New wave-operator identity applied to the study of persistent currents in 1D

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We show that a large class of backward-scattering matrix elements involving $\Delta k \sim \pm 2k_F$ vanish for fermions interacting with two-body attractive forces in one dimension. (These same matrix elements are finite for noninteracting particles and infinite for particles interacting with two-body repulsive forces.) Our results demonstrate the possibility of persistent currents in one dimension at $T = 0$, and are a strong indication of a metal-to-insulator transition at $T = 0$ for repulsive forces. They are obtained by use of a convenient representation of the wave operator in terms of density-fluctuation operators.

INTRODUCTION

It is usual to express the density-fluctuation operators[1] $\rho(p)$ as bilinear forms of the wave operator $\Psi(x)$. We have recently succeeded in inverting the process, expressing the wave operator as an exponential form of the density fluctuation operators, in the special case of Luttinger's soluble model of interacting fermions in one dimension[2]. While certain aspects of our procedure could obviously be used in other applications[3] or even adapted to the case of electrons in three dimensions, we limit the present application to the challenging question, of whether persistent currents (i.e., *supercurrents*) can exist in one dimension despite arbitrary random scattering potentials. The surprising result is that, for sufficiently attractive two-body forces, a current-carrying state at $T = 0$ can have infinite lifetime regardless of the strength of the scattering mechanisms. Therefore, it is proved rigorously that superconductivity can exist, at $T = 0$, in one dimension, despite the well-known lack of long range order. We also find the converse, that for sufficiently repulsive two-body forces, the lifetime of a current-carrying state at $T = 0$ tends to zero, and the system acquires the attributes of an insulator. The nontrivial generalization of these results to finite temperature is the subject of an ongoing, separate, study.

DETAILS OF THE MODEL

We first recall certain aspects of the soluble many-fermion model[2] under scrutiny. It consists of right-going particles (labeled 1) having constant velocity v_0, and left-going particles (labeled 2) with velocity $-v_0$, with interactions characterized by a two-body potential $V(x - x')$ and coupling constant λ, obeying a Hamiltonian:

$$\mathcal{H} = v_0 \sum_k k(n_{1k} - n_{2k}) + (\lambda/L) \sum_p U(p)[\rho_1(p) + \rho_2(p)][\rho_1(-p) + \rho_2(-p)], \quad (1)$$

where p, k refer to wave numbers, $U(p)$ is the Fourier transform of $V(x - x')$ and the various operators are

$$n_{ik} = a_{ik}^{+} a_{ik}, \quad \rho_i(p) = \sum_k a_{i\,k+p}^{+} a_{ik}$$
$$\Psi_i(x) = L^{-1/2} \sum_k a_{ik} e^{ikx} \quad (2)$$

with L = dimension of the space, for purposes of box normalization. The particle-current operator j_{op} takes the form

$$j_{\text{op}} = V_0 \sum_k (n_{1k} - n_{2k}). \quad (3)$$

If, for example, at $T = 0$ we set the coupling constant $\lambda = 0$, we find for the eigenvalue of (3) the value

$$j = v_0 (L/2\pi)(k_{1F} + k_{2F}), \quad (4)$$

for, at $T = 0$, the ground state of the non-interacting particles is described by occupation numbers $n_{1k} = 1$ for $-\infty < k < k_{1F}$ and $n_{1k} = 0$ for $k > k_{1F}$, together with $n_{2k} = 1$ for $k_{2F} < k < +\infty$, and $n_{2k} = 0$ for $k < k_{2F}$. In the ground state, $k_{2F} = -k_{1F}$ and no current flows. In general, however, we can have $k_{1F} \neq -k_{2F}$ and the current eigenvalue j will be nonzero. This conclusion is unaffected by the interactions when $\lambda \neq 0$, for j_{op} commutes with both parts of the Hamiltonian $\mathcal{H}$ separately, and j is therefore a good quantum number until a mechanism for decay of the current is introduced into the Hamiltonian.

