{\boldsymbol{\omega}}$. If $\boldsymbol{\tau} \subset \boldsymbol{t}$ and*

$$f = \sum c_j B_{j,k,\boldsymbol{\tau}} = \sum d_i B_{i,k,\boldsymbol{t}}$$

then

$$d_i = \sum_{j=\mu'-\nu}^{\mu'} c_j \alpha_{j,\nu+1,\boldsymbol{\tau}',\boldsymbol{t}'}(i), \tag{7}$$

where the integer μ' is such that

$$\tau'_{\mu'} \leq t_i = t'_i < \tau'_{\mu'+1}.$$

Proof. By Theorem 2.1

$$\psi_{i,\nu+1,\boldsymbol{t}'}(y) = \sum_{j=\mu'-\nu}^{\mu'} \psi_{j,\nu+1,\boldsymbol{\tau}'}(y)\alpha_{j,\nu+1,\boldsymbol{\tau}',\boldsymbol{t}'}(i). \tag{8}$$

Also, (cf. Lemma 2.6 in Lyche and Mørken[3])

$$\psi_{\boldsymbol{\omega}}(y) = (\tau_{\mu'+1} - y)\cdots(\tau_{\mu'+k-1-\nu} - y), \tag{9}$$

$$\tau_{\mu'} < \xi_1 \leq \ldots \leq \xi_\nu < \tau_{\mu'+k-\nu}. \tag{10}$$

Since $t'_{i+j} = \xi_j$, $j = 1, 2, \ldots \nu$ it follows that $\psi_{i,k,\boldsymbol{t}} = \psi_{\boldsymbol{\omega}}\psi_{\boldsymbol{\xi}} = \psi_{\boldsymbol{\omega}}\psi_{i,\nu+1,\boldsymbol{t}'}$, and by (9) $\psi_{\boldsymbol{\omega}}$ is a factor of $\psi_{j,k,\boldsymbol{\tau}}$ for $j = \mu'-\nu, \ldots, \mu'$ so that $\psi_{j,k,\boldsymbol{\tau}} = \psi_{\boldsymbol{\omega}}\psi_{j,\nu+1,\boldsymbol{\tau}'}$, $j = \mu'-\nu, \ldots, \mu'$.

Therefore, multiplying by $\psi_{\boldsymbol{\omega}}$ on both sides of (8) gives the equation

$$\psi_{i,k,\boldsymbol{t}}(y) = \sum_{j=\mu'-\nu}^{\mu'} \psi_{j,k,\boldsymbol{\tau}}(y)\alpha_{j,\nu+1,\boldsymbol{\tau}',\boldsymbol{t}'}(i).$$

Proceeding as in Theorem 2.2 equation (7) follows. ■

By (4) and (7) $\alpha_{j,k,\boldsymbol{\tau},\boldsymbol{t}}(i) = \alpha_{j,\nu+1,\boldsymbol{\tau}',\boldsymbol{t}'}(i)$ for all values of j. Moreover by (10), equation (7) gives precisely the range of j's for which $\alpha_{j,k,\boldsymbol{\tau},\boldsymbol{t}}(i)$ is positive.

Corollary. *Suppose* $\boldsymbol{t} = \boldsymbol{\tau} \cup \{t\}$, *i.e. only one knot at the location* t *is inserted. If* $\tau_\mu \leq t < \tau_{\mu+1}$ *then the* d_i*'s in (7) are given by*

$$d_i = \begin{cases} c_i, & \text{if } i \leq \mu - k + 1; \\ \omega_{j,k}(t) c_i + (1 - \omega_{j,k}(t)) c_{i-1}, & \text{if } i = \mu - k + 2, \ldots, \mu - m; \\ c_{i-1}, & \text{if } i > \mu - m, \end{cases} \tag{11}$$

where m *is the multiplicity of* τ_μ *in* $\boldsymbol{\tau}$ *if* $t = \tau_\mu$ *and* $m = 0$ *if* $t > \tau_\mu$.

