COMPARISON PRINCIPLES FOR PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

by

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ABSTRACT

We show that a large class of stochastic heat equations can be approximated by systems of interacting stochastic differential equations. We use this fact to build moment comparison principles for stochastic heat equations with smooth spatially homogeneous noises (SHE(1)), and then use them to approximate the solution of stochastic heat equations with spatially homogeneous noise with Riesz kernels (SHE(2)), and obtain moment comparison principles for SHE(2) as well.

CONTENTS

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CHAPTER 1

INTRODUCTION

Consider the stochastic heat equation with multiplicative noise

$$\frac{\partial}{\partial t}u_t(x) = -\nu(-\Delta)^{\alpha/2}u_t(x) + \sigma\big(u_t(x)\big)\eta(t,x) \quad t \geq 0,\ x \in \mathbf{R}^d, \tag{1.1}$$

which satisfies the following assumptions:

1. $-\nu(-\Delta)^{\alpha/2}$, $0 < \alpha \leq 2$ is the fractional Laplacian operator, which is also the generator of an isotropic $\alpha-$stable process. $\nu > 0$.

2. $\sigma : \mathbf{R} \to [0, \infty)$ is Lipschitz continuous and $\sigma(0) = 0$.

3. The initial data $u_0(x)$ is a nonnegative, bounded continuous, and nonrandom function.

By "the solution to the above heat equations with noise η", we mean the space–time random field $u_t(x)$ that satisfies the following **mild form**:

$$u_t(x) = \int_{\mathbf{R}^d} p_t(x-y)u_0(y)\,dy + \int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x)\sigma(u_s(y))\eta(ds,dy). \tag{1.2}$$

We refer the reader to [3, 11, 19] for the background knowledge and motivations of the solutions of the mild form. Later in Chapter 2, we would review the basic properties on the space–time stochastic integrals. Here,

$$p_t(x) := (2\pi)^{-d}\int_{\mathbf{R}^d} e^{-iz\cdot x}e^{-\nu t|z|^\alpha}\,dz, \tag{1.3}$$

which is the transition density for the $\alpha-$stable process X_t with characteristic function $E[e^{iz\cdot X_t}] = e^{-\nu t|z|^\alpha}$.

In this dissertation, we consider stochastic heat equations with two different types of noises for $\eta(t,x)$:

1. Stochastic Heat Equation with smooth spatially homogeneous noise (SHE(1)).

Let $\eta(t,x)$ be a Gaussian random field with covariance,

$$\mathrm{Cov}(\eta(t,x),\,\eta(s,y)) = \delta_0(t-s)f(x-y), \tag{1.4}$$

where f is a bounded continuous, symmetric, and positive definite function on $\mathbf{R}^d$. We say that f is the **covariance kernel** for η. In order to guarantee that a unique solution to (SHE(1)) exists, we need (1.9) in [5] to hold; that,

$$\int_{\mathbf{R}^d} \frac{1}{1+2\nu|\xi|^\alpha}\mathcal{F}[f](\xi)\,d\xi\,ds < \infty. \tag{1.5}$$

Here $\mathcal{F}[f]$ is the Fourier transform of f, and $\mathcal{F}[f](\xi)\,d\xi$ is a positive finite measure on $\mathbf{R}^d$. Therefore, for all $\alpha > 0$, (1.5) is satisfied. As a result, (SHE(1)) has a unique solution for all $0 < \alpha \le 2$ due to [5].

2. Stochastic Heat Equation with spatially homogeneous noise with Riesz kernels (SHE(2)).

Let $\eta(t,x) = \eta_\beta(t,x)$ be a family of Gaussian random fields ($0 < \beta < d$) with covariance

$$\mathrm{Cov}(\eta_\beta(t,x),\,\eta_\beta(s,y)) = \delta_0(t-s)f_\beta(x-y), \tag{1.6}$$

where the covariance kernel of η_β is given by $f_\beta(z) := \mathrm{const}\cdot|z|^{-\beta}$, $0<\beta<d$. We note that

$$f_\beta = h_\beta * h_\beta, \tag{1.7}$$

where

$$h_\beta(x) = \mathrm{const}\cdot|x|^{-(d+\beta)/2}. \tag{1.8}$$

Besides (see Appendix A.1 for details),

$$\mathcal{F}[f_\beta](\xi) = \mathrm{const}\cdot|\xi|^{-(d-\beta)}, \quad \mathcal{F}[h_\beta](\xi) = \mathrm{const}\cdot|\xi|^{-(d-\beta)/2}. \tag{1.9}$$

For (SHE(2)), (1.9) in [5] becomes

$$\int_{\mathbf{R}^d} \frac{1}{1+2\nu|\xi|^\alpha}\cdot\frac{1}{|\xi|^{d-\beta}}\,d\xi\,ds < \infty, \tag{1.10}$$

which holds only when $\alpha > \beta$. Therefore, in order to make sure that a unique solution to (SHE(2)) exists, we need to assume that $\alpha > \beta$.

Let us now consider the system of interacting SDEs indexed by $x \in \mathbf{Z}^d$ given by

$$dU_t(x) = \big(\mathscr{L}U_t\big)(x)dt + \sigma\big(U_t(x)\big)dB_t(x) \tag{SDE}$$

with the following assumptions:

1. $\{B_t(x)\}_{x\in\mathbf{Z}^d}$ is a family of correlated Brownian motions such that

$$\mathrm{Cov}\big(B_s(x),\, B_t(y)\big) = \mathrm{const}\cdot(s\wedge t)\cdot\mathcal{R}(|x-y|), \tag{1.11}$$

   where $\mathcal{R} : [0,\infty) \to [0,\infty)$ is a bounded function.

2. $\mathscr{L}$ is defined by

$$\mathscr{L}g(j) := \nu \sum_{i\in\mathbf{Z}^d} p_{j,i}(g(i)-g(j)), \tag{1.12}$$

   for every $g : \mathbf{Z}^d \to \mathbf{R}$, and for all $i,j \in \mathbf{Z}^d$, $p_{i,j} = p_{j,i} = \mu(i-j)$, where μ is a probability measure on $\mathbf{Z}^d$.

3. $\sigma : \mathbf{R} \to [0,\infty)$ is a Lipschitz continuous function, and $\sigma(0) = 0$.

4. The initial condition $U_0(x) \geq 0$ is a bounded, nonrandom function on $\mathbf{Z}^d$.

Later in Chapter 3, we will prove the following comparison principle for (SDE), which is a generalization of Theorem 1 of [2] (the underlying Brownian motions are no longer independent).

Theorem 1. *Consider two solutions U_t and V_t to (SDE) with the same initial conditions $U_0 \equiv V_0$, but with different $\sigma = \sigma_1, \sigma_2$ such that $\sigma_1 \leq \sigma_2$. Then for every $x_1,\, x_2, \cdots, x_n \in \mathbf{Z}^d$, $t_1,\, t_2 \cdots t_n \geq 0$ and $k_1, \cdots, k_n \in [0,\infty)$,*

$$E\big[U_{t_1}(x_1)^{k_1}\cdots U_{t_n}(x_n)^{k_n}\big] \leq E\big[V_{t_1}(x_1)^{k_1}\cdots V_{t_n}(x_n)^{k_n}\big]. \tag{1.13}$$

We may construct a family of SDEs indexed by $(\epsilon\mathbf{Z})^d$, and the solutions are denoted by $U_t^{(\epsilon)}(x)$. Also, let $u_t(x)$ solve (SHE(1)). We have

$$U_t^{(\epsilon)}(\epsilon[x/\epsilon]) \to u_t(x) \text{ in } L^k(P) \tag{1.14}$$

for every $k \geq [2,\infty)$, uniformly in $x \in \mathbf{R}^d$ and $t \in [T_1, T_2]$, where $T_2 > T_1 > 0$ are arbitrarily given. A complete statement of this result will be given as Theorem 23 in Chapter 4. In the proof, we have used ideas from [10], in which a similar approximation for stochastic heat equation with space–time white noise is presented.

We may now combine the above results with Lemma 22 of Chapter 3, in order to deduce the following.

Theorem 2. *Consider two solutions $u_t(x)$ and $v_t(x)$ to (SHE(1)) with the same initial conditions $u_0(x) \equiv v_0(x)$, but with σ_1 and σ_2, respectively, such that $\sigma_1(x) \le \sigma_2(x)$ for all $x \in \mathbf{R}^d$. Then for any $x_1, x_2, \cdots, x_n \in \mathbf{R}^d$, $t_1, t_2 \cdots t_n \ge 0$, and $k_1, \cdots, k_n \in [0, \infty)$,*

$$E\left[u_{t_1}(x_1)^{k_1} \cdots u_{t_m}(x_m)^{k_m}\right] \le E\left[v_{t_1}(x_1)^{k_1} \cdots v_{t_m}(x_m)^{k_m}\right]. \tag{1.15}$$

In Chapter 5, we will establish the following approximation of (SHE(2)) by (SHE(1)):

Theorem 3. *Let $u_t(x)$ be the solution to (SHE(2)), with covariance kernel $f_\beta(z) := C_1 \cdot |z|^{-\beta}$, and $f_\beta(z) = h_\beta * h_\beta(x)$, where $h_\beta(x) := C_2 \cdot |x|^{-(d+\beta)/2}$. Then there exists a sequence $u_t^\delta(x)$ of solutions to (SHE(1)) with covariance kernels f_β^δ, such that*

$$\lim_{\delta \downarrow 0} \sup_{[0,T]} \sup_{x \in \mathbf{R}^d} \|u_t(x) - u_t^\delta(x)\|_k = 0 \tag{1.16}$$

for all $k \in [2, \infty)$, $T > 0$.

Thanks to Theorem 3, Theorem 2 now holds for (SHE(2)) as well, as the following theorem. The proof will be given in Chapter 5.

Theorem 4. *Consider two solutions $u_t(x)$ and $v_t(x)$ to (SHE(2)) with the same initial conditions $u_0(x) \equiv v_0(x)$, but with σ_1 and σ_2, respectively, such that $\sigma_1(x) \le \sigma_2(x)$ for all $x \in \mathbf{R}^d$. Then for any $x_1, x_2, \cdots, x_n \in \mathbf{R}^d$, $t_1, t_2 \cdots t_n \ge 0$, and $k_1, \cdots, k_n \in [0, \infty)$,*

$$E\left[u_{t_1}(x_1)^{k_1} \cdots u_{t_m}(x_m)^{k_m}\right] \le E\left[v_{t_1}(x_1)^{k_1} \cdots v_{t_m}(x_m)^{k_m}\right]. \tag{1.17}$$

To simplify the notations we define throughout this dissertation,

$$\|X\|_k := E\left[|X|^k\right]^{1/k}, \tag{1.18}$$

for every $k > 0$ and random variable X.

CHAPTER 2

PRELIMINARIES ON SPACE–TIME STOCHASTIC INTEGRALS

This chapter prepares some background knowledge of the space–time stochastic integrals when the noise is as defined in SHE(1) or SHE(2). For the construction of spatially homogeneous noise, we refer the reader to [3].

Throughout this chapter, the probability space is denoted by $(\Omega, \mathscr{F}, P)$. Given a spatially homogeneous noise η from either (1.4) or (1.6), we let $\mathscr{F}_t := \sigma\big(\int_{(0,t)\times\mathbf{R}} h(s,y)\,\eta(ds,dy) : h \in \mathscr{C}\big)$, where $\mathscr{C} := \{h : \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} h(s,x)f(x-y)h(s,y)\,dx\,dy\,dt < \infty\}$. A space–time process $\phi(t,x,\omega)$ is called elementary if $\phi(t,x,\omega) = X(\omega)1_{(a,b]}(t)1_A(x)$, where $X \in \mathscr{F}_a$ and A is a compact set on $\mathbf{R}$. We define the predictable σ-field on $\mathbf{R}\times\mathbf{R}^+\times\Omega$ to be $\sigma\big(\phi^{-1}(B) : B \in \mathscr{B}(\mathbf{R}), \phi \in \mathscr{A}\big)$; here $\mathscr{B}(\mathbf{R})$ is the Borel σ-field on $\mathbf{R}$ and $\mathscr{A}$ is the class of all elementary processes. Any space–time process measurable with respect to the predictable σ-field is called a predictable process.

2.1 Integration against spatially homogeneous noise

Consider the norm $\|\cdot\|_{\beta,2}$ for space–time processes (see [11]), defined by

$$\|g\|_{\beta,2} := \sup_{t\geq 0}\sup_{x\in\mathbf{R}} e^{-\beta t}\|g(t,x)\|_2, \qquad \beta > 0, \tag{2.1}$$

and we let $\mathscr{S}$ be the space of all space–time processes $\phi(t,x,\omega) = \sum_{i=1}^{\infty} X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x)$, where each $X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x)$ is an elementary process, $\{(a_i,b_i]\times A_i\}$ is a disjoint family of sets, and $\|\phi\|_{\beta,2} < \infty$. It is not hard to see that $\mathscr{S}$ is closed under addition. We then define $\mathcal{L}^{\beta,2}$ to be completion of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{\beta,2}$. Note that our definition of $\mathcal{L}^{\beta,2}$ is slightly different from the one in [11].