Accordingly, we introduce a mechanism allowing electrons to be backward scattered from one branch to the other, in order to test the hypothesis of persistent currents. For definiteness, consider a one-body scattering Hamiltonian $\mathcal{H}'$:

$$\mathcal{H}' = \int dx [W(x) \Psi_2^{+}(x) \Psi_1(x) + \text{h.c.}], \quad (5)$$

where $W(x)$ is a random potential. Because j_{op} does not commute with $\mathcal{H}'$, j is no longer a constant of the motion, and generally decays exponentially:

$$j(t) = j(0) \exp(-t/\tau), \quad (6)$$

where τ = lifetime of the current, is a measure of the strength of the scattering potential $W(x)$ and of the effective density of one-particle states. To probe the latter, we compute the matrix element:

$$M(i \to f) \equiv \langle f | \Psi_2^{+}(x) \Psi_1(x) | i \rangle, \quad (7)$$

in which $|i\rangle$ is the initial, exact, eigenstate of both $\mathcal{H}$ and j_{op}, and $|f\rangle$ is the final eigenstate of these operators. It will be appreciated that if the initial eigenvalue of j_{op} is j, the final eigenvalue is $j - 2v_0$. The matrix elements M enable us to compute the structure of $\mathcal{H}'$ in the Hilbert space of the eigenstates of $\mathcal{H}$. Some of them could be finite, as for noninteracting particles, and then we would have normal decay. But if we find that the matrix elements connecting low-lying states are *all* zero, then there can be no scattering, and the existence of persistent currents is demonstrated. If, on the other hand, some matrix elements are infinite, then we may conclude either that the lifetime τ of a current is zero, or, more accurately, that the effects of $\mathcal{H}'$ are too profound to be taken into account by perturbation theory (for it causes an insulator phase to replace the metallic phase, and this should presumably be taken into account before the effects of the two-body forces.) In the following, we shall find all these possibilities to be realizable, depending on the sign and magnitude of $U(0) = \int dx\, V(x)$.

PRELIMINARY COMPUTATIONS

We start by recalling the unitary transformation S which renders $\exp(iS)\mathcal{H}\exp(-iS)$ diagonal. It has the form[2]

$$S = (2\pi i/L)\sum_{\text{all } p} p^{-1}\varphi(p)\,\rho_1(p)\rho_2(-p). \tag{8}$$

We recall that, owing to the peculiarities of a filled Fermi sea, the ρ's do not all commute, but obey the commutation relations

$$[\rho_i(p), \rho_j(-p)] = \epsilon_i \delta_{ij}\, pL/2\pi \tag{9}$$

in which $\epsilon_1 = -1$ and $\epsilon_2 = +1$. The correct value of φ to diagonalize $\mathcal{H}$ is found to be

$$\varphi(p) = -\tfrac{1}{4}\ln[1 + 2\lambda u(p)], \tag{10}$$

where $u(p) \equiv U(p)/\pi v_0$, so that

$$e^{iS}\mathcal{H}e^{-iS} = (2\pi v_0/L)\sum_{p>0}[1 + 2\lambda u(p)]^{1/2} \times [\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)] + W_1. \tag{11}$$

Making use of the commutation relations (9), one sees that $\mathcal{H}$ is reduced to a set of noninteracting harmonic oscillators having characteristic energy $E(p) = v_0 p\,[1 + 2\lambda u(p)]^{1/2}$. W_1 is the vacuum renormalization energy,

$$W_1 = \frac{Lv_0}{2\pi}\int_0^\infty dp\, p\{[1 + 2\lambda u(p)]^{1/2} - 1 - \lambda u(p)\}. \tag{12}$$

To obtain the effect of S on $\mathcal{H}'$, we have found the following new operator identities to be extremely convenient:

$$\Psi_1(x) \Longleftrightarrow \frac{e^{ik_{1F}x}}{L^{1/2}} \exp\left(\frac{-2\pi}{L}\sum_{p>0} p^{-1}\rho_1(p)e^{-ipx}\right) \times \exp\left(\frac{2\pi}{L}\sum_{p>0} p^{-1}\rho_1(-p)e^{ipx}\right) \tag{13a}$$