Proof. If $i \leq \mu - k + 1$ then $\psi_{\boldsymbol{\omega}} = \psi_{i,k,\boldsymbol{\tau}} = \psi_{i,k,\boldsymbol{t}}$ so that $\nu = 0$ and $\boldsymbol{t}' = \boldsymbol{\tau}' = \{\ldots \tau_i, \tau_{i+k}, \ldots\}$. Thus $\mu' = i$, and (11) follows from (7), since $\alpha_{i,1,\boldsymbol{\tau}',\boldsymbol{t}'}(i) = 1$. Similarly, (11) follows if $i > \mu - m$, for then $\psi_{\boldsymbol{\omega}} = \psi_{i-1,k,\boldsymbol{\tau}} = \psi_{i,k,\boldsymbol{t}}$, which implies that $\nu = 0$ and $\mu' = i - 1$. Finally, for the remaining values of i it is easily seen that $\nu = 1$, $\mu' = i$, $\boldsymbol{\tau}' = \{\ldots \tau_i, \tau_{i+k-1}, \ldots\}$, and $\boldsymbol{t}' = \{\ldots, \tau_i, t, \tau_{i+k-1}, \ldots\}$. Therefore, by (7) and (1)

$$d_i = c_i \alpha_{i,2,\boldsymbol{\tau}',\boldsymbol{t}'}(i) + c_{i-1} \alpha_{i-1,2,\boldsymbol{\tau}',\boldsymbol{t}'}(i) = c_i \frac{t'_{i+1} - \tau'_i}{\tau'_{i+1} - \tau'_i} + c_{i-1} \frac{\tau'_{i+1} - t'_{i+1}}{\tau'_{i+1} - \tau'_i}.$$

Since $t'_{i+1} = t$, $\tau'_i = \tau_i$, and $\tau'_{i+1} = \tau_{i+k-1}$, equation (11) follows. ■

References

[1] **Cohen, E, Lyche, T and Riesenfeld, R F** 'Discrete B-splines and subdivision techniques in computer aided geometric design and computer graphics' *Computer Graphics and Image Processing* Vol 14 no 2 (October 1980) pp 87–111.

[2] **de Boor, C and Höllig, K** 'B-splines without divided differences' in *Geometric Modeling: Algorithms and new trends*, G E Farin (ed.), Siam, Philadelphia (1987) pp 21–27.

[3] **Lyche, T and Mørken, K** 'Making the Oslo algorithm more efficient' *Siam J. Numer. Anal.* Vol 23 no 3 (June 1986) pp 663-675.

[4] **Böhm, W** 'Inserting new knots into B-spline curves' *Computer Aided Design* Vol 12 no 4 (July 1980) pp 199–201.

[5] **Prautzsch, H** 'A short proof of the Oslo algorithm' *CAGD* Vol 1 (1984) pp 95–96.

[6] **Lee, E T Y** 'Some remarks concerning B-splines' *CAGD* Vol 2 (1985) pp 307–311.

[7] **Mørken, K** 'On two topics in spline theory: discrete splines and the equioscillating spline' Cand. real. thesis, Institutt for informatikk, Univ of Oslo (May 1984).

[8] **Barry, P J** 'Urn models, recursive curve schemes, and computer aided geometric design' Ph.D. dissertation, Dep. of Math., Univ. of Utah (June 1987).

[9] **Lyche, T, Cohen, E and Mørken, K** 'Knot line refinement algorithms for tensor product B-spline surfaces' *CAGD* Vol 2 (1985) pp 133–139.

[10] **Böhm, W and Prautzsch, H** 'On the efficiency of knot insertion algorithms' *CAGD* Vol 2 (1985) pp 141–143.

[11] **Böhm, W and Prautzsch, H** 'The insertion algorithm' *Computer Aided Design* Vol 17 (1985) pp 58–59.