Given h such that

$$\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} h(s,x)f(x-y)h(s,y)\,dx\,dy\,ds < \infty, \tag{2.2}$$

and $\phi(t,x,\omega)=\sum_{i=1}^{n} X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x)\in\mathscr{S}$, we define

$$\int_{[0,t)\times\mathbf{R}^d} h(s,x)\phi(s,x,\omega)\,\eta(dx,ds):=\sum_{i=1}^{n} X_i(\omega)\cdot\int_{[0,\infty)\times\mathbf{R}^d} h(s,x)1_{(a_i,b_i]\cap(0,t]}(s)1_{A_i}(x)\,\eta(dx,ds). \tag{2.3}$$

It is easily seen

$$M_t:=\int_{[0,t)\times\mathbf{R}^d} h(s,x)\phi(s,x,\omega)\,\eta(dx,ds) \tag{2.4}$$

is a martingale. Also, M_t has a continuous modification, because it is a sum of time-change Brownian motions multiplied by random variables. We define

$$X_t^{(i)}:=X_i(\omega)\cdot\int_{[0,\infty)\times\mathbf{R}^d} h(s,x)1_{(a_i,b_i]\cap(0,t]}(s)1_{A_i}(x)\,\eta(dx,ds), \tag{2.5}$$

and for all $1\le i,j\le n$,

$$\langle X^{(i)},X^{(j)}\rangle_t=X_i\cdot X_j\cdot\int_{[0,t)\times\mathbf{R}^d\times\mathbf{R}^d} h(s,x)h(s,y)1_{(a_i,b_i]\cap(a_j,b_j]}(s)1_{A_i}(x)1_{A_i}(y)f(x-y)\,dx\,dy\,ds. \tag{2.6}$$

It turns out that

$$\begin{aligned}\langle M\rangle_t:=&\int_{[0,t)\times\mathbf{R}^d\times\mathbf{R}^d} h(s,x)h(s,y)\Big(\sum_{i=1}^{n} X_i 1_{(a_i,b_i]}(s)1_{A_i}(y)\Big)\\ &\cdot\Big(\sum_{j=1}^{n} X_j 1_{(a_j,b_j]}(s)1_{A_j}(x)\Big)f(x-y)\,dx\,dy\,ds.\end{aligned} \tag{2.7}$$

Now we consider $\phi(t,x,\omega)=\sum_{i=1}^{\infty} X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x)\in\mathscr{S}$, and define

$$\begin{aligned}&\int_{[0,t)\times\mathbf{R}^d} h(s,x)\phi(s,x,\omega)\,\eta(dx,ds)\\ &:=L^2-\lim_{n\to\infty}\sum_{i=1}^{n} X_i(\omega)\cdot\int_{[0,\infty)\times\mathbf{R}^d} h(s,x)1_{(a_i,b_i]\cap(0,t]}(s)1_{A_i}(x)\,\eta(dx,ds).\end{aligned} \tag{2.8}$$

We would like to show (2.8) is well-defined. Let $\phi_n(t,x,\omega):=\sum_{i=1}^{n} X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x)$, we have for any $n<m$,

$$\begin{aligned}&E\Big[\Big|\int_{[0,t)\times\mathbf{R}^d} h(s,x)\phi_n(s,x,\omega)\,\eta(dx,ds)-\int_{[0,t)\times\mathbf{R}^d} h(s,x)\phi_m(s,x,\omega)\,\eta(dx,ds)\Big|^2\Big]\\ &\le\sum_{i=n+1}^{m}\sum_{j=n+1}^{m} E[|X_iX_j|]\cdot\int_{[0,t)\times\mathbf{R}^d\times\mathbf{R}^d} h(s,x)h(s,y)1_{(a_i,b_i]\cap(a_j,b_j]}(s)1_{A_i}(x)1_{A_i}(y)f(x-y)\,dx\,dy\,ds\\ &\le\text{const}\cdot\|\phi\|_{\beta,2}\\ &\times\sum_{i=n+1}^{m}\sum_{j=n+1}^{m}\int_{[0,t)\times\mathbf{R}^d\times\mathbf{R}^d} h(s,x)h(s,y)1_{(a_i,b_i]\cap(a_j,b_j]}(s)1_{A_i}(x)1_{A_i}(y)f(x-y)\,dx\,dy\,ds.\end{aligned} \tag{2.9}$$

It follows that $\{\phi_n\}_n$ is a Cauchy sequence in $L^2(\Omega)$. Therefore, (2.4) is a continuous $L^2(\Omega)$ martingale if $\phi(t,x,\omega) = \sum_{i=1}^{\infty} X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x) \in \mathscr{S}$, with quadratic variation

$$\int_{[0,t)\times\mathbf{R}^d\times\mathbf{R}^d} h(s,x)h(s,y)\Big(\sum_{i=1}^{\infty} X_i 1_{(a_i,b_i]}(s)1_{A_i}(y)\Big)\cdot\Big(\sum_{j=1}^{\infty} X_j 1_{(a_j,b_j]}(s)1_{A_j}(x)\Big) f(x-y)\,dx\,dy\,ds. \tag{2.10}$$

Remark: the above derivations does not imply $\phi_n(t,x,\omega) \to \phi(t,x,\omega)$ in $\mathcal{L}^{\beta,2}$. So this is why we define $\mathcal{L}^{\beta,2}$ as the completion of infinite sums instead of finite sums. Such revision of definition of $\mathcal{L}^{\beta,2}$ helps statements like Proposition 4.6 of [11] work correctly.

Following similar calculations as (2.9), for any $\phi(t,x), \psi(t,x) \in \mathscr{S}$, we have

$$\begin{aligned} &E\Big[\Big|\int_{[0,t)\times\mathbf{R}^d} h\phi\,\eta(dx,ds) - \int_{[0,t)\times\mathbf{R}^d} h\psi\,\eta(dx,ds)\Big|^2\Big] \\ &\le \text{const}\cdot\|\phi-\psi\|_{\beta,2}^2\cdot\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} h(s,x)f(x-y)h(s,y)\,dx\,dy\,ds. \end{aligned} \tag{2.11}$$

Due to (2.11), we could define $\int_{[0,t)\times\mathbf{R}} h\phi\,\xi(dx,ds)$ for all $\phi \in \mathcal{L}^{\beta,2}$. We then follow the argument from Sec. 4.2 of [11] to show the following proposition holds as well.

Proposition 5. *Let h satisfy (2.2) for all $t \ge 0$ and $\phi \in \mathcal{L}^{\beta,2}$ for some $\beta > 0$, then*

$$M_t := \int_{(0,t)\times\mathbf{R}^d} h(s,x)\phi(s,x,\omega)\,\eta(dx,ds) \tag{2.12}$$

defines a continuous $L^2(\Omega)$-martingale with quadratic variation

$$\langle M\rangle_t := \int_{(0,t)\times\mathbf{R}^d\times\mathbf{R}^d} h(s,x)h(s,y)\phi(s,x)\phi(s,y)f(x-y)\,dx\,dy\,ds. \tag{2.13}$$

By application of Burkholder–Davis–Gundy inequality and Minkowski's integral inequality to the previous proposition, we have

Proposition 6 (The BDG inequality for spatially homogeneous noise integral)**.** *Let h satisfy (2.2) for all $t \ge 0$ and $\phi \in \mathcal{L}^{\beta,2}$ for some $\beta > 0$, then for all real numbers $k \ge 2$ and $t > 0$,*

$$\begin{aligned} &\Big\|\int_{(0,t)\times\mathbf{R}^d} h(s,x)\phi(s,x,\omega)\,\eta(dx,ds)\Big\|_k^2 \\ &\le 4k\cdot\int_{(0,t)\times\mathbf{R}^d\times\mathbf{R}^d} h(s,x)h(s,y)\big\|\phi(s,x)\phi(s,y)\big\|_{k/2}\,dx\,dy\,ds. \end{aligned} \tag{2.14}$$

We remark in the end that in [5], the authors have shown that the solution $u_t(x)$ to (SHE(1)) or (SHE(2)) is predictable, and for all $\beta > 0$, $p \ge 2$,

$$\Big(\sup_{t\ge 0}\sup_{x\in\mathbf{R}} e^{-\beta t}E[|u_t(x)|^p]\Big)^{1/p} < \infty. \tag{2.15}$$

2.2 Coupling of space–time noises

As is done in (3.5) of [1], given a space–time white noise ξ defined on $(0,\infty)\times\mathbf{R}^d$, we may define random fields $\eta_\beta(t,x)$ on $(0,\infty)\times\mathbf{R}^d$, such that

$$\eta_\beta(\phi) := \int_{(0,\infty)\times\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}\phi(s,y)h_\beta(y-x)\,dy\Big)\xi(ds,dx), \tag{2.16}$$

for all ϕ such that

$$\widetilde{\phi}(t,x) = \int_{\mathbf{R}^d}\phi(t,y)h_\beta(y-x)\,dy \in L^2((0,\infty)\times\mathbf{R}^d), \tag{2.17}$$

where h_β is defined in (1.8).

It can be shown that η_β satisfies (1.6) by the following calculations:

$$\begin{aligned}
&\mathrm{Cov}(\eta_\beta(\phi),\eta_\beta(\psi))\\
&=\int_{(0,\infty)\times\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}\phi(s,y)h_\beta(y-x)\,dy\Big)\cdot\Big(\int_{\mathbf{R}^d}\psi(s,y)h_\beta(y-x)\,dy\Big)\,ds\,dx\\
&=\int_{(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d}\phi(s,y)\psi(s,z)\Big(\int_{\mathbf{R}^d}h_\beta(y-x)h_\beta(x-z)\,dx\Big)\,ds\,dy\,dz\\
&=\int_{(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d}\phi(s,y)\psi(s,z)\Big(\int_{\mathbf{R}^d}h_\beta(y-z-x)h_\beta(x)\,dx\Big)\,ds\,dy\,dz\\
&=\int_{(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d}\phi(s,y)\psi(s,z)f_\beta(y-z)\,ds\,dy\,dz,
\end{aligned} \tag{2.18}$$

for all ϕ,ψ satisfying (2.17).

Now, given h satisfying (2.2), and $\phi(t,x,\omega)=\sum_{i=1}^n X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x)\in\mathscr{S}$, we have

$$\begin{aligned}
&\int_{(0,\infty)\times\mathbf{R}^d}h(s,x)\phi(s,x,\omega)\eta_\beta(ds,dx)\\
&=\sum_{i=1}^n X_i(\omega)\int_{(0,\infty)\times\mathbf{R}^d}1_{(a_i,b_i]}(s)\Big(\int_{\mathbf{R}^d}h(s,y)1_{A_i}(y)h_\beta(y-x)\,dy\Big)\,\xi(ds,dx)\\
&=\int_{(0,\infty)\times\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}h(s,y)\phi(s,y,\omega)h_\beta(y-x)\,dy\Big)\,\xi(ds,dx).
\end{aligned} \tag{2.19}$$

When $\phi(t,x,\omega)=\sum_{i=1}^\infty X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x)\in\mathscr{S}$, we may define

$$\phi_n(t,x,\omega)=\sum_{i=1}^n X_i(\omega)1_{(a_i,b_i]}(t)1_{A_i}(x), \tag{2.20}$$

where each ϕ_n satisfies (2.19). Taking $L^2(P)$ limits of both sides of (2.19) with ϕ replaced by ϕ_n, we get the same identity for this new ϕ. We may then repeat the same approximation procedure to show for any $\phi\in\mathcal{L}^{\beta,2}$

$$\begin{aligned}
&\int_{(0,\infty)\times\mathbf{R}^d}h(s,x)\phi(s,x,\omega)\eta_\beta(ds,dx)\\
&=\int_{(0,\infty)\times\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}h(s,y)\phi(s,y,\omega)h_\beta(y-x)\,dy\Big)\,\xi(ds,dx).
\end{aligned} \tag{2.21}$$

2.3 Representation of Itô integrals as space–time integrals

Consider the following family of correlated Brownian motions indexed by $x \in \mathbf{Z}^d$:

$$W_t(x) := \int_{(0,t)\times\mathbf{R}^d} 1_{[x,x+1)^d}(y)\eta(ds,dy), \tag{2.22}$$

We would like to show for any continuous process $X \in L^2_{loc}(W(x))$ (see Definition 4.2.6 of [13]), where $X_t \in \mathscr{F}_t^W$ for all $t \geq 0$, we have

$$\int_0^t X_s\, dW_s(x) = \int_{(0,t)\times\mathbf{R}^d} X_s 1_{[x,x+1)^d}(y)\eta(ds,dy). \tag{2.23}$$

It is easily seen (2.23) is true when X is an elementary process (see the comments after Definition 4.2.3 of [13]). (2.23) then holds by taking limits of elementary processes, which the reader could refer to Proposition 4.2.13 of [13].

CHAPTER 3

COMPARISON THEOREMS FOR INFINITE INTERACTING SDES

Throughout this chapter, (SDE) denote the system of SDEs defined earlier in Chapter 1 on page 3. Before stating the main results of (SDE), we first make a few remarks. We note that $\mathscr{L}$ defined in (1.12) is the infinitesimal generator of a continuous time random walk X_t, with jump matrix $\widetilde{P} := \left(p_{i,j}\right)_{i,j\in\mathbf{Z}^d}$, and the jump rate from state to state is always ν. We would like to show this fact using three lemmas. Let us define $P_t(x,y) := P_x(X_t = y)$.

Lemma 7. *Let $E_1, E_2, \cdots$ be a sequence of i.i.d exp(ν) random variables. Then $P(E_1 + \cdots + E_{n-1} < t \le E_1 + \cdots + E_n) = \frac{e^{-\nu t}(\nu t)^n}{n!}$.*

Proof: Let N_t be a Poisson process with jump rate ν. Then

$$P(E_1 + \cdots + E_{n-1} < t \le E_1 + \cdots + E_n) = P(N_t = n) = \frac{e^{-\nu t}(\nu t)^n}{n!}. \tag{3.1}$$

Q.E.D.

Lemma 8. $P_t(x,y) = \left(e^{-\nu t(I-\widetilde{P})}\right)_{x,y}$.

Proof: By the previous lemma,

$$\begin{aligned} P_t(x,y) &= \sum_{n=1}^{\infty} (\widetilde{P}^n)_{x,y} \cdot P(E_1 + \cdots + E_{n-1} < t, E_1 + \cdots + E_n \ge t) + (\widetilde{P}^0)_{x,y} \cdot P(E_1 \ge t) \\ &= \sum_{n=1}^{\infty} (\widetilde{P}^n)_{x,y} \cdot \frac{e^{-\nu t}(\nu t)^n}{n!} + (\widetilde{P}^0)_{x,y} \cdot e^{-\nu t} \\ &= \sum_{n=0}^{\infty} (\widetilde{P}^n)_{x,y} \cdot \frac{e^{-\nu t}(\nu t)^n}{n!} \\ &= e^{-\nu t} \cdot \left(e^{\nu t \widetilde{P}}\right)_{x,y} \\ &= \left(e^{-\nu t(I-\widetilde{P})}\right)_{x,y}. \end{aligned} \tag{3.2}$$

Q.E.D.

Lemma 9. $\mathscr{L} = -\nu(I - \widetilde{P})$.

Proof: Because $p_{i,j} = p_{j,i}$ for all $i, j \in \mathbf{Z}^d$, by the previous lemma,

$$\begin{aligned}\left(\mathscr{L}\right)_{i,j} := \left(\mathscr{L}\delta_i\right)_j &= \nu \sum_{k\in\mathbf{Z}^d} p_{j,k}(\delta_i(k) - \delta_i(j)) \\ &= \nu \sum_{k\in\mathbf{Z}^d} p_{k,j}(\delta_i(k) - \delta_i(j)) \\ &= \nu \cdot (p_{i,j} - \delta(i,j)) = -\nu(I - \widetilde{P})_{i,j}. \end{aligned} \tag{3.3}$$

Q.E.D.