and

$$\Psi_2(x) \Longleftrightarrow \frac{e^{ik_{2F}x}}{L^{1/2}} \exp\left[i\pi\int dx'\,\Psi_1^\dagger(x')\Psi_1(x')\right] \times \exp\left(\frac{-2\pi}{L}\sum_{p>0} p^{-1}\rho_2(-p)e^{ipx}\right) \times \exp\left(\frac{2\pi}{L}\sum_{p>0} p^{-1}\rho_2(p)e^{-ipx}\right) \tag{13b}$$

with $p = \text{integer} \times 2\pi/L$. The double arrows indicate that the identities hold in a special sense only; supposing $|F,N\rangle$ to be the ground state Fermi sea corresponding to N particles of type 1 and Ω_1 to be an arbitrary function of the $\rho_1(\pm p)$ operators, we have

$$\Psi_1(x)\Omega_1|F, N+1\rangle = \frac{e^{ik_{1F}x}}{L^{1/2}} \exp\left(\frac{-2\pi}{L}\sum_{p>0} p^{-1}\rho_1(p)e^{-ipx}\right) \times \exp\left(\frac{2\pi}{L}\sum_{p>0} p^{-1}\rho_1(-p)e^{ipx}\right)\Omega_1|F,N\rangle \tag{14}$$

and similarly for $\Psi_2(x)$ and for Hermitean conjugate operators $\Psi_i^\dagger(x)$. Since any state in our Hilbert space can be written in the form $\Omega_1|F,N\rangle$, Eqs. (13), and their Hermitean conjugate relations, are operator identities in every practical sense. We note also that they are kinematical identities, independent of the nature of the dynamical interactions or of the magnitude of the coupling constant.

Applying the unitary transformation S to the wave operators,

$$\Psi_i(x) \to \exp(iS)\Psi_i(x)\exp(-iS), \tag{15}$$

we obtain expressions that are readily evaluated using the bosonlike commutation relations, Eq.(9). We cast such expressions into normal ordering [$\rho_1(-p)$ to the right of $\rho_1(+p)$, $\rho_2(p)$ to the right of $\rho_2(-p)$, with $p>0$]. As an example, consider the bilinear form:

$$\begin{aligned}\Psi_1^\dagger(x')\Psi_1(x) \to \frac{1}{L}&\exp[ik_{1F}(x-x')]\exp\left(\frac{2\pi}{L}\sum_{p>0} p^{-1}e^{ip(x'-x)}\right)\\ &\times \exp\left(-\frac{4\pi}{L}\sum_{p>0} p^{-1}[1-\cos p(x-x')]\sinh^2\varphi_p\right)\\ &\times \exp\left(\frac{2\pi}{L}\sum_{p>0} p^{-1}\rho_2(-p)(e^{ipx}-e^{ipx'})\sinh\varphi_p\right)\\ &\times \exp\left(-\frac{2\pi}{L}\sum_{p>0} p^{-1}\rho_2(p)(e^{-ipx}-e^{-ipx'})\cosh\varphi_p\right)\\ &\times \exp\left(-\frac{2\pi}{L}\sum_{p>0} p^{-1}\rho_1(p)(e^{-ipx}-e^{-ipx'})\cosh\varphi_p\right)\\ &\times \exp\left(\frac{2\pi}{L}\sum_{p>0} p^{-1}\rho_1(-p)(e^{ipx}-e^{ipx'})\cosh\varphi_p\right)\end{aligned} \tag{16}$$

This generalizes an earlier result,[4] the calculation of the ground-state expectation value by an entirely different and more laborious technique:

$$\langle F|\Psi_1^\dagger(x')\Psi_1(x)|F\rangle = \frac{1}{L}e^{ik_{1F}(x-x')}\textstyle\sum(x'-x) \times \exp\left(\frac{-4\pi}{L}\sum_{p>0} p^{-1}[1-\cos p(x-x')]\sinh^2\varphi_p\right). \tag{17}$$

Here, and elsewhere, the following identity proves helpful:

$$\textstyle\sum(R) \equiv \exp\left[\frac{2\pi}{L}\sum_{p>0} p^{-1}e^{ipR}\right] = \sum_{p>0}^{\infty} e^{ipR} = [1 - e^{i2\pi R/L}]^{-1} \tag{18}$$

We have denoted this quantity $\sum(R)$ for typographical convenience.

