Implications of a cosmological constant varying as R^{-2}

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We advocate the possibility that the (effective) cosmological constant Λ varies in time as R^{-2}, R being the scale factor of our expanding Universe. This behavior can be obtained under some simple and general assumptions in conformity with quantum cosmology. After pointing out several advantages worth noticing, we show that such a time-varying Λ leads to no conflict with existing observations. However, it does change the predictions of the standard cosmology in the matter-dominated epoch and alleviates some problems in reconciling observations with the inflationary scenario. In particular, this "medium" time variation of Λ leads to creation of matter with a rate at present which is comparable to that in the steady-state cosmology.

Einstein's theory of gravity contains two parameters: Newton's gravitational constant G and the cosmological constant.[1] Normally they are considered as fundamental constants. A possible time variation of G has been suggested by Dirac[2] and extensively discussed in the literature.[3,4] What about a possible cosmological time variation of the cosmological constant?

According to modern quantum field theory, the energy-momentum tensor of the vacuum is generally nonvanishing and of the form[5] $\langle T_{\mu\nu}\rangle = -\langle\rho\rangle g_{\mu\nu}$. Therefore, the observed or effective cosmological constant Λ receives an extra contribution from $\langle T_{\mu\nu}\rangle$:

$$\Lambda=\lambda+8\pi G\langle\rho\rangle\ , \qquad (1)$$

where λ is the "bare" cosmological constant. The "natural" naive expectation[6] for $\langle\rho\rangle$ is that $\langle\rho\rangle \approx M_{\rm Pl}^4 \approx 2\times10^{71}$ GeV^4, where $M_{\rm Pl}$ is the Planck mass. However, a crude experimental upper bound on the present value Λ_0 is provided by measurements of the Hubble constant H_0: $|\Lambda_0| \le H_0^2$ and numerically reads[6,7]

$$|\Lambda_0|/8\pi G \le 10^{-29}\ {\rm g/cm^3} \simeq 10^{-47}\ {\rm GeV^4}\ . \qquad (2)$$

(The subscript 0 denotes the present value of a physical quantity; in this paper the variation of G, if acceptable, is unimportant and ignored.) How to understand the amazingly delicate cancellation between the two terms in Eq. (1) to achieve the observed upper bound (2) is the essence of the cosmological-constant problem.[6,8]

Normally a "natural" way to solve the problem is trying to invent a mechanism which simply renders Λ exactly or almost exactly vanishing. Several recent attempts in quantum cosmology are along this line.[9–12] (For attempts in other approaches, see the reviews Refs. 6 and 8.) However, if Λ is a dynamical variable or vacuum parameter, similar to the θ-vacuum parameter in axion theory, it seems more "natural" that in an expanding universe Λ relaxes to the present small value by some relaxation mechanism.[13] Such a mechanism may be provided by a time-varying "vacuum" with a rolling scalar field.[14,15] Whether an effective Λ varying on the cosmological time scale is phenomenologically acceptable has been extensively discussed in the literature.[16–19] Among various possibilities, we would like to advocate in this paper a particular R-dependent behavior, i.e., $\Lambda \propto R^{-2}$. Theoretically it can be obtained from some simple and general assumptions in line with quantum cosmology and has several distinct features. Observationally it is not in conflict with present data and may alleviate some problems in reconciling data with the inflationary scenario.

First we point out that one may argue in favor of the R^{-2} behavior of Λ from some very general arguments in line with quantum cosmology. From dimensional consideration and in the spirit of quantum cosmology, one can always write Λ as $M_{\rm Pl}^4$ times a dimensionless product of quantities. For an ansatz for the evoluting behavior of Λ, as in the common practice in quantum cosmology, it is more convenient to use the scale factor R instead of the age t of the Universe. Supposing that no other parameters are relevant here, the natural ansatz is that Λ varies according to a power law in R. Therefore, we can write

$$\Lambda(R) \propto M_{\rm Pl}^4 (r_{\rm Pl}/R)^n \quad (\text{with } \hbar=c=1)\ . \qquad (3)$$

Let us argue that $n=2$ is a preferred choice. First of all, let us try to recover the Planck constant $\hbar$. It is easy to verify that $n<2$ (or $n>2$) will lead to a negative (or positive) power of $\hbar$ appearing explicitly in the right-hand side of Eq. (3). One would feel very uncomfortable with such an $\hbar$ dependence Λ in the classical Einstein equation for cosmology much later than the Planck time. But $n=2$ is just right to survive the semiclassical limit $\hbar\to0$. With $n=2$ in Eq. (3), both $\hbar$ and G disappear and we have (with $c=1$ only)

$$\Lambda(R)=\gamma R^{-2}\ , \qquad (4)$$

where γ is a pure number of the order 1 which should be calculable in a model for the time variation of Λ. To recapitulate the spirit of the above argument, let us point out that the argument is very similar to the following one in elementary quantum mechanics for the ground-state energy E of the hydrogen atom. From dimensional reasons one can write E as the rest mass mc^2 of the electron times a function of the dimensionless fine-structure constant, $\alpha \equiv e^2/\hbar c$; then remembering that this is a nonrelativistic problem, the velocity of light c should not appear at all, and this requires the function of α is a power function with power $n=2$.