The previous lemma shows

$$\frac{d}{dt}P_t(x,y) = \left(\mathscr{L}P_t\right)_{x,y}, \tag{3.4}$$

$$\frac{d}{dt}P_t(x,y)\Big|_{t=0} = \left(\mathscr{L}\right)_{x,y}. \tag{3.5}$$

Now we define

$$P_t(x) := P_0(X_t = x). \tag{3.6}$$

The characteristic function ψ of probability measure P_t equals

$$\begin{aligned}\psi(z) = \sum_{x\in\mathbf{Z}^d} P_t(x)e^{iz\cdot x} &= \sum_{x\in\mathbf{Z}^d}\sum_{n=0}^{\infty}(\widetilde{P}^n)_{0,x}\cdot\frac{e^{-\nu t}(\nu t)^n}{n!}e^{iz\cdot x} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{x\in\mathbf{Z}^d} p_{0,x}e^{iz\cdot x}\Big)^n\cdot\frac{e^{-\nu t}(\nu t)^n}{n!} \\ &= \sum_{n=0}^{\infty}(\widehat{\mu}(z))^n\cdot\frac{e^{-\nu t}(\nu t)^n}{n!} \\ &= e^{-\nu t(1-\widehat{\mu}(z))}. \end{aligned} \tag{3.7}$$

3.1 Existence and uniqueness of (SDE)

We would like to show that (SDE) has a unique solution. References [2], [15] treat this problem in the case that the underlying Brownian motions are independent. As a side remark, the assumption $\sigma(0) = 0$ is not used when we prove the existence and uniqueness of (SDE).

Before we get started, we first establish a BDG inequality. A milder inequality for independent Brownian motions can be found in Lemma 2.1 of [8].

Lemma 10 (BDG inequality)**.** *Let $Z := \{Z_t(x)\}_{t\geq 0, x\in\mathbf{Z}^d}$ be a predictable random field with respect to the Brownian motions $\{B_t(x)\}_{t\geq 0, x\in\mathbf{Z}^d}$ as in assumption (1) of (SDE). We also assume that*

$$\sum_{y\in\mathbf{Z}^d} E\Big[\int_0^t Z_s(y)^2\,ds\Big] < \infty. \tag{3.8}$$

Then, the following Itô integral

$$\int_0^t Z_s \cdot dB_s := \sum_{y\in\mathbf{Z}^d} \int_0^t Z_s(y)dB_s(y) \tag{3.9}$$

exists in $L^2(P)$. Furthermore, for any $k \in [2,\infty)$, we have

$$E\Big[\Big|\int_0^t Z_s\cdot dB_s\Big|^k\Big] \leq \Big|4k\cdot(1+C_{\mathcal{R}})\cdot \sum_{y\in\mathbf{Z}^d}\int_0^t \|Z_s(y)\|_k^2\,ds\Big|^{k/2} \tag{3.10}$$

for any $t\geq 0$, where $\mathcal{R} \leq C_{\mathcal{R}} < \infty$ and $\mathcal{R}$ is defined in (1.11).

Proof: First we enumerate the elements of $\mathbf{Z}^d$ as $x_1, x_2, \cdots$, and then define $F_n := \{x_1, x_2, \cdots, x_n\}$ $\forall n\geq 1$. For $n > m$,

$$\begin{aligned}\Big\|\sum_{y\in F_n}\int Z_s(y)dB_s(y) - \sum_{y\in F_m}\int Z_s(y)dB_s(y)\Big\|_2 &= \Big\|\sum_{y\in F_n\setminus F_m}\int_0^t Z_s(y)dB_s(y)\Big\|_2 \\ &= \left(\sum_{y\in F_n\setminus F_m}\int_0^t E[Z_s(y)^2]\,ds\right)^{1/2}.\end{aligned}$$

This shows that $\{\sum_{y\in F_n}\int Z_s(y)dB_s(y)\}_n$ is a Cauchy sequence in $L^2(P)$. In particular, $\sum_{y\in\mathbf{Z}^d}\int Z_s(y)dB_s(y) := \lim_{n\to\infty}\sum_{y\in F_n}\int Z_s(y)dB_s(y)$.

Next we compute the quadratic variation of the preceding as follows:

$$\begin{aligned}\left\langle \sum_{y\in F_N}\int_0^\cdot Z_s(y)dB_s(y)\right\rangle_t &= \sum_{y\in F_N}\int_0^t Z_s(y)^2\,ds + \sum_{x,y\in F_N}\int_0^t Z_s(x)Z_s(y)\mathcal{R}(|x-y|)\,ds \\ &\leq \sum_{y\in F_N}\int_0^t Z_s(y)^2\,ds + \frac{C_{\mathcal{R}}}{2}\cdot\sum_{x,y\in F_N}\int_0^t \left(Z_s(x)^2 + Z_s(y)^2\right)\,ds \\ &= (1+C_{\mathcal{R}})\cdot\sum_{y\in F_N}\int_0^t Z_s(y)^2\,ds.\end{aligned} \tag{3.11}$$

A more standard form of the BDG inequality (see, for example, Theorem B.1 in [11]) implies that for any $k\in[2,\infty)$,

$$E\left(\Big|\int Z_s dB_s(y)\Big|^k\right) \leq (4k)^{k/2}\cdot E\left(\Big|\int_0^t Z_s(y)^2\,ds\Big|^{k/2}\right). \tag{3.12}$$

We apply Minkowski's integral inequality to see that

$$E\left(\left|\sum_{y\in F_n}\int Z_s dB_s(y)\right|^k\right) \leq (4k)^{k/2}\cdot(1+C_{\mathcal{R}})^{k/2}\cdot E\left(\left|\sum_{y\in F_n}\int_0^t Z_s(y)^2\,ds\right|^{k/2}\right)$$
$$\leq (4k)^{k/2}\cdot(1+C_{\mathcal{R}})^{k/2}\cdot\left(\sum_{y\in F_n}\int_0^t \|Z_s(y)\|_k^2\,ds\right)^{k/2}$$
$$\leq \left(4k\cdot(1+C_{\mathcal{R}})\cdot\sum_{y\in\mathbf{Z}^d}\int_0^t \|Z_s(y)\|_k^2\,ds\right)^{k/2}. \tag{3.13}$$

Because $\sum_{y\in F_n}\int Z_s dB_s(y) \to \sum_{y\in\mathbf{Z}^d}\int Z_s dB_s(y)$ in $L^2(P)$, we may let $n\to\infty$ along a subsequence which $\sum_{y\in F_n}\int Z_s dB_s(y) \to \sum_{y\in\mathbf{Z}^d}\int Z_s dB_s(y)$ a.s. in (3.13). The lemma is then proved using Fatou's lemma. **Q.E.D.**

Theorem 11. *(SDE) has a solution $U_t(x)$ that is continuous almost surely in the variable t for every $x\in\mathbf{Z}^d$. Moreover, there exists a constant C depending only on $k\in[2,\infty)$, U_0, $T>0$, and σ, such that*

$$\sup_{t\in[0,T]}\sup_{x\in\mathbf{Z}^d}\|U_t(x)\|_k \leq C. \tag{3.14}$$

Furthermore, the solution $U_t(x)$ is unique among all other solutions that satisfy

$$\sup_{t\in[0,T]}\sup_{x\in\mathbf{Z}^d}\|U_t(x)\|_2 < \infty \qquad \forall T>0. \tag{3.15}$$

Proof: Let $U_t^0(x) := U_0(x)$, for $n\in\mathbf{N}\cup\{0\}$, and define iteratively

$$U_t^{(n+1)}(x) := \int_0^t \big(\mathscr{L}U_s^{(n)}\big)(x)ds + \int_0^t \sigma\big(U_s^{(n)}(x)\big)dB_s(x). \tag{3.16}$$

We first note that

$$\left|\big(\mathscr{L}U_t^{(n)}\big)(x) - \big(\mathscr{L}U_t^{(n-1)}\big)(x)\right|$$
$$= \left|\nu\sum_{y\in\mathbf{Z}^d}p_{x,y}(U_t^{(n)}(y)-U_t^{(n)}(x)) - \nu\sum_{y\in\mathbf{Z}^d}p_{x,y}(U_t^{(n-1)}(y)-U_t^{(n-1)}(x))\right|$$
$$\leq \nu\cdot\left|U_t^{(n)}(x)-U_t^{(n-1)}(x)\right| + \nu\cdot\left|\sum_{y\in\mathbf{Z}^d}p_{x,y}\Big(U_t^{(n)}(y)-U_t^{(n-1)}(y)\Big)\right|, \tag{3.17}$$

for all $t\in[0,T]$, $n\geq 1$, and $x\in\mathbf{Z}^d$. Also,

$$
\begin{aligned}
&\sup_{x\in\mathbf{Z}^d}\left\|U_t^{(1)}(x)-U_t^{(0)}(x)\right\|_k\\
&=\sup_{x\in\mathbf{Z}^d}\left(\left\|\int_0^t \nu\sum_{y\in\mathbf{Z}^d}p_{x,y}(U_0(y)-U_0(x))ds-U_0(x)\right\|_k+\left\|\int_0^t\sigma\big(U_0(x)\big)dB_s(x)\right\|_k\right)\\
&=\sup_{x\in\mathbf{Z}^d}\left(\left\|\int_0^t \nu\sum_{y\in\mathbf{Z}^d}p_{x,y}(U_0(y)-U_0(x))ds\right\|_k+\|U_0(x)\|_k+\left\|\sigma\big(U_0(x)\big)\cdot B_t6\right\|_k\right)\\
&\le c_1:=(1+2\nu T)\cdot\sup_{x\in\mathbf{Z}^d}U_0(x)+\sqrt{T}\cdot\sup_{x\in\mathbf{Z}^d}\sigma\big(U_0(x)\big)\cdot\|N\|_k<\infty, \qquad (3.18)
\end{aligned}
$$

where N is a standard normal variable.

Therefore, by Lemma 10, (3.17), and Minkowski's integral inequality, together imply that for all $n\in\mathbf{N}$, $k\in[2,\infty)$, and $x\in\mathbf{Z}^d$,

$$
\begin{aligned}
&\left\|U_t^{(n+1)}(x)-U_t^{(n)}(x)\right\|_k\\
&\le\left\|\int_0^t\big(\mathscr{L}U_s^{(n)}\big)(x)-\big(\mathscr{L}U_s^{(n-1)}\big)(x)ds\right\|_k+\left\|\int_0^t\sigma\big(U_s^{(n)}(x)\big)-\sigma\big(U_s^{(n-1)}(x)\big)dB_s(x)\right\|_k\\
&\le\int_0^t\left\|\big(\mathscr{L}U_s^{(n)}\big)(x)-\big(\mathscr{L}U_s^{(n-1)}\big)(x)\right\|_k ds\\
&+(4k)^{1/2}\cdot\mathrm{Lip}_\sigma\cdot\left(\int_0^t\left\|U_s^{(n)}(x)-U_s^{(n-1)}(x)\right\|_k^2ds\right)^{1/2}\\
&\le 2\nu\int_0^t\sup_{x\in\mathbf{Z}^d}\left\|U_s^{(n)}(x)-U_s^{(n-1)}(x)\right\|_k ds\\
&+(4k)^{1/2}\cdot\mathrm{Lip}_\sigma\cdot\left(\int_0^t\sup_{x\in\mathbf{Z}^d}\left\|U_s^{(n)}(x)-U_s^{(n-1)}(x)\right\|_k^2ds\right)^{1/2}. \qquad (3.19)
\end{aligned}
$$

This implies

$$
\sup_{x\in\mathbf{Z}^d}\left\|U_t^{(n+1)}(x)-U_t^{(n)}(x)\right\|_k^2\le c_2\cdot\int_0^t\sup_{x\in\mathbf{Z}^d}\left\|U_s^{(n)}(x)-U_s^{(n-1)}(x)\right\|_k^2ds, \qquad (3.20)
$$

where $c_2:=4k\cdot\mathrm{Lip}_\sigma^2+8\nu^2t$.

We iterate (3.20) to deduce from (3.18) that

$$
\sup_{t\in[0,T]}\sup_{x\in\mathbf{Z}^d}\left\|U_t^{(n+1)}(x)-U_t^{(n)}(x)\right\|_k\le\left(\frac{1}{n!}\cdot c_1^2\cdot c_2^n\cdot T^n\right)^{1/2}. \qquad (3.21)
$$

Due to (3.21), we define $U_t(x)=\lim_{n\to\infty}U_t^{(n)}(x)$, and we have

$$
\sup_{t\in[0,T]}\sup_{x\in\mathbf{Z}^d}\left\|U_t(x)\right\|_k\le\sum_{n=0}^{\infty}\left(\frac{1}{n!}\cdot c_1^2\cdot c_2^n\cdot T^n\right)^{1/2}<\infty. \qquad (3.22)
$$

In particular,

$$
\sup_{t\in[0,T]}\sup_{x\in\mathbf{Z}^d}\left\|U_t(x)-U_t^{(n)}(x)\right\|_k\to 0 \text{ as } n\to\infty. \qquad (3.23)
$$

Now we return to (3.16). For each $x \in \mathbf{Z}^d$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} E\left(\left|\int_0^t \sigma\big(U_s(x)\big)dB_s(x) - \int_0^t \sigma\big(U_s^{(n)}(x)\big)dB_s(x)\right|^2\right) \\
&\leq \text{const}\cdot \limsup_{n\to\infty} \int_0^t E\left(\left|U_s(x) - U_s^{(n)}(x)\right|^2\right) ds \\
&\leq \text{const}\cdot \limsup_{n\to\infty} \sup_{s\in[0,t]} \sup_{y\in\mathbf{Z}^d} \left\|U_s(y) - U_s^{(n)}(y)\right\|_2^2 = 0. \qquad (3.24)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\limsup_{n\to\infty} E\left(\left|\int_0^t \big(\mathscr{L}U_s\big)(x)ds - \int_0^t \big(\mathscr{L}U_s^{(n)}\big)(x)ds\right|^2\right) \\
&\leq \text{const}\cdot \limsup_{n\to\infty} E\left[\left(\int_0^t |U_s(x) - U_s^{(n)}(x)| + \sum_{y\in\mathbf{Z}^d} p_{x,y}|U_s(y) - U_s^{(n)}(y)|\, ds\right)^2\right] \\
&\leq \text{const}\cdot \limsup_{n\to\infty} \int_0^t E\left(\left|U_s(x) - U_s^{(n)}(x)\right|^2\right) + E\left[\left(\sum_{y\in\mathbf{Z}^d} p_{x,y}|U_s(y) - U_s^{(n)}(y)|\right)^2\right] ds \\
&\leq \text{const}\cdot \limsup_{n\to\infty} \int_0^t E\left(\left|U_s(x) - U_s^{(n)}(x)\right|^2\right) + \sum_{y\in\mathbf{Z}^d} p_{x,y}E\left(\left|U_s(y) - U_s^{(n)}(y)\right|^2\right) ds \\
&\leq \text{const}\cdot \limsup_{n\to\infty} \sup_{s\in[0,t]} \sup_{y\in\mathbf{Z}^d} \left\|U_s(y) - U_s^{(n)}(y)\right\|_2^2 = 0. \qquad (3.25)
\end{aligned}$$

Let $n \to \infty$ in (3.16) as follows: By (3.24) and (3.25), for all $x \in \mathbf{Z}^d$,

$$U_t(x) = \int_0^t \big(\mathscr{L}U_s\big)(x)ds + \int_0^t \sigma\big(U_s(x)\big)dB_s(x). \qquad (3.26)$$

(3.26) also implies that $U_t(x)$ has a continuous modification for every $x \in \mathbf{Z}^d$.