SCATTERING MATRIX ELEMENT

The state of lowest energy carrying a current j is denoted the ground state for current j, and symbolized $|F;j\rangle$. At $T = 0$ one may always assume the initial state to be a state of this type.

We therefore calculate the transition matrix element from an initial state, the ground state of current $j > 0$, to a final state, which can be either the ground state of current $j - 2v_0$, or any excited state of the same current. The total rate of decay out of the initial state into the final states, subject to the requirement of conservation of energy, determines the lifetime τ of the current j. It shall, however, not be necessary for us to calculate τ in any detail in cases when $U(0) \neq 0$, for we shall find $\tau = 0$ when $U(0) > 0$ and $\tau = \infty$ when $U(0) < 0$.

We apply (15) to the right-hand sides of Eqs.(13) and their Hermitean conjugates, to obtain

$$\Psi_2^+(x)\Psi_1(x) \to (e^{i(k_{1F}-k_{2F})x}/L)e^{-\alpha}e^{B_2^+}e^{-B_2}e^{-A_1^+}e^{A_1}, \qquad (19)$$

in which we note that the phase factor $k_{1F} - k_{2F} \sim 2k_F$ corresponds to backward scattering across the Fermi surface, and

$$\alpha = \frac{2\pi}{L}\sum_{p>0} p^{-1}(e^{2\varphi(p)}-1) = \int_0^\infty dp\, p^{-1}(e^{2\varphi(p)}-1)$$
$$= \int_0^\infty dp\, p^{-1}\{[1+2\lambda u(p)]^{-1/2}-1\},$$
$$A_1 = \frac{2\pi}{L}\sum_{p>0} p^{-1}\rho_1(-p)e^{ipx}e^{\varphi(p)}, \qquad (20)$$
$$B_2 = \frac{2\pi}{L}\sum p^{-1}\rho_2(p)e^{-ipx}e^{\varphi(p)}.$$

Therefore the ground state-ground state matrix element, which we write $M(F\to F)$ in an obvious notation, takes on the value:

$$M(F\to F) = (1/L)e^{i(k_{1F}-k_{2F})x}e^{-\alpha}. \qquad (21)$$

The magnitude of $M(F\to F)$ depends on α, and this in turn depends sensitively on $\varphi(0) \equiv \lim_{p\to 0}\varphi(p)$. A two-body interaction which is attractive on the whole has $U(0) < 0$, hence, by Eq. (10), $\varphi(0) > 0$. Such an interaction implies a positive α which is logarithmically divergent $(+\infty)$, and thus a vanishing matrix element. Similarly, a two-body interaction which is repulsive on the whole implies a negatively divergent value of α, hence an infinite matrix element. When both $\varphi(0)$ and $\varphi(\infty)$ are zero, the integral defining α is well-behaved and the matrix element is finite. It should be noted that any two-body interaction $V(x-x')$, the spatial integral of which is nonzero, corresponds to a Fourier transform $U(p\to 0)\neq 0$, hence to a divergent α (negatively or positively divergent according as to whether the interaction is repulsive or attractive). In all such cases the matrix element $M(F\to F)$ is nonanalytic in the coupling constant λ at $\lambda = 0$, despite the persistence of a "sharp Fermi surface" to finite values of λ (cf. discussion in Ref. 4). In the case of potentials which are neither repulsive nor attractive on the whole, $U(p\to 0) = 0$, α is finite and is a continuous function of λ. In such cases only is the decay of an induced current qualitatively the same as for noninteracting particles.

STRUCTURED FINAL STATES

Concerning the divergence in α arising primarily from long wavelengths ($p\to 0$), it is legitimate to wonder whether it is not possible to cancel this divergence through an appropriate linear combination of low-lying excited states. We shall examine two typical compound final states in some detail:

$$\langle Q^{(1)}| \equiv \langle F; j-2v_0|a_{1k_{1F}}a^+_{1(k_{1F}-Q)}, \qquad (22a)$$

$$\langle Q^{(2)}| \equiv \langle F; j-2v_0|a^+_{2k_{2F}}a_{2(k_{2F}-Q)}. \qquad (22b)$$

These have the advantage of being eigenstates of the free-fermion Hamiltonian ($\lambda = 0$). It is of interest to see whether the matrix elements $M(F\to Q^{(i)})$ vanish or diverge under the same conditions as $M(F\to F)$. We start the analysis under the supposition that the forces are essentially attractive $[\varphi(p) > 0]$; a separate analysis will follow in the case of essentially repulsive forces.

