Improved Convergence Analysis of Stochastic Gradient Adaptive Filters Using the Sign Algorithm

V. JOHN MATHEWS, MEMBER, IEEE, AND SUNG HO CHO

Abstract—**Convergence analysis of stochastic gradient adaptive filters using the sign algorithm is presented in this paper. The methods of analysis currently available in literature assume that the input signals to the filter are white. This restriction is removed for Gaussian signals in our analysis. Expressions for the second moment of the coefficient vector and the steady-state error power are also derived. Simulation results are presented, and the theoretical and empirical curves show a very good match.**

I. Introduction

Stochastic gradient adaptive filters using nonlinear correlation multipliers have received a great deal of attention recently [1], [2], [4]–[6], [11], [14], [15]. In this paper we are concerned only with adaptation using the sign algorithm, where the coefficient vector is updated using

$$H(n+1) = H(n) + \mu X(n) \operatorname{sign}(e(n)), \quad (1)$$

where $H(n)$ is the vector of N coefficient values at time n, $X(n)$ is the primary input vector to the filter, μ is a time-invariant convergence parameter, and $e(n)$ is the error in estimating the reference input $d(n)$ using the primary input vector $X(n)$, i.e.,

$$e(n) = d(n) - H^T(n)\, X(n), \quad (2)$$

where $(\cdot)^T$ denotes the transpose of $(\cdot)$.

Earlier analyses of the system described by (1) and (2) assume that the input signals are zero mean and white [4], [6]. In particular, the input correlation matrix is assumed to be diagonal, with

$$R_{XX} = E\{X(n)\, X^T(n)\} = \sigma_x^2 I. \quad (3)$$

In many practical situations, this assumption is grossly violated and the convergence curves using the above model tend to show faster than true convergence, especially when the eigenvalue spread for the input autocorrelation matrix is large (see Fig. 1).

In this paper we will undo the whiteness assumption for the input data signals. We will assume that the primary and reference input signals are jointly Gaussian, zero mean signals. As in many convergence analyses of this

Manuscript received May 13, 1986; revised October 24, 1986.
The authors are with the Department of Electrical Engineering, University of Utah, Salt Lake City, UT 84112.
IEEE Log Number 8612883.

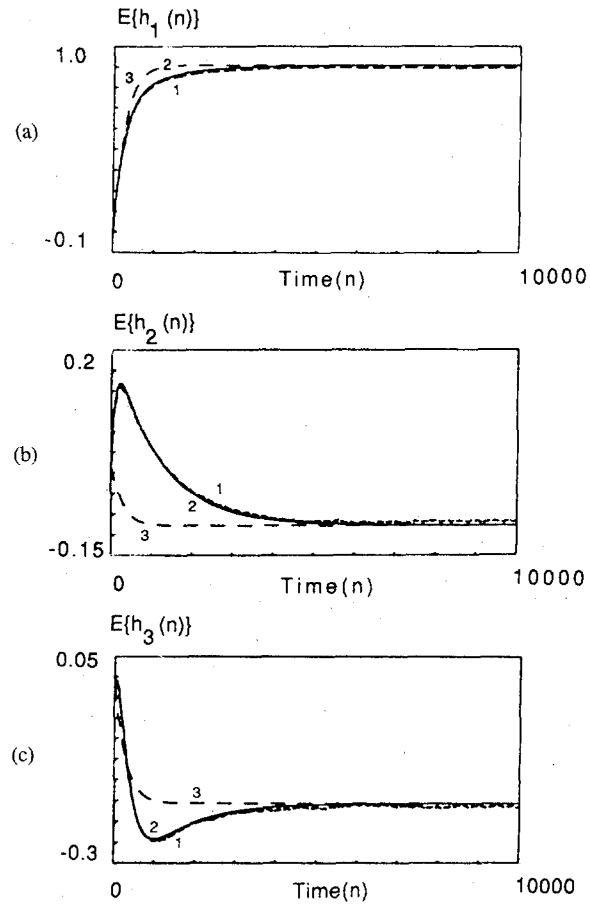


Fig. 1. Comparison of the theoretical and simulation results for the mean behavior of the filter coefficients: (1) simulation results, (2) mean values predicted by (15), and (3) mean values predicted by white signal assumption.

type [10], [12], we will also assume that the input pairs $\{X(n), d(n)\}$ are mutually uncorrelated for different values of n. (Note that we are not restricting the nature of the input autocorrelation matrix R_{XX}.) $H(n)$ is then uncorrelated with $\{X(n), d(n)\}$, since $H(n)$ depends only on inputs at time $n - 1$ and before. This assumption is not true in general, but it produces results that are very close to the true behavior of the system if μ is chosen to be small [10]. The analyses that do not use this assumption can be found in [3] and [7] for least mean square (LMS) adaptive filters.