Now we prove uniqueness of (SDE). Let $U_t(x), V_t(x)$ be two different solutions of (SDE) that satisfy (3.15), such that $U_0 \equiv V_0$. We carry out the same calculations as we did for (3.19) to see that

$$\sup_{x\in\mathbf{Z}^d} \left\|U_t(x) - V_t(x)\right\|_2^2 \leq c_1 \cdot \int_0^t \sup_{x\in\mathbf{Z}^d} \left\|U_s(x) - V_s(x)\right\|_2^2 ds. \qquad (3.27)$$

By Gronwall's inequality (see Appendix A.2), $U_t(x) = V_t(x)$ a.s. **Q.E.D.**

Now we define **mild solution** $U_t(x)$ to (SDE):

$$U_t(x) = \sum_{y\in\mathbf{Z}^d} P_t(x-y)\cdot U_0(y) + \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)\sigma(U_s(y))\, dB_s(y), \qquad (3.28)$$

where $P_t(x)$ is defined in (3.6).

It was shown in [8] that, when the underlying Brownian motions of (SDE) are independent, (SDE) has a unique mild solution. Here we would like to show that a unique mild solution to (SDE) exists in the present setting when the Brownian motions are correlated.

Theorem 12. *(SDE) has a **mild solution** $U_t(x)$ that is continuous in the variable t for each $x \in \mathbf{Z}^d$. Moreover, there exists $C < \infty$ depending only on $k \in [2, \infty)$, U_0, $T > 0$, and σ, such that*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} E[|U_t(x)|^k] < C. \tag{3.29}$$

In addition, the mild solution to (SDE) is unique among all solutions that satisfy

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} E[|U_t(x)|^k] < \infty. \tag{3.30}$$

Proof: We define iteratively for $n \in \mathbf{N} \cup \{0\}$,

$$U_t^{(n+1)}(x) := \sum_{y\in\mathbf{Z}^d} P_t(x-y) \cdot U_0(y) + \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)\sigma\big(U_s^{(n)}(y)\big)\, dB_s(y). \tag{3.31}$$

Because for any $x \in \mathbf{Z}^d$ and $t \geq 0$,

$$\begin{aligned}\sum_{y\in\mathbf{Z}^d} E\left[\int_0^t \Big(P_{t-s}(y-x)\sigma(U_0(y))\Big)^2 ds\right] &\leq \text{const} \cdot \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)^2\, ds \\ &\leq \text{const} \cdot \int_0^t \sum_{y\in\mathbf{Z}^d} P_{t-s}(y-x)\, ds < \infty,\end{aligned}$$

it follows that the random field $Z_s^{(0)}(y) := P_{t-s}(y-x)\sigma(U_0(y))$ satisfies (3.8). By Lemma 10, for any $0 \leq t \leq T$,

$$\begin{aligned}\|U_t^{(1)}(x)\|_k &\leq \sup_{y\in\mathbf{Z}^d} U_0(y) + \left(4k \cdot (1+C_{\mathcal{R}}) \cdot \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)^2 \|U_0(y)\|_k^2\, ds\right)^{1/2} \\ &\leq \sup_{y\in\mathbf{Z}^d} U_0(y) + \left(4k \cdot (1+C_{\mathcal{R}}) \cdot \sum_{y\in\mathbf{Z}^d} \int_0^T P_{T-s}(y-x)^2 \|U_0(y)\|_k^2\, ds\right)^{1/2} \\ &\leq \sup_{y\in\mathbf{Z}^d} U_0(y) + \left(4k \cdot (1+C_{\mathcal{R}}) \cdot \sup_{y\in\mathbf{Z}^d} U_0(y)^2 \cdot \int_0^T \sum_{y\in\mathbf{Z}^d} P_s(y-x)\, ds\right)^{1/2} \\ &\leq c_1 < \infty,\end{aligned} \tag{3.32}$$

where the constant c_1 is independent of the choice of $x \in \mathbf{Z}^d$ and $t \in [0,T]$. Next, for $n \in \mathbf{N}$ and any $x \in \mathbf{Z}^d$, if $\sup_{s\in[0,T]} \sup_{x\in\mathbf{Z}^d} E[|U^{(n)}(x)|^k] < \infty$ for some $k \in [2,\infty)$, then for any $t \in [0,T]$ we have

$$\begin{aligned}
&\sum_{y\in\mathbf{Z}^d} E\left[\int_0^t \left(P_{t-s}(y-x)\sigma(U_s^{(n)}(y))\right)^2 ds\right] \\
&\leq \text{const} \cdot \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)^2 \left(1 + E[U_s^{(n)}(y)^2]\right) ds < \infty.
\end{aligned} \tag{3.33}$$

Therefore, Lemma 10 implies that

$$\begin{aligned}
\|U_t^{(n+1)}(x)\|_k &\leq \sup_{y\in\mathbf{Z}^d} U_0(y) + \left(4k \cdot (1+C_{\mathcal{R}}) \cdot \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)^2 \|\sigma(U_s^{(n)}(y))\|_k^2 \, ds\right)^{1/2} \\
&\leq \sup_{y\in\mathbf{Z}^d} U_0(y) + \left(\text{const} \cdot \sum_{y\in\mathbf{Z}^d} \int_0^T P_{T-s}(y-x)^2 \, ds\right)^{1/2} \leq c_2,
\end{aligned}$$

where the constant c_2 is finite and independent of the choice of $x \in \mathbf{Z}^d$ and $t \in [0,T]$. Due to the above discussions, for any $n \in \mathbf{N} \cup \{0\}$, $T > 0$, and $k \in [2,\infty)$,

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \left\|U_t^{(n+1)}(x) - U_t^{(n)}(x)\right\|_k^2 < \infty. \tag{3.34}$$

By (3.34) and Lemma 10, for any $x \in \mathbf{Z}^d$ we have

$$\begin{aligned}
&\|U_t^{(n+1)}(x) - U_t^{(n)}(x)\|_k^2 \\
&\leq 4k \cdot (1+C_{\mathcal{R}}) \cdot \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)^2 \left\|\sigma(U_s^{(n)}(y)) - \sigma(U_s^{(n-1)}(y))\right\|_k^2 ds \\
&\leq 4k \cdot (1+C_{\mathcal{R}}) \cdot \mathrm{Lip}_\sigma^2 \cdot \int_0^t \sup_{y\in\mathbf{Z}^d} \left\|U_s^{(n)}(y) - U_s^{(n-1)}(y)\right\|_k^2 ds.
\end{aligned} \tag{3.35}$$

This, and Gronwall's lemma, together imply that for any $t \in [0,T]$,

$$\begin{aligned}
&\sup_{x\in\mathbf{Z}^d} \left\|U_t^{(n+1)}(x) - U_t^{(n)}(x)\right\|_k^2 \\
&\leq \frac{1}{n!} \cdot \sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \left\|U_t^{(1)}(x) - U_t^{(0)}(x)\right\|_k \cdot \left(4k \cdot (1+C_{\mathcal{R}}) \cdot \mathrm{Lip}_\sigma^2 \cdot T\right)^n.
\end{aligned} \tag{3.36}$$

By (3.36), there exists a space–time random field $U_t(x)$ such that for all $k \in [2,\infty)$,

$$\begin{aligned}
&\sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \left\|U_t(x)\right\|_k \\
&\leq \sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \|U_t^{(1)}(x) - U_t^{(0)}(x)\|_k^2 \cdot \sum_{n=0}^\infty \frac{1}{(n!)^{1/2}} \left(4k \cdot (1+C_{\mathcal{R}}) \cdot \mathrm{Lip}_\sigma^2 \cdot T\right)^{n/2} < \infty,
\end{aligned} \tag{3.37}$$

and as $n \to \infty$,

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \left\| U_t(x) - U_t^{(n)}(x) \right\|_k \to 0. \tag{3.38}$$

For each $x \in \mathbf{Z}^d$, by Lemma 10, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left\| \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)\big(\sigma(U_s(y)) - \sigma(U_s^{(n)}(y))\big)\, ds \right\|_k^2 \\
&\le 4k \cdot (1 + C_{\mathcal{R}}) \cdot \limsup_{n\to\infty} \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)^2 \left\| \sigma(U_s(y)) - \sigma(U_s^{(n)}(y)) \right\|_k^2 ds \\
&\le 4k \cdot (1 + C_{\mathcal{R}}) \cdot \mathrm{Lip}_\sigma^2 \cdot \limsup_{n\to\infty} \int_0^t \sup_{y\in\mathbf{Z}^d} \left\| U_s(y) - U_s^{(n)}(y) \right\|_k^2 ds = 0.
\end{aligned} \tag{3.39}$$

Therefore, if we let $n \to \infty$ in (3.31), then we obtain (3.28). Also, (3.28) shows that $U_t(x)$ has a continuous modification for every $x \in \mathbf{Z}^d$.

Now we prove the uniqueness of the mild solution of (SDE). Let $U_t(x), V_t(x)$ be two different mild solutions of (SDE) satisfying (3.30) such that $U_0 \equiv V_0$. By Lemma 10, for any $x \in \mathbf{Z}^d$,

$$\|U_t(x) - V_t(x)\|_k^2 \le 4k \cdot (1 + C_{\mathcal{R}}) \cdot \mathrm{Lip}_\sigma^2 \cdot \limsup_{n\to\infty} \int_0^t \sup_{y\in\mathbf{Z}^d} \|U_s(y) - V_s(y)\|_k^2 \, ds. \tag{3.40}$$

Gronwall's inequality (see Appendix A.2) implies that $\sup_{x\in\mathbf{Z}^d} \|U_t(x) - V_t(x)\|_k = 0$, and hence $U_t(x) = V_t(x)$ a.s. **Q.E.D.**

A natural question arises: Are these two different 'solutions' to (SDE) that are actually the same? The answer is affirmative. We state it as the following theorem.

Theorem 13. *Let $U_t(x)$ be the unique solution to (SDE) satisfying (3.14) with initial data $U_0(x)$, and $V_t(x)$ be the unique mild solution to (SDE) with the same initial data $U_0(x)$, and $V_t(x)$ satisfies (3.30). Then $U_t(x) = V_t(x)$ a.s.*

Proof: The proof is complete if we could show $U_t(x)$ is also the mild solution to (SDE). For each $x, y \in \mathbf{Z}^d$, because $U_t(x)$ satisfies (SDE), by the associativity property of stochastic integrals,

$$\int_0^t P_{t-s}(y-x)\sigma(U_s(y))\, dB_s(y) = \int_0^t P_{t-s}(y-x)\, dU_s(y) - \int_0^t P_{t-s}(y-x)(\mathscr{L}U_s)(y)\, ds. \tag{3.41}$$

We enumerate $\mathbf{Z}^d$ as $x_1, x_2, \cdots$, and define $F_n := \{x_1, \cdots, x_n\}$ $\forall n \geq 1$. We sum over $y \in F_n$ in (3.41), in order to obtain the following: For any $x \in F_n$,

$$\begin{aligned}
&\sum_{y\in F_n} \int_0^t P_{t-s}(y-x)\sigma(U_s(y))\,dB_s(y) \\
&= \sum_{y\in F_n} \int_0^t P_{t-s}(y-x)\,dU_s(y) - \sum_{y\in F_n} \int_0^t P_{t-s}(y-x)(\mathscr{L}U_s)(y)\,ds \\
&= U_t(x) - \sum_{y\in F_n} P_t(y-x)\cdot U_0(y) - \sum_{y\in F_n} \int_0^t U_s(y)\,dP_{t-s}(y-x) \\
&- \sum_{y\in F_n} \int_0^t P_{t-s}(y-x)(\mathscr{L}U_s)(y)\,ds \\
&= U_t(x) - \sum_{y\in F_n} P_t(y-x)\cdot U_0(y) + \sum_{y\in F_n} \int_0^t U_s(y)(\mathscr{L}P_{t-s})(y-x)\,ds \\
&- \sum_{y\in F_n} \int_0^t P_{t-s}(y-x)(\mathscr{L}U_s)(y)\,ds. \qquad (3.42)
\end{aligned}$$

Here we have used (3.41) and the integration by parts formula for the Itô integrals (see Proposition IV. 3.1 of [13]). Because $U_t(x)$ satisfies (3.14), the left-hand side of (3.42) converges in $L^2(P)$ and the right-hand side of (3.42) converges in $L^1(P)$, as $n \to \infty$:

$$\begin{aligned}
&\sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)\sigma(U_s(y))\,dB_s(y) - U_t(x) + \sum_{y\in\mathbf{Z}^d} P_t(y-x)\cdot U_0(y) \\
&= \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)\sigma(U_s(y))\,dB_s(y) - U_t(x) + \sum_{y\in\mathbf{Z}^d} P_t(x-y)\cdot U_0(y) \\
&= \sum_{y\in\mathbf{Z}^d} \int_0^t U_s(y)(\mathscr{L}P_{t-s})(y-x)\,ds - \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x)(\mathscr{L}U_s)(y)\,ds \\
&= \sum_{y\in\mathbf{Z}^d} \int_0^t U_s(y) \sum_{z\in\mathbf{Z}^d} p_{y-x,z}(P_{t-s}(z) - P_{t-s}(y-x))\,ds \\
&- \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x) \sum_{z\in\mathbf{Z}^d} p_{y,z}(U_s(z) - U_s(y))\,ds \\
&= \sum_{y\in\mathbf{Z}^d} \int_0^t U_s(y) \sum_{z\in\mathbf{Z}^d} p_{y-x,z}P_{t-s}(z)\,ds - \sum_{y\in\mathbf{Z}^d} \int_0^t P_{t-s}(y-x) \sum_{z\in\mathbf{Z}^d} p_{y,z}U_s(z)\,ds \\
&= \sum_{y\in\mathbf{Z}^d}\sum_{z\in\mathbf{Z}^d} \int_0^t U_s(y)p_{y-x,z}P_{t-s}(z)\,ds - \sum_{y\in\mathbf{Z}^d}\sum_{z\in\mathbf{Z}^d} \int_0^t P_{t-s}(z-x)p_{y,z}U_s(y)\,ds = 0. \qquad (3.43)
\end{aligned}$$

In the above calculations, we have used the fact $p_{i,j} = p_{j,i}$ and $p_{i+k,j+k} = p_{i,j}$ for all $i, j, k \in \mathbf{Z}^d$. **Q.E.D.**

3.2 Approximation of (SDE) by other SDEs under simplifications

In this section, we are going to show that the solution to (SDE) can be approximated in $L^k(P)$ by the solutions to a sequence of SDEs, where each of them is indexed by a finite set of $\mathbf{Z}^d$ instead of $\mathbf{Z}^d$ itself, and σ is twice differentiable and compactly supported on $[a,b]$, $a>0$.