Second, if we estimate the present R_0 by $R_0 = ct_0$ in terms of the present age t_0, then $n \leq 1$ would lead to a too big Λ_0 violating the upper bound (2), but $n \geq 3$ would lead to a too small Λ_0 compared to the bound (2); the latter case is uninteresting to us in this paper, in the sense that there would be no essential difference from the case with a constant $\Lambda=0$. However, numerically $n=2$ again is just right to give a Λ_0 around the upper bound (2).

Incidentally we note that Strominger's semiclassical Lorentzian analysis[20] of quantum cosmological tunneling leads to exactly the same formula (4) for the most probable value of Λ at given R. This is not too surprising since his assumptions are consistent with our simple and general arguments. However, we should emphasize that the tunneling in quantum cosmology as analyzed by Strominger[20] does not give us a time-varying Λ since in his work once Λ is measured, the wave function of the Universe is collapsed and Λ is time independent. So we need some other mechanism, yet to be found, to generate a genuinely $R^{-2}(t)$-varying Λ.

Phenomenologically the parameter γ in Eq. (4) is a new cosmological parameter to be determined from observations, replacing the usual Λ. If $\gamma=0$, then this ansatz covers the $\Lambda=0$ case. If $\gamma \neq 0$, we call the ansatz (4) a "medium" time variation in contrast with the $n=1$ and $n=3$ cases. Later we will see that if our Universe is flat, observationally γ should be positive.

According to the ansatz (4), the would-be "most natural" value $\Lambda \approx M_{\rm Pl}^4$ is actually the value of Λ at the Planck time when R was of the order of the Planck length $r_{\rm Pl}$. Theoretically, this ansatz does not directly solve the cosmological-constant problem. But with it the latter is reduced to, or becomes related to, the interesting problem of why our Universe can be so old aged or can have a "radius" R much larger than the Planck scale, a problem very mysterious from the point of view of quantum cosmology. In other words, with the ansatz (4), two well-known problems, the cosmological-constant problem and the problem of the age of the Universe, are reduced to one and the same mystery: Why our Universe could have escaped the death at the Planck time, which seems to be the most natural fate of a baby universe in quantum cosmology?

Obviously, with a time variation such as (4), the values of Λ in the early Universe could be several tens of orders of magnitude bigger than the present Λ_0. It might be large enough to drive various symmetry breakings (except inflation) which we believe have occurred in the early Universe. Note that an always vanishing Λ would not be able to do so. Furthermore, the fine-tuning problem for the value of Λ before symmetry breakings now gets much alleviated by several tens of orders of magnitude, if not completely solved. Later we will see the famous $\Omega_0=1$ problem also getting alleviated.

Nevertheless, would the much bigger value of Λ at early times have disturbed the well-known predictions such as the observed helium abundance from the standard cosmology? To analyze this problem, we need to determine the ratio of Λ to the energy density ρ of matter or radiation at early times by solving the Einstein equations

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} - \Lambda g^{\mu\nu} = -8\pi G T^{\mu\nu} . \tag{5}$$

(By definition, the effective cosmological constant Λ is the coefficient of the additional $g^{\mu\nu}$ term in the Einstein equations. So we start with the latter but not from an action principle.) With the Robertson-Walker metric and the ansatz (4) for Λ, one has

$$R\ddot{R} + 2\dot{R}^2 + 2k - \gamma = 4\pi G(\rho - p) R^2 , \tag{6}$$

$$\dot{R}^2 + k - \gamma/3 = (8\pi G/3)\rho R^2 . \tag{7}$$

Note that now the associated (total) energy-conservation law reads

$$\frac{d}{dR}(\rho R^3) + 3pR^3 = -\frac{1}{8\pi G}\left[\frac{d}{dR}(\Lambda R^3) - 3\Lambda R^2\right] \equiv \frac{\gamma}{4\pi G} . \tag{8}$$

One can view $-(8\pi G)^{-1}\Lambda g^{\mu\nu}$ as the effective vacuum energy-momentum tensor with $\rho^{\rm vac} = -p^{\rm vac} = \Lambda/8\pi G$. When Λ is time varying, what is conserved is the total energy-momentum tensor, instead of the usual $T^{\mu\nu}$ of matter and radiation with energy density ρ and pressure p. In Eqs. (6) and (7), k is the space curvature.