The rest of the paper is organized as follows. In Section II, we present the convergence analysis of the sign algorithm assuming jointly Gaussian input signals. In this section, we also derive expressions for the second moment of the coefficient vector and also the steady-state error power. A simulation example that verifies the expressions derived is presented in Section III. Concluding remarks are made in Section IV.

II. Convergence Analysis

We start by evaluating the statistical expectation of both sides of (1). This gives

$$E\{H(n+1)\} = E\{H(n)\} + \mu E\{X(n)\ \mathrm{sign}\ (e(n))\}. \tag{4}$$

Since $d(n)$ and $X(n)$ are zero mean and jointly Gaussian, the error sequence, conditioned on the coefficient vector $H(n)$, is also zero mean and Gaussian [say, with variance $\sigma^2_{e|H}(n)$]. We can then use the fact [9] that for an arbitrary Borel function $G(\cdot)$, and Gaussian X and e

$$E\{XG(e)\} = E\{Xe\}\ E^{-1}\{e^2\}\ E\{eG(e)\}, \tag{5}$$

and express $E\{X(n)\ \mathrm{sign}\ (e(n))\}$ as

$$\begin{aligned} &E\{X(n)\ \mathrm{sign}\ (e(n))\} \\ &\quad = E\{E[X(n)\ \mathrm{sign}\ (e(n)) | H(n)]\} \\ &\quad = E\left\{\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_{e|H}(n)}\ E\{X(n)\ e(n) | H(n)\}\right\}, \end{aligned} \tag{6}$$

where we have made use of the fact that the mean absolute value of a Gaussian random variable with zero mean and variance σ^2 is $(\sqrt{2/\pi})\ \sigma$. In this analysis we will approximate the standard deviation of the error sequence, conditioned on the coefficient vector by the unconditional standard deviation of the error sequences, i.e.,

$$\sigma_{e|H}(n) \approx \sigma_e(n). \tag{7}$$

This approximation is valid for small values of μ. Now (6) can be rewritten as

$$E\{X(n)\ \mathrm{sign}\ (e(n))\} = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)}\ E\{X(n)\ e(n)\}. \tag{8}$$

Expanding for $e(n)$ using (2),

$$\begin{aligned} E\{X(n)\ e(n)\} &= E\{X(n)\ d(n)\} \\ &\quad - E\{X(n)\ X^T(n)\ H(n)\}. \end{aligned} \tag{9}$$

Under the assumption that $X(n)$ is uncorrelated with $H(n)$, equation (9) becomes

$$E\{X(n)\ e(n)\} = R_{Xd} - R_{XX}E\{H(n)\}, \tag{10}$$

where R_{Xd} is the cross-correlation vector of $X(n)$ and $d(n)$. Substituting (10) and (8) in (4) we get

$$\begin{aligned} &E\{H(n+1)\} \\ &\quad = E\{H(n)\} + \frac{\mu}{\sigma_e(n)}\sqrt{\frac{2}{\pi}}\ (R_{Xd} - R_{XX}E\{H(n)\}). \end{aligned} \tag{11}$$

Let us define the misalignment vector $V(n)$ as

$$V(n) = H(n) - H_{\mathrm{opt}}, \tag{12}$$

where

$$H_{\mathrm{opt}} = R_{XX}^{-1}R_{Xd} \tag{13}$$

is the optimum coefficient vector. Also, let

$$K(n) = E\{V(n)\ V^T(n)\} \tag{14}$$

define a second moment of the misalignment vector. At this point it is convenient to rewrite (11) using the misalignment vector $V(n)$. Using (12), (11) can be equivalently expressed as

$$E\{V(n+1)\} = \left(I - \frac{\mu}{\sigma_e(n)}\sqrt{\frac{2}{\pi}}\ R_{XX}\right) E\{V(n)\}. \tag{15}$$