We define a family of finite systems of SDEs as follows:

$$\begin{aligned} &dU_t^{(N)}(x) = \big(\mathscr{L}^{(N)}U_t^{(N)}\big)(x)dt + \sigma\big(U_t^{(N)}(x)\big)dB_t(x), \\ &x \in K_N := \{-N, -N+1, \cdots, N-1, N\}^d \end{aligned} \tag{3.44}$$

where

1. The Brownian motions $\{B_t(x)\}_{x\in K_N}$ are the same as in the assumption of (SDE);

2. $\mathscr{L}^{(N)}$ is defined by

$$\mathscr{L}^{(N)}g(j) := \nu \sum_{i\in K_N} p_{j,i}(g(i)-g(j)), \tag{3.45}$$

   for any $g: K_N \to \mathbf{R}$. $p_{i,j}$ is the same as the ones in (SDE);

3. $\sigma: \mathbf{R} \to [0,\infty)$ is the same function as we define (SDE);

4. The initial condition $U_0^{(N)}(x) = U_0(x)$ for all $x \in K_N$.

We start with the following approximation result.

Theorem 14. *Let $U_t^{(N)}(x)$ solve (3.44), and $U_t(x)$ solve (SDE). Then,*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sup_{x\in K_N} \|U_t^{(N)}(x) - U_t(x)\|_k = 0 \tag{3.46}$$

for every $k \in [2,\infty)$.

Proof: We note that by (3.14), there exists $C<\infty$ such that $\sup_{t\in[0,T]}\sup_{x\in\mathbf{Z}^d}\|U_s(x)\|_k \le C$. Moreover, by using the same techniques as in Theorem 11, it can be shown that

$$\sup_{t\in[0,T]} \sup_{x\in K_N} \|U_s^{(N)}(x)\|_k < \infty, \tag{3.47}$$

for every $N \in \mathbf{N}$. Now for any $x \in K_N$, by Minkowski's integral inequality and Lemma 10, we have

$$\begin{aligned}
&\|U_t^{(N)}(x) - U_t(x)\|_k \\
&\le \left\| \int_0^t (\mathscr{L}^{(N)} U_t^{(N)})(x) - (\mathscr{L} U_s)(x) ds \right\|_k + \left\| \int_0^t \sigma(U_s^{(N)}(x)) - \sigma(U_s(x)) dB_s(x) \right\|_k \\
&\le \int_0^t \left\| (\mathscr{L}^{(N)} U_t^{(N)})(x) - (\mathscr{L} U_s)(x) \right\|_k ds + \mathrm{Lip}_\sigma \cdot (4k)^{1/2} \cdot \left(\int_0^t \left\| U_s^{(N)}(x) - U_s(x) \right\|_k^2 ds \right)^{1/2}.
\end{aligned} \tag{3.48}$$

Therefore, for any $t \in [0, T]$,

$$\begin{aligned}
&\|U_t^{(N)}(x) - U_t(x)\|_k^2 \\
&\le 2 \left(\int_0^t \left\| (\mathscr{L}^{(N)} U_t^{(N)})(x) - (\mathscr{L} U_s)(x) \right\|_k ds \right)^2 + 2\mathrm{Lip}_\sigma^2 \cdot (4k) \cdot \int_0^t \|U_s^{(N)}(x) - U_s(x)\|_k^2 ds \\
&\le 2\nu^2 \left(\int_0^t \sum_{y \in \mathbf{Z}^d \setminus K_N} p_{x,y} \|U_s(y) - U_s(x)\|_k \right. \\
&\quad \left. + \left\| \sum_{y \in K_N} p_{x,y} \big((U_s(y) - U_s(x)) - (U_s^{(N)}(y) - U_s^{(N)}(x)) \big) \right\|_k ds \right)^2 \\
&\quad + 8k \cdot \mathrm{Lip}_\sigma^2 \cdot \int_0^t \|U_s^{(N)}(x) - U_s(x)\|_k^2 ds \\
&\le 2\nu^2 \Big(2Ct \cdot \sum_{y \in \mathbf{Z}^d \setminus K_N} p_{x,y} + 2 \int_0^t \sup_{x \in \mathbf{Z}^d} \|U_s^{(N)}(x) - U_s(x)\|_k ds \Big)^2 \\
&\quad + 8k \cdot \mathrm{Lip}_\sigma^2 \cdot \int_0^t \sup_{x \in \mathbf{Z}^d} \|U_s^{(N)}(x) - U_s(x)\|_k^2 ds \\
&\le 16C^2\nu^2 t^2 \cdot \left(\sum_{y \in \mathbf{Z}^d \setminus K_N} p_{x,y} \right)^2 + \big(16\nu^2 t + 8k \cdot \mathrm{Lip}_\sigma^2 \big) \cdot \int_0^t \sup_{x \in \mathbf{Z}^d} \|U_s^{(N)}(x) - U_s(x)\|_k^2 ds.
\end{aligned} \tag{3.49}$$

By (3.49) and Gronwall's inequality (see Appendix), for any $t \in [0, T]$ we have

$$\sup_{x \in K_N} \|U_t^{(N)}(x) - U_t(x)\|_k^2 \le 16C^2\nu^2 T^2 \cdot \left(\sum_{y \in \mathbf{Z}^d \setminus K_N} p_{x,y} \right)^2 \cdot e^{\left(16\nu^2 T + 8k \cdot \mathrm{Lip}_\sigma^2\right) \cdot T}. \tag{3.50}$$

Let $N \to \infty$ in (3.50) to complete the proof. **Q.E.D.**

Next we would like to present Theorem 1.2 from [6], which would be used to prove the nonnegativity of the solution to (3.44).

Theorem 15 (Geiß, Manthey). *Consider two systems of SDEs*

$$X_j(t) = X_j(0) + \int_0^t a_j(s, X(s))\, ds + \sum_{k=1}^{r} \int_0^t \sigma_{jk}(s, X(s))\, dW_k(s)$$
$$Y_j(t) = Y_j(0) + \int_0^t b_j(s, Y(s))\, ds + \sum_{k=1}^{r} \int_0^t \sigma_{jk}(s, Y(s))\, dW_k(s),$$

where $1 \le j \le n$, which satisfy

(1) $X_j(0) \le Y_j(0)$ for $1 \le j \le n$,

(2) $a_j(t, x) \le b_j(t, x)$ for $1 \le j \le n$,

(3) For any $1 \le j \le n$, $a_j(t, x) \le a_j(t, y)$ and $b_j(t, x) \le b_j(t, y)$, whenever $x_j = y_j$ and $x_l \le y_l$, $l \ne j$.

(4) There exists a strictly increasing function $\rho : [0, \infty) \to [0, \infty)$ with $\rho(0) = 0$ and $\int_0^1 [\rho(u)]^{-2}\, du = \infty$, such that for each $1 \le j \le n$, $\sum_{k=1}^{r} |\sigma_{jk}(t, x) - \sigma_{jk}(t, y)| \le \rho(|x_j - y_j|)$.

(5) $W_1, \cdots, W_k$ are standard Brownian motions.

Then we have $P(X(t) \le Y(t), t \in [0, \theta_X \wedge \theta_Y)) = 1$, where θ_X, θ_Y denote the explosion times of X, Y, respectively.

We remark in the above theorem that the Brownian motions $W_1, \cdots, W_k$ are not required to be independent of each other from its proof.

Corollary 16. *Let $U_t^{(N)}(x)$ denote the solution to (3.44). If there exists $m \in \mathbf{R}$ such that $\sigma(m) = 0$ and $\inf_{x \in K_N} U_0(x) \ge m$, then $U_t^{(N)}(x) \ge m$ for every $t \ge 0$ and $x \in K_N$, a.s. If there exists $M \in \mathbf{R}$ such that $\sigma(M) = 0$ and $\sup_{x \in K_N} U_0(x) \le M$, then $U_t^{(N)}(x) \le M$ for every $t \ge 0$ and $x \in K_N$ a.s.*

Proof: For each N, set $n = (2N+1)^d$. To see that assumption (1) of Theorem 15 is true, because $U_t(x) \equiv m$ is the solution to (SDE) with $U_0(x) \equiv m$, the result follows by comparing $U^N(t)$ to m using the previous theorem. It suffices to check if assumptions (2), (3), (4) of the previous theorem hold. Let $a_j(x) = b_j(x) = \nu \sum_{i \in K_N} p_{j,i} \cdot (x_i - x_j)$ for all $1 \le j \le d$, it is easy to check (2), (3) are both true. Assumption (4) holds because r in the previous theorem equals 1, $\sigma_{j1}(x) = \sigma(x_j)$ for all j, and we let $\rho(x) := \mathrm{Lip}_\sigma \cdot x$. For the second assertion, we compare $U^N(t)$ to M. **Q.E.D.**

Corollary 17. *Let $U_t(x)$ denote the solution to (SDE). If there exists $m \in \mathbf{R}$ such that $\sigma(m) = 0$ and $\inf_{x \in \mathbf{Z}^d} U_0(x) \ge m$, then $U_t(x) \ge m$ for every $t \ge 0$ and $x \in \mathbf{Z}^d$ a.s. If there exists $M \in \mathbf{R}$*

such that $\sigma(M) = 0$ and $\sup_{x\in\mathbf{Z}^d} U_0(x) \le M$, then $U_t(x) \le M$ for every $t \ge 0$ and $x \in \mathbf{Z}^d$ a.s.

Proof: The proof follows from Corollary 16 and Theorem 14. **Q.E.D.**

Theorem 18. *Let $\sigma^{(N)}$ be a Lipschitz function constructed from σ so that*

$$\sigma^{(N)}(x) = \begin{cases} 2N \cdot (x - \frac{1}{2N}) \cdot \sigma(\frac{1}{N}), & x \in [\frac{1}{2N}, \frac{1}{N}), \\ \sigma(x), & x \in [\frac{1}{N}, N], \\ (N+1-x) \cdot \sigma(N), & x \in (N, N+1), \\ 0, & \textit{otherwise}. \end{cases} \tag{3.51}$$

Let $U_t(x)$ solve (SDE). Then there exists a sequence of solutions $U_t^{(N)}(x)$ solving (SDE) with σ replaced by $\sigma^{(N)}$, such that

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \|U_t^{(N)}(x) - U_t(x)\|_k = 0 \tag{3.52}$$

for every $k \in [2, \infty)$.

Proof: We note that $\sigma^{(N)}$ is a Lipschitz continuous function with compact support $\text{supp}(\sigma) \subset [\frac{1}{2N}, N]$. Besides, for all $x \in (N, N+1)$,

$$\begin{aligned} |\sigma^{(N)}(x) - \sigma(x)| &\le (N+1-x) \cdot \sigma(N) + \sigma(x) \\ &\le \text{Lip}_\sigma \cdot N + \text{Lip}_\sigma \cdot x \\ &\le 2\text{Lip}_\sigma \cdot x. \end{aligned} \tag{3.53}$$

Also, for all $x \in [\frac{1}{2N}, \frac{1}{N})$,

$$\begin{aligned} |\sigma^{(N)}(x) - \sigma(x)| &\le 2N \cdot \left(x - \frac{1}{2N}\right) \cdot \sigma\left(\frac{1}{N}\right) + \sigma(x) \\ &\le \text{Lip}_\sigma \cdot \frac{1}{N} + \text{Lip}_\sigma \cdot \frac{1}{N} \\ &\le 2\text{Lip}_\sigma \cdot \frac{1}{N}. \end{aligned} \tag{3.54}$$

We write $U_t^{(N)}(x) - U_t(x) = \text{(I)} + \text{(II)}$, where

$$\text{(I)} = \int_0^t \sum_{y\in\mathbf{Z}^d} P_{t-s}(y-x) \cdot \left[\sigma^{(N)}\big(U_s^{(N)}(y)\big) - \sigma\big(U_s^{(N)}(y)\big)\right] dB_s(y) \tag{3.55}$$

$$\text{(II)} = \int_0^t \sum_{y\in\mathbf{Z}^d} P_{t-s}(y-x) \cdot \left[\sigma\big(U_s^{(N)}(y)\big) - \sigma\big(U_s(y)\big)\right] dB_s(y). \tag{3.56}$$

By the Cauchy–Schwarz inequality and Theorem 11,

$$\|U_s^{(N)}(y)\cdot 1_{\{U_s^{(N)}(y)\geq N\}}\|_k \leq E\left[(U_s^{(N)}(y))^{2k}\right]^{1/2k}\cdot P(U_s^{(N)}(y)\geq N)^{1/2k}$$
$$\leq \|U_s^{(N)}(y)\|_{2k}\cdot\|U_s^{(N)}(y)\|_{2k}\cdot\frac{1}{N}\leq \text{const}\cdot\frac{1}{N}. \tag{3.57}$$