Given the equation of state, it is a simple matter to solve Eq. (8).

(1) For the radiation-dominated epoch ($p=\rho/3$),

$$\rho^{\rm rad} = A_1 R^{-4} + \gamma R^{-2}/8\pi G . \tag{9}$$

(2) For the matter-dominated epoch ($p=0$),

$$\rho^{\rm matt} = A_2 R^{-3} + 2\gamma R^{-2}/8\pi G . \tag{10}$$

A_1 and A_2 have to be positive, since at early times it is the first term rather than the second term that dominates, given that at present the second term is at most of the order of the first term, in accordance to the bound (1). Substituting Eq. (9) or (10) into Eqs. (7) and (6), one has

$$\dot{R}^2 + k^{\rm eff} = (8\pi G/3)\rho^{\rm cons} R^2 , \tag{11}$$

$$R\ddot{R} + 2\dot{R}^2 + 2k^{\rm eff} = 4\pi G \rho^{\rm cons} R^2 , \tag{12}$$

where $k^{\rm eff} = k - 2\gamma/3$, $k-\gamma$ for the radiation- and matter-dominated epoch, respectively, $\rho^{\rm cons}$ is the conserving part of ρ, i.e., the first term in either Eq. (9) or (10), which is of the same R dependence as in the standard model. The second term in Eq. (9) or (10) is nonconserving in time, resulting from the time variation of Λ. Therefore, compared to the standard cosmology (with $\Lambda=0$), our time-varying Λ gives rise to two effects. (i) It shifts the three-space curvature parameter k by a constant of order of 1 in the Einstein equations. (ii) It leads to creation of matter or radiation.

It is well known in standard cosmology[3] that the three-space curvature parameter k can be neglected in the radiation-dominated epoch because of the dominance of $\rho^{\rm rad}$. So in our model, by the same argument, $k^{\rm eff}$ can be ignored too and the Einstein equations become exactly the same as in the standard model. Thus at early times, especially at the instant when the proton-neutron ratio just got frozen, the usual conserving energy density was much bigger than the effective vacuum energy density. We conclude that the latter can be neglected and could not have disturbed any predictions of the standard model in the radiation-dominated epoch, including those for the helium abundance.

However, our time-varying Λ will change the predictions of the standard cosmology for the matter-dominated epoch, especially those about the fate of our Universe: Our Einstein equations (11) and (12) are of the same form as in the standard model, but with the space curvature k shifted by the amount of $-\gamma$. As is well known, the standard model with $\Lambda=0$ predicts that a flat ($k=0$) or open ($k=-1$) universe will expand forever, but a closed universe ($k=1$) will collapse from some moment on in the future. With our ansatz (4) these predictions about the fate of the Universe have to change accordingly, depending on the value of γ: If $\gamma \geq k$ (or $\gamma < k$), then the Universe will expand forever (or will collapse in the future). In particular, if $\gamma > 1$, even a closed universe (with $k=1$) will expand forever; and the $k=0$ case is no longer critical for the collapse of our Universe.

For the present value of the deceleration parameter $q=-R\ddot{R}/\dot{R}^2$, one has

$$(k-\gamma)/R_0^2=(2q_0-1)H_0^2 , \qquad (13)$$

$$2q_0=\rho_0^{\text{cons}}/\rho_0^{\text{cr}} , \qquad (14)$$

where $H=\dot{R}/R$ is the Hubble constant; $\rho^{\text{cr}}=3H^2/8\pi G$ is the critical density. The sign of $(2q_0-1)$ is seen to be determined by the sign of $k-\gamma$. If $\gamma>0$ or $\gamma<0$, a flat universe $k=0$ favored by the inflationary scenario[21] now requires $0<q_0<\frac{1}{2}$ or $q_0>\frac{1}{2}$ instead of the stringent requirement $q_0=\frac{1}{2}$ in the standard model. Related to this is the alleviation of the $\Omega_0=1$ problem for the ratio $\Omega=\rho^{\text{matt}}/\rho^{\text{cr}}$: In the standard model one has $\Omega_0=2q_0=1$ if $k=0$; but observationally one has $\Omega_0=0.2\mp 0.1$ (Ref. 22). (See also Ref. 23.) From our Eq. (11) with $k=0$,

$$\rho^{\text{matt}}+\rho^{\text{vac}}=\rho^{\text{cr}} . \qquad (15)$$

Since observationally $\rho_0^{\text{matt}}<\rho_0^{\text{cr}}$, we know $\rho_0^{\text{vac}}>0$ or $\gamma>0$. It follows from Eq. (10) that $\rho^{\text{matt}}=\rho^{\text{cons}}+2\rho^{\text{vac}}$. Thus, Ω_0 should satisfy

$$\tfrac{2}{3}<\Omega_0<1 \quad \text{and} \quad 2q_0=\Omega_0-2\rho_0^{\text{vac}}/\rho_0^{\text{cr}}<\Omega_0 . \qquad (16)$$