It is easy to show that the misalignment vector will converge to the zero vector if the convergence constant μ is chosen so that

$$0 < \mu < \sqrt{2\pi}\ \frac{\sigma_e(n)}{\lambda_i}, \tag{16}$$

where $\lambda_i (i = 1, 2, \cdots, N)$ are the eigenvalues of the autocorrelation matrix R_{XX}. A more restrictive, but sufficient and simpler, condition for the convergence (in mean) of the system is

$$0 < \mu < \sqrt{2\pi}\ \frac{\sqrt{\xi_{\min}}}{\mathrm{tr}\ \{R_{XX}\}}, \tag{17}$$

where

$$\xi_{\min} = E\{d^2(n)\} - R_{Xd}^T H_{\mathrm{opt}} \tag{18}$$

is the minimum mean-squared estimation error, and $\mathrm{tr}\{\cdot\}$ denotes the trace of $\{\cdot\}$. An inspection of (16) will immediately show that if the convergence does occur, the root mean-squared estimation error $\sigma_e(n)$ at time n is such that

$$\sigma_e(n) > \frac{\mu\lambda_{\max}}{\sqrt{2\pi}}. \tag{19}$$

We now need an expression for $\sigma_e(n)$ to complete the analysis. For this,

$$\begin{aligned} &\sigma_e^2(n) \\ &\quad = E\{e^2(n)\} \\ &\quad = E\{(d(n) - H^T(n)\ X(n))(d(n) - H^T(n)\ X(n))^T\} \\ &\quad = E\{d^2(n)\} - 2E\{H^T(n)\}\ R_{Xd} + E\{H^T(n)\ X(n) \\ &\qquad \cdot X^T(n)\ H(n)\} \\ &\quad = \xi_{\min} + E\{V^T(n)\ X(n)\ X^T(n)\ V(n)\} \\ &\quad = \xi_{\min} + \mathrm{tr}\ \{R_{XX}K(n)\}, \end{aligned} \tag{20}$$

where we have once again made use of the fact that $H(n)$ and $\{X(n), d(n)\}$ are uncorrelated. To evaluate $\sigma_e(n)$, we also need to know $K(n)$. An expression for $K(n)$ is derived next.

A. Second Moment Behavior of the Weights

From (1)

$$\begin{aligned} E\left\{(H_{\text{opt}} + V(n+1))(H_{\text{opt}} + V(n+1))^T\right\} &= E\left\{(H_{\text{opt}} + V(n))(H_{\text{opt}} + V(n))^T\right\} + \mu^2 R_{XX} \\ &+ \mu E\left\{(H_{\text{opt}} + V(n))\, X^T(n) \operatorname{sign}(e(n))\right\} \\ &+ \mu E\left\{X(n)(H_{\text{opt}} + V(n))^T \operatorname{sign}(e(n))\right\}. \end{aligned} \tag{21}$$

Equation (21) can be simplified as

$$\begin{aligned} K(n+1) = K(n) &+ \mu^2 R_{XX} \\ &+ \mu E\left\{V(n)\, X^T(n) \operatorname{sign}(e(n))\right\} \\ &+ \mu E\left\{X(n)\, V^T(n) \operatorname{sign}(e(n))\right\}. \end{aligned} \tag{22}$$

The expectations in the third and fourth terms of the right-hand side of (22) can be evaluated as follows:

$$\begin{aligned} E&\left\{V(n)\, X^T(n) \operatorname{sign}(e(n))\right\} \\ &= E\left\{E\left[V(n)\, X^T(n) \operatorname{sign}(e(n)) \mid V(n)\right]\right\} \\ &= E\left\{V(n)\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_{e|H}(n)} E\left\{X^T(n)\, e(n) \mid V(n)\right\}\right\} \\ &= E\left\{V(n)\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_{e|H}(n)} \left[R_{Xd}^T - (H_{\text{opt}} + V(n))^T R_{XX}\right]\right\} \\ &= -E\left\{V(n)\, V^T(n) R_{XX}\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_{e|H}(n)}\right\} \end{aligned} \tag{23}$$

$$\approx -\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)} K(n) R_{XX}. \tag{24}$$

Similarly,

$$E\left\{X(n)\, V^T(n) \operatorname{sign}[e(n)]\right\} \approx -\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)} R_{XX} K(n). \tag{25}$$

Substituting (24) and (25) in (22) will yield

$$K(n+1) = K(n)\left[I - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)} R_{XX}\right] + R_{XX}\left[\mu^2 I - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)} K(n)\right]. \tag{26}$$

Note that in going from (23) to (24), we have once again made use of the approximation in (7). Closed-form expressions for the limiting (steady-state) values of the second moment matrix and error power are derived next.