Therefore, Corollary 17, (3.53), (3.54), (3.57), Theorem 11, and Lemma 10 together imply that

$$\begin{aligned}
\|(\mathrm{I})\|_k^2 &\leq \text{const}\cdot\int_0^t\sum_{y\in\mathbf{Z}^d}P_{t-s}^2(y-x)\left\|\sigma^{(N)}\big(U_s^{(N)}(y)\big)-\sigma\big(U_s^{(N)}(y)\big)\right\|_k^2 ds\\
&\leq \text{const}\cdot\int_0^t\sum_{y\in\mathbf{Z}^d}P_{t-s}^2(y-x)\left(\left\|U_s^{(N)}(y)\cdot 1_{\{U_s^{(N)}(y)\geq N\}}\right\|_k^2+\left\|\frac{1}{N}\cdot 1_{\{\frac{1}{N}\geq U_s^{(N)}(y)\geq 0\}}\right\|_k^2\right) ds\\
&\leq \text{const}\cdot\int_0^t\sum_{y\in\mathbf{Z}^d}P_{t-s}^2(y-x)\cdot\frac{1}{N^2}\,ds\\
&= \text{const}\cdot t\cdot\frac{1}{N^2},
\end{aligned}\tag{3.58}$$

By Theorem 11 and Lemma 10, we have

$$\begin{aligned}
\|(\mathrm{II})\|_k^2 &\leq \text{const}\cdot\int_0^t\sum_{y\in\mathbf{Z}^d}P_{t-s}^2(y-x)\left\|\sigma\big(U_s^{(N)}(y)\big)-\sigma\big(U_s(y)\big)\right\|_k^2 ds\\
&\leq \text{const}\cdot\int_0^t\sum_{y\in\mathbf{Z}^d}P_{t-s}^2(y-x)\left\|U_s^{(N)}(y)-U_s(y)\right\|_k^2 ds.
\end{aligned}\tag{3.59}$$

Now define

$$\mathscr{D}_k(t) := \sup_{x\in\mathbf{Z}^d}\left\|U_t(x)-U_t^{(N)}(x)\right\|_k^2. \tag{3.60}$$

Theorem 11 ensures that $\mathscr{D}_k$ is bounded in $t\in[0,T]$ for every $k\in[2,\infty)$ and $N\in\mathbf{N}$. As a result, (3.58) and (3.59) together imply that

$$\begin{aligned}
\mathscr{D}_{k,N}(t) &\leq \text{const}\cdot t\cdot\frac{1}{N^2}+\text{const}\cdot\int_0^t\sum_{y\in\mathbf{Z}^d}P_{t-s}(y)^2\mathscr{D}_{k,N}(s)\,ds\\
&\leq \text{const}\cdot t\cdot\frac{1}{N^2}+\text{const}\cdot\int_0^t\mathscr{D}_{k,N}(s)\,ds \qquad \forall t\geq 0.
\end{aligned}\tag{3.61}$$

By Gronwall's inequality (see Appendix A.2), we have

$$\mathscr{D}_{k,N}(t)\leq \text{const}\cdot t\cdot\frac{1}{N^2}\cdot e^{\text{const}\cdot t} \qquad \forall N\geq 1, t\geq 0. \tag{3.62}$$

This completes the proof. **Q.E.D.**

Theorem 19. *Let $U_t(x)$ solve (SDE) with compactly supported σ such that supp$(\sigma) \subset [a,b]$, $a > 0$. Then there exists a sequence of $U_t^{(N)}(x)$ solving (SDE) with σ replaced by $\sigma^{(N)}$, where $\sigma^{(N)} \in \mathbf{C}_c^\infty(\mathbf{R})$ for all $N \in \mathbf{N}$ and supp$(\sigma^{(N)}) \subset [a_N, b_N]$, $a_N > 0$ for all N large, such that*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sup_{x\in K_N} \left\| U_t^{(N)}(x) - U_t(x) \right\|_k = 0 \tag{3.63}$$

for every $k \in [2, \infty)$.

Proof: Let $\phi \in \mathbf{C}_c^\infty\big((0,1)\big)$ with $\int_{\mathbf{R}} \phi(x)\,dx = 1$ and define

$$\sigma^{(N)}(x) := N \int_{\mathbf{R}} \phi\big(N(y-x)\big)\sigma(y)\,dy. \tag{3.64}$$

As we did in the proof of Theorem 18, we write $U_t^{(N)}(x) - U_t(x) = \text{(I)} + \text{(II)}$, where

$$\text{(I)} = \int_0^t \sum_{y\in\mathbf{Z}^d} P_{t-s}(y-x) \cdot \left[\sigma^{(N)}\big(U_s^{(N)}(y)\big) - \sigma\big(U_s^{(N)}(y)\big)\right] dB_s(y) \tag{3.65}$$

$$\text{(II)} = \int_0^t \sum_{y\in\mathbf{Z}^d} P_{t-s}(y-x) \cdot \left[\sigma\big(U_s^{(N)}(y)\big) - \sigma\big(U_s(y)\big)\right] dB_s(y). \tag{3.66}$$

Note that

$$\begin{aligned}\left|\sigma^{(N)}(U_s(y)) - \sigma(U_s(y))\right| &\leq N \int_{\mathbf{R}} \phi\big(N(z - U_s(y))\big) \left|\sigma(z) - \sigma(U_s(y))\right| dz \\ &\leq \text{const} \cdot N \int_{\mathbf{R}} \phi\big(N(z - U_s(y))\big) \cdot \frac{1}{N}\,dz \\ &\leq \text{const} \cdot \frac{1}{N}.\end{aligned} \tag{3.67}$$

Therefore, Theorem 11 and Lemma 10 imply that

$$\begin{aligned}\|\text{(I)}\|_k^2 &\leq \text{const} \cdot \int_0^t \sum_{y\in\mathbf{Z}^d} P_{t-s}^2(y-x) \left\|\sigma^{(N)}\big(U_s^{(N)}(y)\big) - \sigma\big(U_s^{(N)}(y)\big)\right\|_k^2 ds \\ &\leq \text{const} \cdot \int_0^t \sum_{y\in\mathbf{Z}^d} P_{t-s}^2(y-x) \cdot \frac{1}{N^2}\,ds \\ &= \text{const} \cdot t \cdot \frac{1}{N^2},\end{aligned} \tag{3.68}$$

Because $\sup_{x\in\mathbf{Z}^d} \left\|U_t(x) - U_t^{(N)}(x)\right\|_k^2$ is bounded in $t \in [0,T]$ for every $k \geq 1$ and $N \in \mathbf{N}$ (see Theorem 14), the rest of the proof follows exactly from the one in Theorem 18.

Q.E.D.

We conclude this section with the following result, which is a combination of several theorems presented in this section.

Theorem 20. *Let $U_t(x)$ solve (SDE). Then there exists a sequence of solutions $U_t^{(N)}(x)$ solving (3.44) with σ replaced by $\sigma^{(N)}$, where every $\sigma^{(N)} \in \mathbf{C}_c^\infty(\mathbf{R})$ for all $N \in \mathbf{N}$ and supp$(\sigma^{(N)}) \subset [a_N, b_N]$, $a_N > 0$ for all N large, such that*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sup_{x\in\mathbf{Z}^d} \left\| U_t^{(N)}(x) - U_t(x) \right\|_k = 0, \tag{3.69}$$

for every $k \in [2, \infty)$, $T > 0$.

Proof: By Theorem 18, the solution $U_t(x)$ to (SDE) can be approximated by the ones with compactly supported σ. By Theorem 19, the solution to (SDE) with compactly supported σ can be further approximated by the ones with smooth and compactly supported σ. Finally, we can approximate the solution $U_t(x)$ to (SDE) with smooth and compactly supported σ, by the solutions to (3.44) with the same σ, due to Theorem 14. **Q.E.D.**

3.3 Comparison principles for (SDE)

The goal of this section is to prove Theorem 1. We start with the comparison result under simplifications.

Theorem 21. *Consider two solutions $U_t^{(N)}$ and $V_t^{(N)}$ to (3.44) with the same initial conditions $U_0^{(N)} \equiv V_0^{(N)}$, but with different $\sigma = \sigma_1, \sigma_2$ such that $\sigma_1 \le \sigma_2$, and both σ_1 and σ_2 are twice continuously differentiable. In addition, we suppose supp$(\sigma) \subset [0, a]$, $a > 0$, and we define $I := [0, a]$. We write $K_N = \{x_1, \cdots, x_m\}$, and let $\mathbf{F_0}$ to be the class of functions $f : I^{K_N} \to \mathbf{R}$ such that f is twice differentiable in $x_1, \cdots, x_m$ with bounded continuous first and second derivatives, $\frac{\partial^2 f}{\partial x_i \partial x_j} \ge 0$ for all $1 \le i, j \le m$, and f is nondecreasing in each x_i, $1 \le i \le m$. Then for any $f_1, \cdots, f_n \in \mathbf{F_0}$ and $t_n > t_{n-1} > \cdots > t_1 \ge 0$,*

$$E\left[\prod_{j=1}^n f_j(U_{t_j}^{(N)})\right] \le E\left[\prod_{j=1}^n f_j(V_{t_j}^{(N)})\right]. \tag{3.70}$$

Proof: When both $U_t^{(N)}$ and $V_t^{(N)}$ are solutions to (3.44) and the underlying Brownian motions are independent, this theorem is a special case of Theorem 1 of [2]. We would like to demonstrate here how we follow the proof in [2], to prove the comparison result for (SDE) with a few adjustments.

By Corollary 17, we know that $U_t^{(N)}(x), V_t^{(N)}(x) \in I$ for every $x \in K_N$. Define two semigroups S^{σ_1} and S^{σ_2} associated with $U_t^{(N)}$ and $V_t^{(N)}$ by

$$S_t^{\sigma_1} f(x) = E_z \left[f(U_t^{(N)}(x_1), \cdots, U_t^{(N)}(x_m)) \right] \tag{3.71}$$

$$S_t^{\sigma_2} f(x) = E_z \left[f(V_t^{(N)}(x_1), \cdots, V_t^{(N)}(x_m)) \right]. \tag{3.72}$$

for any $t \geq 0, z \in I^{K_N}$, and Borel measurable function $f \geq 0$.

It is known that $U_t^{(N)}$ and $V_t^{(N)}$ are Feller processes. For a proof the reader could refer to, for example, Theorem 19.9 of [14], of which the proof also applies to the correlated Brownian motion case. Furthermore, given $f \in \mathbf{C}^2(I^{K_N})$, following the same proof as how Theorem 8.4.3 is done in [9], we have $S_t^{\sigma} f \in \mathbf{C}^2(I^{K_N})$ for $\sigma = \sigma_1, \sigma_2$.

When $\nu = 0$ in (3.44), the same proof of Proposition 16 in [2] also shows $S_t^{\sigma} f \in \mathbf{F_0}$ when $f \in \mathbf{F_0}$, for our case. When $\sigma \equiv 0$ in (3.44), the solution $X_t^{(N)}$ to it is given by

$$X_t^{(N)}(x) = \sum_{y \in K_N} \left(e^{\nu t(P-A)}\right)_{x,y} \cdot X_0^{(N)}(y), \tag{3.73}$$

where $x \in K_N$, $P := (p_{ij})_{i,j \in K_N}$, $A = (a_{ij})_{i,j \in K_N}$ such that $a_{ij} := \left(\sum_{k \in K_N} p_{ik}\right) \cdot \delta_{ij}$. The semigroup S associated with $X_t^{(N)}$ is given by

$$\begin{aligned} S_t f(z) &= E_z \left[f(X_t^{(N)}(x_1), \cdots, X_t^{(N)}(x_m)) \right] \\ &= f\left(\sum_{1 \leq i \leq d} \left(e^{\nu t(P-A)}\right)_{x_1, x_i} \cdot z_i, \cdots, \sum_{1 \leq i \leq d} \left(e^{\nu t(P-A)}\right)_{x_m, x_i} \cdot z_i \right). \end{aligned} \tag{3.74}$$

If $f \in \mathbf{F_0}$, then

$$\frac{\partial^2}{\partial z_i \partial z_j} (S_t f)(z) = \sum_{1 \leq i', j' \leq m} S_t \left(\frac{\partial^2}{\partial z_i \partial z_j} f \right)(z) \cdot \left(e^{\nu t(P-A)}\right)_{x'_i, x_i} \cdot \left(e^{\nu t(P-A)}\right)_{x'_j, x_j} \geq 0. \tag{3.75}$$

In (3.75), we have used the fact that $e^{\nu t(P-A)} = e^{\nu t P} \cdot e^{-\nu t A}$ is a nonnegative matrix because P is nonnegative and A is a diagonal matrix. The monotonicity of $S_t f(z)$ for each z_i also holds due to this fact. Therefore, $S_t f \in \mathbf{F_0}$.

To see that $S_t^{\sigma_1} f, S_t^{\sigma_2} f, \in \mathbf{F_0}$ when $f \in \mathbf{F_0}$, we first use Trotter product formula (see, for example, Corollary 1.6.7 of [4]), which shows

$$S_t^{\sigma} f = \lim_{n \to \infty} \left[S_{t/n}^{\sigma, \nu=0} S_{t/n}^{\sigma=0} \right]^n f, \tag{3.76}$$

where the limit exists in $C_0(I^{K_N})$. The Trotter product formula is applicable, because the infinitesimal generators G^σ, $G^{\sigma,\nu=0}$, and $G^{\sigma=0}$ for S_t^σ, $S_t^{\sigma,\nu=0}$, and $S_t^{\sigma=0}$ are given respectively by

$$G^\sigma := \nu \sum_{1\le j\le m} (p_{j,i} - \delta_{i,j}) z_j \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{1\le i,j\le m} \sigma(z_i)\mathcal{R}(i-j)\sigma(z_j)\frac{\partial^2}{\partial z_i \partial z_j}, \tag{3.77}$$

$$G^{\sigma,\nu=0} := \frac{1}{2} \sum_{1\le i,j\le m} \sigma(z_i)\mathcal{R}(i-j)\sigma(z_j)\frac{\partial^2}{\partial z_i \partial z_j}, \tag{3.78}$$

$$G^{\sigma=0} := \nu \sum_{1\le j\le m} (p_{j,i} - \delta_{i,j}) z_j \frac{\partial}{\partial z_i}, \tag{3.79}$$

where $z \in I^{K_N}$. See, for instance, Theorem 19.9 of [14] for reference.