After some elementary manipulations based on Eqs. (2) and (13), we obtain

$$M(F\to Q^{(1)}) = (1/L)e^{-\alpha}e^{i(k_{1F}-k_{2F}-Q)x}$$
$$\times(1/L)\int dR\, e^{-iQR}\sum(R)$$
$$\times \exp\Big((2\pi/L)\sum_{p>0}p^{-1}e^{ipR}(e^{\varphi(p)}-1)\Big) \qquad (23a)$$

and

$$M(F\to Q^{(2)}) = e^{2iQx}M(F\to Q^{(1)}). \qquad (23b)$$

It is therefore sufficient to study the behavior of $M(F\to Q^{(1)})$. If $\varphi(0)\neq 0$ the sum in the exponential is logarithmically divergent, and we manipulate it so as to combine it with the divergent expression in α.

Thus

$$\frac{2\pi}{L}\sum_{p>0}p^{-1}e^{ipR}(e^{\varphi(p)}-1)$$
$$= \frac{2\pi}{L}\sum_{p>0}p^{-1}(e^{\varphi(p)}-1) - \frac{2\pi}{L}\sum_{p>0}p^{-1}(1-e^{ipR})(e^{\varphi(p)}-1) \qquad (24)$$

Finally,

$$M(F\to Q^{(1)}) = \frac{1}{L}e^{-\alpha'}e^{i(k_{1F}-k_{2F}-Q)x}\ \frac{1}{L}\int dR\ e^{-iQR}\sum(R)$$
$$\times \exp\Big(-\int_0^\infty dp\, p^{-1}(1-e^{ipR})(e^{\varphi(p)}-1)\Big) \qquad (25)$$
$$\equiv \frac{1}{L}e^{-\alpha'}e^{i(k_{1F}-k_{2F}-Q)x}I(Q), \qquad (25a)$$

where

$$\alpha' \equiv \int_0^\infty dp\, p^{-1}e^{\varphi(p)}(e^{\varphi(p)}-1). \qquad (26)$$

We note that although $\alpha' < \alpha$ [for the case under consideration, viz., $\varphi(p) > 0$], it is nonetheless infinite when $\varphi(0)\neq 0$. It remains only to study the behavior of $I(Q)$, and to verify that all quantities reduce to the appropriate value when the interaction is turned off. For this purpose, it is most convenient to expand the exponential in a power series about $R = 0$, retaining up to quadratic terms. Thus,

$$\int_0^\infty dp\, p^{-1}(1-e^{ipR})(e^{\varphi(p)}-1) = -i\gamma R + \tfrac{1}{2}\delta R^2 + O(R^3), \qquad (27)$$

Where

$$\gamma \equiv \int_0^\infty dp(e^{\varphi(p)}-1), \qquad \delta \equiv \int_0^\infty dp\, p(e^{\varphi(p)}-1), \qquad (28)$$

both positive quantities in the case under consideration. Then,

$$I(Q) = \frac{1}{L}\int dR\ e^{-iQR}\sum(R)e^{i\gamma R-\delta R^2/2}$$
$$= \frac{1}{L}\sum_{p>0}\int_{-\infty}^{\infty} dR\, e^{i(p-Q+\gamma)R}e^{-\delta R^2/2}$$
$$= \frac{1}{L}\sum_{p>0}\Big(\frac{2\pi}{\delta}\Big)^{1/2}e^{-(p+\gamma-Q)^2/2\delta}$$
$$= \frac{1}{(2\pi\delta)^{1/2}}\int_0^\infty dp\, e^{-(p+\gamma-Q)^2/2\delta}. \qquad (29)$$

[In the limit $\lambda\to 0$, both γ and δ vanish and, for any finite positive Q, $I = 1$, and $M(F\to Q^{(1)})$ tends to what is obviously the correct value for free particles.] For any $\lambda > 0, I$ is finite and in the case of $\varphi(0) > 0$ *the matrix element* $M(F \to Q^{(i)})$ *vanishes just as did* $M(F \to F)$.