Let $\sigma_e(\infty)$ and $K(\infty)$ denote the limiting values of $\sigma_e(n)$ and $K(n)$, respectively. We will now show that $K(\infty)$ exists for sufficiently small values of μ. Using (20), since $K(n)$ converges, $\sigma_e^2(n)$ also does.

Let Q be an orthonormal matrix that diagonalizes R_{XX}. Pre- and postmultiplying both sides of (26) by Q^T and Q, respectively, we get

$$K'(n+1) = K'(n)\left[I - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)}\Lambda\right] + \Lambda\left[\mu^2 I - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)} K'(n)\right], \tag{27}$$

where

$$K'(n) = Q^T K(n) Q, \tag{28}$$

$$\Lambda = Q^T R_{XX} Q = \operatorname{diag}[\lambda_1, \lambda_2, \cdots, \lambda_N]. \tag{29}$$

We are then able to decompose the matrix equation (27) into the scalar form as

$$K'_{ij}(n+1) = \left(1 - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)}[\lambda_i + \lambda_j]\right) \cdot K'_{ij}(n) + \mu^2 \lambda_i \delta(i-j), \tag{30}$$

where

$$\delta(i-j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases} \tag{31}$$

and K'_{ij} is the (i, j)th element of K'. Note that $K'(n)$ is a symmetric matrix.

Mean square convergence of the weight vector will occur if μ belongs to the range

$$0 < \mu < \sqrt{2\pi}\,\frac{\sigma_e(n)}{\lambda_i + \lambda_j}. \tag{32}$$

As before, a sufficient, but more stringent, condition for mean square convergence is

$$0 < \mu < \sqrt{\frac{\pi}{2}}\,\frac{\sqrt{\xi_{\min}}}{\operatorname{tr}\{R_{XX}\}}. \tag{33}$$

Once again, it is easy to see that if mean-squared convergence does occur,

$$\sigma_e(n) > \sqrt{\frac{2}{\pi}}\,\mu\lambda_{\max} \tag{34}$$

for all n.

To find the steady-state error, we can solve for $\sigma_e(\infty)$ and $K(\infty)$ from (20) and (26) after taking the limits as n

goes to ∞. In the limit, (26) becomes

$$K(\infty) = K(\infty)\left[I - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(\infty)}R_{XX}\right] + R_{XX}\left[\mu^2 I - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(\infty)}K(\infty)\right], \quad (35)$$

which gives

$$K(\infty) = \mu\sqrt{\frac{\pi}{2}}\frac{\sigma_e(\infty)}{2}I. \quad (36)$$

Using (36) in (20) after taking the limits, we get

$$\sigma_e^2(\infty) = \xi_{\min} + \operatorname{tr}\{R_{XX}K(\infty)\} = \xi_{\min} + \alpha\sigma_e(\infty), \quad (37)$$

where

$$\alpha = \frac{\mu}{2}\sqrt{\frac{\pi}{2}}\left(\sum_{i=1}^{N}\lambda_i\right). \quad (38)$$

Solving for $\sigma_e(\infty)$ in (37) and retaining the positive root, we get

$$\sigma_e(\infty) = \frac{\alpha + \sqrt{\alpha^2 + 4\xi_{\min}}}{2}. \quad (39)$$

If μ is very small, the steady-state error power can be approximated as

$$\sigma_e^2(\infty) \approx \xi_{\min} + \frac{\mu}{2}\sqrt{\frac{\pi}{2}}\left(\sum_{i=1}^{N}\lambda_i\right)\sqrt{\xi_{\min}}. \quad (40)$$

Equation (40) is obtained by squaring (39) and ignoring all terms containing μ^2.

Remarks:

1) For very small values of μ, we can approximate $E\{H^T(n)X(n)X^T(n)H(n)\}$ in (20) as

$$E\{H^T(n)X(n)X^T(n)H(n)\} \approx E\{H^T(n)\}R_{XX}E\{H(n)\}. \quad (41)$$

In (41), the basic assumption is that the variations in the weight vector are much smaller than the mean value of the weight vector itself. Under this assumption, the expression for the error power (20) can be approximated by

$$\sigma_e^2(n) \approx \xi_{\min} + E\{V^T(n)\}R_{XX}E\{V(n)\}. \quad (42)$$

If we substitute (42) in (15) and also assume that the input signals to the adaptive filter are white, the expression derived for the convergence of the weight vector (15) is the same as that given by [6].