If we define for any $f \in \mathbf{F_0}$,

$$S_{n,t}^\sigma f := \left[S_{t/n}^{\sigma,\nu=0} S_{t/n}^{\sigma=0} \right]^n f, \tag{3.80}$$

then we have $S_{n,t}^\sigma f \in \mathbf{F_0}$ from the previous discussions in this proof. We now follow the arguments in Proposition 16 of [2]: Let

$$\begin{aligned} u^0 &= z; \\ u^i &= z + h_i e_i; \\ u^j &= z + h_i e_j; \\ u^{ij} &= z + h_i ei + h_j e_j; \end{aligned} \tag{3.81}$$

for any $i \neq j, 1 \le i, j \le m$. Then,

$$S_{n,t}^\sigma f(u^{ij}) - S_{n,t}^\sigma f(u^i) - S_{n,t}^\sigma f(u^j) + S_{n,t}^\sigma f(u^0) \ge 0, \tag{3.82}$$

for any $i \neq j, 1 \le i, j \le m$, and

$$S_{n,t}^\sigma f(u^i) - S_{n,t}^\sigma f(u^0) \ge 0, \tag{3.83}$$

for any $1 \le i \le m$. Due to (3.76), (3.82) and (3.83) holds for $S_t^\sigma f$ as well. Because $S_t^\sigma f \in \mathbf{C}^2(I^{K_N})$, it turns out that $S_t^\sigma f \in \mathbf{F_0}$.

We would like to show

$$\left(S_t^{\sigma_1} f\right)(z) \le \left(S_t^{\sigma_2} f\right)(z), \tag{3.84}$$

for $t \ge 0$ and $f \in \mathbf{F_0}$. By the fundamental theorem of calculus on $\mathbf{C}(I^{K_N})$, we have

$$S_s^{\sigma_1} S_{t-s}^{\sigma_2} f\Big|_{s=0}^{s=t} = \int_0^t \left[S_s^{\sigma_1}(-G^{\sigma_2} S_{t-s}^{\sigma_2})f + (S_s^{\sigma_1} G^{\sigma_1}) S_{t-s}^{\sigma_2} f\right] ds. \tag{3.85}$$

Thus, we have the following integration by parts formula (as is done in [2]):

$$(S_t^{\sigma_1})f - (S_t^{\sigma_2})f = \int_0^t (S_s^{\sigma_1}(G^{\sigma_1} - G^{\sigma_2}) S_{t-s}^{\sigma_2}) f \, ds. \tag{3.86}$$

(3.84) then follows from (3.86) because $(G^{\sigma_1} - G^{\sigma_2}) \le 0$ by (3.77).

To show that (3.70) is true, we apply the Markov property of $U^{(N)}$ to see that

$$\begin{aligned} & E\left[\Pi_{j=1}^n f_j(U_{t_j}^{(N)})\right] \\ =& E\left[f_1(U_{t_1}^{(N)}) E_{U_{t_1}^{(N)}}\left[\Pi_{j=2}^n f_j(U_{t_j-t_1}^{(N)})\right]\right] \\ =& E\left[f_1(U_{t_1}^{(N)}) E_{U_{t_1}^{(N)}}\left[f_2(U_{t_2-t_1}^{(N)}) E_{U_{t_2-t_1}^{(N)}}\left[\Pi_{j=3}^n f_j(U_{t_j-t_2}^{(N)})\right]]\right]\right]] \\ =& \cdots = E\left[f_1(U_{t_1}^{(N)}) E_{U_{t_1}^{(N)}}\left[f_2(U_{t_2-t_1}^{(N)}) \cdots E_{U_{t_{n-1}}^{(N)}}\left[f_n(U_{t_n-t_{n-1}}^{(N)})\right]\right]\right]. \end{aligned} \tag{3.87}$$

Because $\mathbf{F_0}$ is closed under multiplication, and $S_t^{\sigma} f \in \mathbf{F_0}$ if $f \in \mathbf{F_0}$, by (3.87), we may find some $g \in \mathbf{F_0}$ such that

$$E\left[\prod_{j=1}^n f_j(U_{t_j}^{(N)})\right] = E\left[g(U_{t_j}^{(N)})\right]. \tag{3.88}$$

This reduces (3.70) to (3.84), which is shown already. So (3.70) is proved and hence the theorem. **Q.E.D.**

Lemma 22. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous function and let $\{X_1^{(N)}, \cdots, X_n^{(N)}\}_{N \in \mathbf{N}}$ be a family of random variables such that for all $1 \le i \le n$, $X_i^{(N)} \to X_i$ in probability. Also, assume that there exists $M < \infty$ so that $\forall n \ge 1$,*

$$\left\|f(X_1^{(N)}, \cdots, X_n^{(N)})\right\|_{k_1} \le M, \tag{3.89}$$

$$\left\|f(X_1, \cdots, X_n)\right\|_{k_2} \le M, \tag{3.90}$$

$$\max_{1 \le i \le N} E\left(|X_i^{(N)}|^{k_3}\right) \le M, \tag{3.91}$$

$$\max_{1 \le i \le N} E\left(|X_i|^{k_4}\right) \le M, \tag{3.92}$$

for some $k_1, k_2 \in (1,\infty)$ and $k_3, k_4 \in (0,\infty)$. Then we have

$$\lim_{N\to\infty} E\left[f(X_1^{(N)},\cdots,X_n^{(N)})\right] = E[f(X_1,\cdots,X_n)]. \tag{3.93}$$

Proof: Fix $A>0$, and let $X_i^{(N,A)} := X_i^{(N)} 1_{\{|X_i^{(N)}|\le A\}}$ and $X_i^{(A)} := X_i 1_{\{|X_i|\le A\}}$. Then,

$$\begin{aligned}
&\left|E[f(X_1^{(N)},\cdots,X_n^{(N)})] - E[f(X_1,\cdots,X_n)]\right| \\
&\qquad \le \left|E[f(X_1^{(N)},\cdots,X_n^{(N)})] - E[f(X_1^{(N,A)},\cdots,X_n^{(N,A)})]\right| \\
&\qquad + \left|E[f(X_1^{(N,A)},\cdots,X_n^{(N,A)})] - E[f(X_1^{(A)},\cdots,X_n^{(A)})]\right| \\
&\qquad + \left|E[f(X_1^{(A)},\cdots,X_n^{(A)})] - E[f(X_1,\cdots,X_n)]\right| \\
&\qquad = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).
\end{aligned} \tag{3.94}$$

To approximate (I), we note that

$$\begin{aligned}
&\left|E[f(X_1^{(N)},\cdots,X_n^{(N)})] - E[f(X_1^{(N,A)},\cdots,X_n^{(N,A)})]\right| \\
&\qquad \le E\left[|f(X_1^{(N)},\cdots,X_n^{(N)})| \cdot \left(1_{\bigcup_{i=1}^n \{X_i^{(N)} > A\}}\right)\right] \\
&\qquad \le \left\|f(X_1^{(N)},\cdots,X_n^{(N)})\right\|_{k_1} \cdot \left(\sum_{i=1}^n P(X_i^{(N)} > A)\right)^{\frac{k_1-1}{k_1}} \\
&\qquad \le M \cdot \left(\sum_{i=1}^n \frac{M}{A^{k_3}}\right)^{(k_1-1)/k_1},
\end{aligned} \tag{3.95}$$

which is small when A is large. To approximate (II), we note that for each fixed $A>0$, $f(z_1 1_{\{|z_1|\le A\}},\cdots,z_n 1_{\{|z_n|\le A\}})$ is a bounded continuous function on $\mathbf{R}^n$. So (II) is small when N is large, by the convergence in probability of each $X_i^{(N)}$ to X_i as $N\to\infty$ $(1\le i\le n)$.

To approximate (III), we use the same technique as the one for (I). Namely,

$$\begin{aligned}
&\left|E[f(X_1,\cdots,X_n)] - E[f(X_1^{(A)},\cdots,X_n^{(A)})]\right| \\
&\le E\left[|f(X_1,\cdots,X_n)| \cdot \left(1_{\bigcup_{i=1}^n \{X_i > A\}}\right)\right] \\
&\le \left\|f(X_1,\cdots,X_n)\right\|_{k_2} \cdot \left(\sum_{i=1}^n P(X_i > A)\right)^{\frac{k_2-1}{k_2}} \\
&\le M \cdot \left(\sum_{i=1}^n \frac{M}{A^{k_4}}\right)^{\frac{k_2-1}{k_2}}.
\end{aligned} \tag{3.96}$$

We first pick $A>0$ large so that (I) and (III) are both small. Then we fix this A and then let N go to infinity. The lemma is thus proved. **Q.E.D.**

3.3.1 Proof of Theorem 1

Given $t_n > t_{n-1} > \cdots > t_1 \geq 0$, we define $f_i(x) = x_{i,1}^{k_{i,1}} \cdots x_{i,n_i}^{k_{i,n_i}}$, where $x \in I^{K_N}$, $x_1, \cdots, x_{i,n_i} \in K_N$, and $k_{i,j} \geq 0$ for $1 \leq i \leq n$ and $1 \leq j \leq n_i$. It can be seen that $f_i \in \mathbf{F_0}$ for all $1 \leq i \leq n$.

By Theorem 21, if $U_t^{(N)}$ and $V_t^{(N)}$ are solutions to (3.44) with the same initial conditions $U_0^{(N)} \equiv V_0^{(N)}$, such that $\sigma_1 \leq \sigma_2$, then

$$E\left[\prod_{i=1}^{n} U_{t_i}^{(N)}(x_{i,1})^{k_{i,1}} \cdots U_{t_i}^{(N)}(x_{i,n_i})^{k_{i,n_i}}\right] \leq E\left[\prod_{i=1}^{n} V_{t_i}^{(N)}(x_{i,1})^{k_{i,1}} \cdots U_{t_i}^{(N)}(x_{i,n_i})^{k_{i,n_i}}\right]. \quad (3.97)$$

Now we apply Lemma 22. Let $f = f_1 \cdots f_n$, assumptions (3.89), (3.90), (3.91), and (3.92) are satisfied, due to (3.14) and Theorem 20. Also, theorem 20 implies the convergence in probability of $U_t^{(N)}(x)$ to $U_t(x)$. Therefore, we let $N \to \infty$ in (3.97) to obtain

$$E\left[\prod_{i=1}^{n} U_{t_i}(x_{i,1})^{k_{i,1}} \cdots U_{t_i}(x_{i,n_i})^{k_{i,n_i}}\right] \leq E\left[\prod_{i=1}^{n} V_{t_i}(x_{i,1})^{k_{i,1}} \cdots V_{t_i}(x_{i,n_i})^{k_{i,n_i}}\right]. \quad (3.98)$$

Relabel all t_i's and x_i's to conclude the proof of Theorem 1. **Q.E.D.**

CHAPTER 4

FROM INTERACTING SDES TO SHE(1): $L^K(P)$ APPROXIMATION

Throughout this chapter, SHE(1) denotes the stochastic heat equation defined on page 2. Let $u_t(x)$ be the solution to (SHE(1)) with initial data $u_0(x)$. We would like to define a family of SDE systems accordingly.

We say that $V_t^{(\epsilon)}(x)$ solves **(SDE(ϵ))** if

$$dV_t^{(\epsilon)}(x) = \big(\mathscr{L}^{(\epsilon)} V_t^{(\epsilon)}\big)(x)dt + \sigma\big(V_t^{(\epsilon)}(x)\big)dB_t^{(\epsilon)}(x), \qquad x \in (\epsilon \mathbf{Z})^d, d \geq 1, \tag{4.1}$$

such that:

1. $\{B_t(x) := \epsilon^{-d} \int_{(0,t)\times C^{(\epsilon)}(x)} \eta(ds, dy)\}_{x \in (\epsilon\mathbf{Z})^d}$ is a family of correlated Brownian motions, where $C^{(\epsilon)}(x) := \Pi_{j=1}^d [x_j, x_j + \epsilon)$ for $x = (x_1, \cdots, x_d)$.

2. $\mathscr{L}^{(\epsilon)} g(j) := \nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \sum_{i \in (\epsilon\mathbf{Z})^d} p_{j,i}^{(\epsilon,\alpha,d)}(g(i) - g(j))$ for all $j \in (\epsilon\mathbf{Z})^d$ and $g : (\epsilon\mathbf{Z})^d \to \mathbf{R}$, where

$$n_{\alpha,d} := 2\zeta(\alpha+1) \qquad \text{when } 0 < \alpha < 2, d = 1, \tag{4.2}$$

$$n_{\alpha,d} := \sum_{\substack{\mathbf{n}=(n_1,\cdots,n_d)\in\mathbf{Z}^d \\ n_1\neq 0,\cdots,n_d\neq 0}} |\mathbf{n}|^{-\alpha-d} + 2d \qquad \text{when } 0 < \alpha < 2, d > 1, \tag{4.3}$$

$$C_{\alpha,d} := n_{\alpha,d} \cdot \left(\int_{\mathbf{R}^d} \frac{1-\cos(x\cdot e_1)}{|x|^{\alpha+d}}\, dx \right)^{-1} \qquad \text{when } 0 < \alpha < 2, \tag{4.4}$$

$$C_{\alpha,d} := 2d \qquad \text{when } \alpha = 2. \tag{4.5}$$

Here in (4.2) $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function. When $0 < \alpha < 2$, $d = 1$,

$$p_{i,j}^{(\epsilon,\alpha,d)} = p_{j,i}^{(\epsilon,\alpha,d)} = n_{\alpha,d}^{-1} \cdot \frac{1}{2|i/\epsilon - j/\epsilon|^{\alpha+1}} \tag{4.6}$$

for all $i \neq j$, $i, j \in \epsilon\mathbf{Z}$, and $p_{i,i}^{(\epsilon,\alpha)} = 0$ for all $i \in \epsilon\mathbf{Z}$.

When $0 < \alpha < 2$, $d > 1$,

$$p_{i,j}^{(\epsilon,\alpha,d)} = p_{j,i}^{(\epsilon,\alpha,d)} = n_{\alpha,d}^{-1} \cdot \frac{1}{|i/\epsilon - j/\epsilon|^{\alpha+d}} \tag{4.7}$$

for all $i = (i_1, \cdots, i_d), j = (j_1, \cdots, j_d) \in (\epsilon \mathbf{Z})^d$ such that $i_k \neq j_k$ for all $1 \leq k \leq d$, or $\sum_{k=1}^d |i_k - j_k| = 1$. $p_{i,j}^{(\epsilon,\alpha,d)} = 0$ otherwise.