We now verify that, for repulsive forces, the matrix element diverges. It must first be understood that when $\varphi(p) < 0$, the main contribution to the spatial integral in (25) is from a region near $R = \pm\frac{1}{2}L$. Because of periodic boundary conditions, we have

$$e^{ip(R\pm L/2)} = -e^{ipR}, \quad \sum(R\pm L/2) = (1 + e^{i2\pi R/L})^{-1}. \tag{30}$$

Therefore we cast (25) in the form

$$M(F \to Q^{(1)}) = (1/L)\, e^{-\alpha''}\, e^{i(k_{1F}-k_{2F}-Q)x} J(Q), \tag{25b}$$

where

$$\alpha'' \equiv \int_0^\infty dp\, p^{-1}(e^{\varphi(p)} - 1)(e^{\varphi(p)} + 2)$$

and

$$J(Q) \equiv -\frac{1}{L}\int dR\, e^{-iQR}(1 + e^{i2\pi R/L})^{-1} \times \exp\left(\int_0^\infty dp\, p^{-1}(1 - e^{ipR})(e^{\varphi(p)}-1)\right) \tag{31}$$

We can evaluate $J(Q)$ by the same methods in (27-29), and show it is finite. *Thus, the divergence* [$\alpha'' \to -\infty$ whenever $\varphi(0) < 0$] *is again confirmed.*

RECAPITULATION AND FUTURE APPLICATIONS

We have found a representation for fermion wave operators in a specific one-dimensional model, in terms of density fluctuation operators, which enables the exact evaluation of rather complicated matrix elements. In applying this to the problem of persistent current we observed that in the case of repulsive two-body forces, $U(0) > 0$, the scattering matrix elements due to impurities become infinite, and in the case of attractive two-body forces, $U(0) < 0$ they vanish. It should be noted that neither the ground state energy W_1, Eq. (12), nor the sharpness of the Fermi surface[4], are so singularly dependent on $U(0)$.

Our finding of what is tantamount to superconductivity, for electrons interacting with attractive forces, is in harmony with the well-known results of the BCS theory of superconductivity for three-dimensional systems. Recently, Heeger and his collaborators[5] have found anomalously large conductivity in certain linear chain molecules (TTF-TCNQ) near a finite temperature ~ 58°K, followed by a rapid decrease in conductivity as the temperature is further decreased. For these experimental facts to be explained on the basis of any one-dimensional model requires a calculation at finite temperature, and, possibly also, considerations of the electron spin and the electron-phonon interactions.

Note added in proof: We have now succeeded in evaluating τ at finite temperature and in taking the electron-phonon forces explicitly into account. (Full details have been submitted for publication elsewhere).

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[1]Also often denoted "current operators" as in "current algebras." We reserve that nomenclature for the quantity defined in Eq. (3).

[2]A model introduced by J. M. Luttinger, J. Math. Phys. **4**, 1154 (1963), given an exact solution by D. Mattis and E. Lieb, J. Math. Phys. **6**, 304 (1965), and which is analogous to the Thirring model; see discussion in E. Lieb and D. Mattis, *Mathematical Physics in One Dimension* (Academic, New York, 1966), Chap. 4.

[3]Such as the Tomonoga model; see discussion, references and reprints in Lieb and Mattis, Ref. 2 (1966) and subsequent work by H. Gutfreund and M. Sehick, Phys. Rev. **168**, 418 (1968).

[4]D. Mattis and E. Lieb, J. Math. Phys. **6**, 304 (1965), Sec. V. Eqs. (5.3) *et seq.*

[5]L. B. Coleman, J. J. Cohen, D. J. Sandman, F. G. Yamagashi, A. F. Garito and A. J. Heeger, Solid State Comm. 12, 1125 (1973).