2) For the LMS algorithm [12], $E\{V(n+1)\}$ and $\sigma_e^2(\infty)$ are given by [13]

$$E\{V(n+1)\} = (I - \mu R_{XX})E\{V(n)\}, \quad (43)$$

and

$$\sigma_e^2(\infty) = \xi_{\min} + \frac{\mu}{2}\xi_{\min}\left(\sum_{i=1}^{N}\lambda_i\right). \quad (44)$$

To have the same steady-state mean-squared error for both the LMS and sign algorithms, we must choose the convergence constant μ_S for the sign algorithm to be

$$\mu_S = \mu_L\sqrt{\frac{2}{\pi}}\sqrt{\xi_{\min}}, \quad (45)$$

where μ_L is the convergence constant for the LMS algorithm. For $\mu = \mu_S = \mu_L\sqrt{2/\pi}\sqrt{\xi_{\min}}$, (15) becomes

$$E\{V(n+1)\} = \left(I - \frac{2}{\pi}\frac{\sqrt{\xi_{\min}}}{\sigma_e(n)}\mu_L R_{XX}\right)E\{V(n)\}. \quad (46)$$

Since $\sqrt{\xi_{\min}} \le \sigma_e(n)$, we have that $(2/\pi)(\sqrt{\xi_{\min}}/\sigma_e(n)) < 1$, implying that the sign algorithm will always converge slower than the LMS algorithm when the steady-state errors are the same. This result agrees with those in [4] and [6]. However, for the same value of μ, it is possible that the sign algorithm can converge faster than the LMS algorithm [14], [15]. Also, if μ is very small, the difference in steady-state errors between the two methods may not be very large.

III. A Simulation Example

For verifying the expressions derived in Section II, we chose a third-order predictor for a third-order autoregressive signal described by

$$x(n) = 0.9x(n-1) - 0.1x(n-2) - 0.2x(n-3) + \xi(n), \quad (47)$$

where $\xi(n)$ is a white, zero mean Gaussian signal with variance such that the variance of $x(n)$ is 1. Note that the eigenvalue ratio of the signal is 16.32. The results presented are comparisons of the theoretical curves with ensemble averages of 900 independent simulations using 10 000 data samples each. The convergence constant was chosen to be 0.005. Fig. 1 shows plots of the mean values of the weights. For comparison, we have also plotted the theoretical curves obtained using the white input signal assumption [6]. We can see that the results of Section II agree with those of the simulations fairly closely, while the white noise assumption gives misleading results in this case. The diagonal elements of the $K(n)$ matrix and their steady-state values obtained using our theoretical model [(36) and (39)] and also simulation experiments are compared in Fig. 2 and Table I, respectively. Here, the steady-state values for the simulation are obtained as the mean values of the last 5000 samples of the corresponding curves. Once again, the theoretical and empirical results show very close match. (Note that in Fig. 2 both curves are always positive. We used negative values for the vertical axis only for ease of displaying the results.)

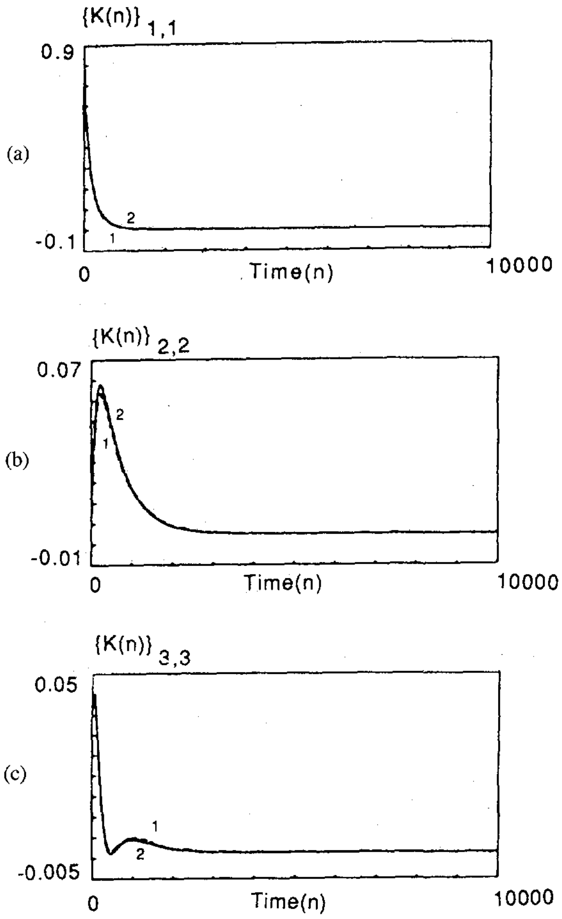


Fig. 2. Comparison of the theoretical and simulation results for the mean-squared behavior of the filter coefficients: (1) simulation curve and (2) theoretical curve.