The value of Ω_0 depends on the ratio $\rho_0^{\text{vac}}/\rho_0^{\text{matt}}$. The first inequality is still not satisfied by observation, but the problem becomes less serious. We note that Ref. 23 provides a value of Ω_0 which is not far from the lower limit $\frac{2}{3}$. Using Λ_0 to provide $1-\Omega_0$ has been proposed in the literature;[24] here the inequalities in (16) are the predictions from our particular ansatz (4).

In our model, the expression of the age t_0 in terms of H_0 and q_0,

$$t_0=\frac{1}{H_0}\int_0^{R/R_0}\left[1-2q_0+\frac{2q_0}{x}\right]^{-1/2}dx , \qquad (17)$$

is exactly the same as in the standard model,[4] and so are the inequalities

$$\tfrac{2}{3}H_0^{-1}<t_0<H_0^{-1} . \qquad (18)$$

More interestingly, our time-varying Λ predicts creation of matter at present with an rate of creation per unit volume given by

$$\left.\frac{d(\rho^{\text{matt}}R^3)}{R^3dt}\right|_0=(4\pi G)^{-1}\Lambda_0H_0=2\rho_0^{\text{vac}}H_0 \qquad (19)$$

from Eq. (8). It is well known[25] that the creation rate in the steady-state cosmology is $3\rho_0^{\text{matt}}H_0$. It is amusing to note that if ρ_0^{vac} is of the order of ρ_0^{matt}, then the present rate of creation in our model is of the same order as in the steady-state cosmology, which is about 10^{-41} $\text{g cm}^{-3}\text{s}^{-1}$ and is certainly inaccessible to test in the laboratory. However, the astrophysical consequences would crucially depend on the hypothesis about the form of the created matter or about the mechanism which converts energy from vacuum to matter or radiation. For example, it is unclear whether the creation takes place uniformly in space or around local inhomogeneities of density. Some hypotheses may be subject to test and have been ruled out by existing data. For example, if it is baryon-antibaryon pairs which are created uniformly in space with the above-cited rate, then the cascade processes

$$p+\bar{p}\to\pi^++\pi^-+\pi^0 \quad \text{and} \quad \pi^0\to 2\gamma$$

would provide an isotropic cosmic γ-ray background,[26] whose intensity is much higher than what has been observed.[27] However, it is very hard to generally rule out the possibility of creation of matter or radiation with an average rate given above. Recall that[25] the fatal blows received by the steady-state model in the mid 1960s were from the realization that the observed helium abundance must be of a primordial origin and from the discovery of the universal microwave background, which conflicted with other main aspects of the model but not with the creation of matter. With our time-varying Λ, it becomes justifiable to revive the interests in creation of matter or radiation, which is perhaps one of the most interesting open problems in physics. Also we note that in our model the ratio of the nonconserving part of $\rho^{\text{matt}}/\rho^{\text{rad}}$ to the conserving part is negligibly small in the early Universe.

Finally, we observe that compared with the literature[16–19] on the phenomenological cosmological time variation of Λ, all the postulated time dependences of Λ in them are different from the ansatz (4). For example, in Ref. 16 the ratio ρ^{vac}/ρ is postulated to be constant in time. But in our model the same ratio is negligibly small in the early Universe. The possibility of creation of matter or radiation is also discussed in the same reference, but Eq. (19) is our particular result from the ansatz (4).

In summary, we have advocated the speculative possibility that the effective cosmological constant Λ is time varying according to an inverse-square law in the scale factor. We have shown that it actually can be obtained from some simple and general arguments in the spirit of quantum cosmology. In the early Universe Λ could be several tens of orders bigger than the present Λ_0, which results in a few advantages in relation to the cosmological-constant problem. But Λ was still not big enough to have disturbed the physics in the radiation-dominated epoch in the standard cosmology. In the matter-dominated epoch, such a time-varying Λ shifts the three-space curvature parameter k by a constant, which changes the predictions of the standard cosmology in a way that alleviates some problems in reconciling observations with the inflationary scenario. Finally, such a time variation of Λ leads to creation of matter with an rate at present which is comparable to that in the steady-state cosmology, and thus justifies the revival of interests in the problem of creation of matter and/or radiation, which might eventually provide an experimental test for the suggested time variation (4) of Λ. Theoretically, it would be interesting to find a model (or a mechanism) for particle production which gives rise to creation of matter or radiation in consistency with an R^{-2}- varying Λ.

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