TABLE I
COMPARISON OF THE THEORETICAL AND SIMULATION RESULTS FOR THE STEADY-STATE MEAN-SQUARED VALUES OF THE MISALIGNMENT VECTOR

	Theoretical Result	Simulation Result
$\{K(\infty)\}_{1,1}$	0.002006	0.002110
$\{K(\infty)\}_{2,2}$	0.002006	0.002023
$\{K(\infty)\}_{3,3}$	0.002006	0.001998

IV. Conclusions

In this paper, we presented convergence analysis for the sign algorithm when the signals involved are Gaussian. Expressions for the expected value of the weight vector, its second moments, and also the steady-state error power were derived. These derivations were done without making any assumptions on the nature of the autocorrelation matrix of the input vector and, because of this, the expressions developed show a much better match with simulation results than previously published results [4], [6] which assumed that the input signals were white.

References

[1] N. J. Bershad, "Comments on 'Comparison of convergence of two algorithms for adaptive FIR digital filters,'" *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-33, pp. 1604–1606, Dec. 1985.

[2] ——, "On the optimum data nonlinearity in LMS adaptation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 69–76, Feb. 1986.

[3] R. R. Bitmead and B. D. O. Anderson, "Performance of adaptive estimation algorithms in dependent random environment," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 788–794, Aug. 1980.

[4] T. A. C. M. Claasen and W. F. G. Mecklenbrauker, "Comparison of the convergence of two algorithms for adaptive FIR digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 670–678, June 1981.

[5] ——, "Authors' reply to 'Comments on "Comparison of the convergence of two algorithms for adaptive FIR digital filters,"'", *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 202–203, Feb. 1986.

[6] D. L. Duttweiler, "Adaptive filter performance with nonlinearities in the correlation multiplier," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 578–586, Aug. 1982.

[7] D. C. Farden, "Stochastic approximation with correlated data," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 105–113, Jan. 1981.

[8] Y. Iiguni, H. Sakai, and H. Tokumaru, "Convergence properties of simplified gradient adaptive lattice algorithms," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-33, pp. 1427–1434, Dec. 1985.

[9] T. Koh and E. J. Powers, "Efficient methods to estimate correlation functions of Gaussian processes and their performance analysis," submitted for publication, 1985.

[10] J. E. Mazo, "On the independence theory of equalizer convergence," *Bell Syst. Tech. J.*, vol. 58, no. 5, pp. 963–993, May–June, 1979.

[11] N. A. M. Verhoeckx and T. A. C. M. Claasen, "Some considerations on the design of adaptive digital filters equipped with the sign algorithm," *IEEE Trans. Commun.*, vol. COM-32, pp. 258–266, Mar. 1984.

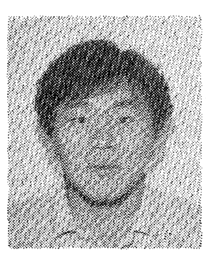
[12] B. Widrow *et al.*, "Adaptive noise cancelling: Principles and applications," *Proc. IEEE*, vol. 63, pp. 1692–1716, Dec. 1975.

[13] ——, "Stationary and nonstationary learning characteristics of the LMS adaptive filter," *Proc. IEEE*, vol. 64, pp. 1151–1162, Aug. 1976.

[14] D. H. Youn, V. J. Mathews, and S. H. Cho, "An efficient lattice predictor algorithm for instantaneous frequency estimation," *Signal Processing*, vol. 10, pp. 75–81, 1986.

[15] ——, "An efficient algorithm for lattice filter/predictor," in *Proc. Int. Conf. Acoust., Speech, Signal Processing*, Tampa, FL, Mar. 1985, pp. 1181–1184.

V. John Mathews (S'82–M'85), for a photograph and biography, see p. 349 of the March 1987 issue of this Transactions.



Sung Ho Cho was born in Seoul, Korea, on February 21, 1960. He received the B.E. degree in electronic engineering from Hanyang University, Seoul, Korea, in 1982, and the M.S. degree in electrical and computer engineering from the University of Iowa, Iowa City, in 1984.

He is currently a Teaching Assistant at the University of Utah, Salt Lake City, working toward the Ph.D. degree. His research interests include adaptive filtering, spectrum estimation, and stochastic processes.