When $\alpha = 2$,

$$p_{i,j}^{(\epsilon,\alpha,d)} = p_{j,i}^{(\epsilon,\alpha,d)} = \frac{1}{2d} \tag{4.8}$$

for all $i, j \in (\epsilon \mathbf{Z})^d$ such that $|i - j| = \epsilon$, and $p_{i,j}^{(\epsilon,\alpha,d)} = 0$ otherwise.

3. σ, ν, α, and $\eta(t, x)$ that appeared in (SDE(ϵ)) are the same as the ones in (SHE(1)).

4. The initial condition $U_0^{(\epsilon)}(x) = u_0(x)$ for all $x \in (\epsilon \mathbf{Z})^d$.

The goal of this chapter is to prove the following result:

Theorem 23. *Let $U_t^{(\epsilon)}(x)$ be the solution to (SDE(ϵ)), and let $u_t(x)$ be the solution to (SHE(1)). Then,*

$$\lim_{\epsilon \downarrow 0} \sup_{t \in [T_1, T_2]} \sup_{x \in \mathbf{R}^d} \left\| U_t^{(\epsilon)}(\epsilon[x/\epsilon]) - u_t(x) \right\|_k = 0 \tag{4.9}$$

for every $k \in [2, \infty)$ and $T_2 > T_1 > 0$. If, in particular, $u_0 \equiv c$ for some constant $c \geq 0$, then,

$$\lim_{\epsilon \downarrow 0} \sup_{t \in [0,T]} \sup_{x \in \mathbf{R}^d} \left\| U_t^{(\epsilon)}(\epsilon[x/\epsilon]) - u_t(x) \right\|_k = 0 \tag{4.10}$$

for every $k \in [2, \infty)$ and $T > 0$.

Following the arguments in Section 2.1 and 2.2.1, the solution $U_t^{(\epsilon)}(x)$ to (SDE(ϵ)) satisfies the following form:

$$U_t^{(\epsilon)}(x) = \sum_{y \in (\epsilon \mathbf{Z})^d} P_t^{(\epsilon)}(y - x) \cdot u_0(y) + \sum_{y \in (\epsilon \mathbf{Z})^d} \int_0^t P_{t-s}^{(\epsilon)}(y - x) \sigma(U_s^{(\epsilon)}(y)) \, dB_s^{(\epsilon)}(y), \tag{4.11}$$

where

$$P_t^{(\epsilon)}(x) := \sum_{n=0}^{\infty} (\widetilde{P}_\epsilon)_{0,x}^n \frac{e^{-\nu \cdot C_\alpha \cdot \epsilon^{-\alpha} \cdot t} (\nu \cdot C_\alpha \cdot \epsilon^{-\alpha} \cdot t)^n}{n!}, \tag{4.12}$$

where $\widetilde{P}_\epsilon := \big(p_{i,j}^{(\epsilon,\alpha,d)}\big)_{i,j\in(\epsilon\mathbf{Z})^d}$ is a probability transition matrix. When $0<\alpha<2$, $d>1$, the characteristic function $\psi_{\epsilon,\alpha,d}$ of $P_t^{(\epsilon)}$ equals

$$
\begin{aligned}
\psi_{\epsilon,\alpha,d}(z) &= \sum_{x\in(\epsilon\mathbf{Z})^d} P_t^{(\epsilon)}(x) e^{iz\cdot x} \\
&= \sum_{n=0}^{\infty}\left(\sum_{x\in(\epsilon\mathbf{Z})^d} p_{0,x}^{(\epsilon,\alpha)} e^{iz\cdot x}\right)^n \cdot \frac{e^{-\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t}(\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t)^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(n_{\alpha,d}^{-1}\cdot \sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in(\epsilon\mathbf{Z})^d\\ x_1\neq 0,\cdots,x_d\neq 0 \text{ or}\\ |x_1|+\cdots+|x_d|=\epsilon}} \frac{1}{|x/\epsilon|^{\alpha+d}}\cdot e^{iz\cdot x}\right)^n \cdot \frac{e^{-\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t}(\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t)^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(n_{\alpha,d}^{-1}\cdot \sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d\\ x_1\neq 0,\cdots,x_d\neq 0 \text{ or}\\ |x_1|+\cdots+|x_d|=1}} \frac{1}{|x|^{\alpha+d}}\cdot e^{i\epsilon z\cdot x}\right)^n \cdot \frac{e^{-\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t}(\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t)^n}{n!} \\
&= \sum_{n=0}^{\infty}(\phi_{\alpha,d}(\epsilon z))^n\cdot \frac{e^{-\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t}(\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t)^n}{n!} \\
&= e^{-\nu\cdot C_\alpha\cdot\epsilon^{-\alpha}\cdot t(1-\phi_{\alpha,d}(\epsilon z))},
\end{aligned}
\tag{4.13}
$$

where

$$
\phi_{\alpha,d}(z) := n_{\alpha,d}^{-1}\cdot\left(\sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d\\ x_1\neq 0,\cdots,x_d\neq 0}} \frac{1}{|x|^{\alpha+d}}\cdot e^{iz\cdot x} + \sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d\\ |x_1|+\cdots+|x_d|=1}} \frac{1}{|x|^{\alpha+d}}\cdot e^{iz\cdot x}\right). \tag{4.14}
$$

When $0<\alpha<2$, $d=1$, (4.13) holds with

$$
\phi_{\alpha,d}(z) := n_{\alpha,d}^{-1}\cdot\sum_{x\in\mathbf{Z},x\neq 0}\frac{1}{|x|^{\alpha+d}}\cdot e^{iz\cdot x}. \tag{4.15}
$$

When $\alpha=2$, (4.13) still holds with

$$
\phi_{\alpha,d}(z) := \sum_{j=1}^{d}\frac{1}{d}\cos(z_j). \tag{4.16}
$$

Lemma 24. *Let $\phi_{\alpha,d}(z)$ be as defined in (4.14), (4.15), and (4.16), and $C_{\alpha,d}$ be as defined in (4.4) and (4.5). Write*

$$
1-\phi_{\alpha,d}(z) - C_{\alpha,d}^{-1}|z|^\alpha = R_{\alpha,d}(z). \tag{4.17}
$$

Then:

(1) There exists $C > 0$ *such that for all* $|z| \le 1$*,*

$$R_{\alpha,d}(z) \le C|z|^{1+\alpha} \qquad \text{when } 0 < \alpha < 1,$$
$$R_{\alpha,d}(z) \le C|z|^2 \ln(|z|^{-1}) \qquad \text{when } \alpha = 1,$$
$$R_{\alpha,d}(z) \le C|z|^2 \qquad \text{when } 1 < \alpha < 2,$$
$$R_{\alpha,d}(z) \le C|z|^3 \qquad \text{when } \alpha = 2; \tag{4.18}$$

(2) For every constant $0 < c_1 < \pi$*, there exists* $c_2 > 0$ *such that*

$$1 - \phi_{\alpha,d}(z) > c_2 \tag{4.19}$$

for all $z \in [-\pi, \pi]^d \setminus [-c_1, c_1]^d$*,*

Proof: First we prove (1). This is done in two cases:

Case 1. $\mathbf{0} < \alpha < \mathbf{2}$, $\mathbf{d} > \mathbf{1}$.

We have

$$\begin{aligned} & n_{\alpha,d} \cdot \left(1 - \phi_{\alpha,d}(z)\right) \\ &= \sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d \\ |x_1|+\cdots+|x_d|=1}} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}} + \sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d \\ x_1\neq 0,\cdots,x_d\neq 0}} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}} - \int_{\mathbf{R}^d} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}}\,dx \\ &+ \int_{\mathbf{R}^d} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}}\,dx, \end{aligned} \tag{4.20}$$

where

$$\int_{\mathbf{R}^d} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}}\,dx = \int_{\mathbf{R}^d} \frac{1-\cos(|z|e_1\cdot x)}{|x|^{\alpha+d}}\,dx = \int_{\mathbf{R}^d} \frac{1-\cos(x\cdot e_1)}{|x|^{\alpha+d}}\,dx \cdot |z|^\alpha. \tag{4.21}$$

Also,

$$\sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d \\ |x_1|+\cdots+|x_d|=1}} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}} \le \sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d \\ |x_1|+\cdots+|x_d|=1}} \frac{|z|^2|x|^2}{|x|^{\alpha+d}} \le \text{const} \cdot |z|^2. \tag{4.22}$$

Now we define

$$R := \{(x_1, \ldots, x_d) : x_1 > 0, \ldots, x_d > 0\}, \tag{4.23}$$

$$D := \{(x_1, \ldots, x_d) : 0 \le x_j \le 1 \text{ for all } 1 \le j \le d\}, \tag{4.24}$$

$$\mathbf{1} := (1, \ldots, 1), \tag{4.25}$$

$$[x] := ([x_1], \ldots, [x_d]) \text{ for } x = (x_1, \ldots, x_d). \tag{4.26}$$

To estimate

$$\sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d\\ x_1\neq 0,\cdots,x_d\neq 0}} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}} - \int_{\mathbf{R}^d} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}}\,dx, \tag{4.27}$$

it suffices to consider

$$Q := \Big|\sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{N}^d\\ x_1\neq 0,\cdots,x_d\neq 0}} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}} - \int_{R} \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}}\,dx\Big|. \tag{4.28}$$

(the estimates on the other quadrants are basically the same)

Now

$$\begin{aligned}
Q &= \Big|\int_R \frac{1-\cos(z\cdot([x]+\mathbf{1}))}{|[x]+\mathbf{1}|^{\alpha+d}}\,dx - \int_R \frac{1-\cos(z\cdot x)}{|x|^{\alpha+d}}\,dx\Big| \\
&\le \int_{R\setminus D}\Big|(1-\cos(z\cdot([x]+\mathbf{1})))\cdot\Big(\frac{1}{|[x]+\mathbf{1}|^{\alpha+d}}-\frac{1}{|x|^{\alpha+d}}\Big)\Big|\,dx \\
&+ \int_{R\setminus D}\Big|\frac{\cos(z\cdot([x]+\mathbf{1}))-\cos(z\cdot x)}{|x|^{\alpha+d}}\Big|\,dx + \frac{1-\cos(z\cdot\mathbf{1})}{|\mathbf{1}|^{\alpha+d}} + \int_D \frac{|1-\cos(z\cdot x)|}{|x|^{\alpha+d}}\,dx \\
&= \text{(I)}+\text{(II)}+\text{(III)}+\text{(IV)}.
\end{aligned} \tag{4.29}$$

Estimate of (I). When $|z|\le 1$ we have

$$\begin{aligned}
\text{(I)} &\le \text{const}\cdot\int_{R\setminus D} 2\sin^2\Big(\frac{z\cdot([x]+\mathbf{1})}{2}\Big)\cdot\frac{1}{|x|^{\alpha+d+1}}dx \\
&\le \text{const}\cdot\int_{R\setminus D}\Big(2\wedge\frac{1}{4}|([x]+\mathbf{1})|^2|z|^2\Big)\cdot\frac{1}{|x|^{\alpha+d+1}}dx \\
&\le \text{const}\cdot\int_{R\setminus D}(2\wedge 2|x|^2|z|^2)\cdot\frac{1}{|x|^{\alpha+d+1}}dx \\
&\le \text{const}\cdot\int_1^\infty (1\wedge r^2|z|^2)\cdot\frac{1}{r^{\alpha+d+1}}r^{d-1}dr \\
&= \text{const}\cdot\int_1^{|z|^{-1}} r^2|z|^2\cdot\frac{1}{r^{\alpha+2}}dr + \text{const}\cdot\int_{|z|^{-1}}^\infty \frac{1}{r^{\alpha+2}}\,dr \\
&\le \text{const}\cdot\left(\int_1^{|z|^{-1}}\frac{1}{r^\alpha}dr\right)\cdot|z|^2 + \text{const}\cdot|z|^{\alpha+1},
\end{aligned} \tag{4.30}$$

where (4.30) $\le$ const$\cdot|z|^{\alpha+1}$ when $0<\alpha<1$, (4.30) $\le$ const$\cdot|z|^2$ when $1<\alpha<2$, and (4.30) $\le$ const$\cdot|z|^2\ln(|z|^{-1})$ when $\alpha=1$.

Estimate of (II). When $|z| \leq 1$ we have

$$
\begin{aligned}
(\text{II}) &\leq \text{const} \cdot \int_{R\setminus D} 2\Big| \sin\big(\frac{([x]+x+\mathbf{1})\cdot z}{2}\big) \sin\big(\frac{([x]-x+\mathbf{1})\cdot z}{2}\big)\Big| \cdot \frac{1}{|x|^{\alpha+d}} dx \\
&\leq \text{const} \cdot \int_{R\setminus D} \big(1 \wedge |[x]+x+\mathbf{1}| \cdot |z|\big) \cdot |z| \cdot \frac{1}{|x|^{\alpha+d}} dx \\
&\leq \text{const} \cdot \int_{R\setminus D} \big(1 \wedge |x| \cdot |z|\big) \cdot |z| \cdot \frac{1}{|x|^{\alpha+d}} dx \\
&\leq \text{const} \cdot \int_{1}^{\infty} \big(1 \wedge r|z|\big) \cdot |z| \cdot \frac{1}{r^{\alpha+d}} \cdot r^{d-1} dr \\
&\leq \text{const} \cdot \big(\int_{1}^{|z|^{-1}} \frac{1}{r^{\alpha}} dr\big) \cdot |z|^2 + \text{const} \cdot \big(\int_{|z|^{-1}}^{\infty} \frac{1}{r^{\alpha+1}} dr\big) \cdot |z|,
\end{aligned}
\tag{4.31}
$$

where (4.31) $\leq$ const$\cdot|z|^{\alpha+1}$ when $0 < \alpha < 1$, (4.31) $\leq$ const$\cdot|z|^2$ when $1 < \alpha < 2$, and (4.31) $\leq$ const$\cdot|z|^2 \ln(|z|^{-1})$ when $\alpha = 1$.

Estimate of (III) and (IV). Because $|1 - \cos(z \cdot \mathbf{1})| \leq \text{const} \cdot |z|^2$, (III) $\leq \text{const} \cdot |z|^2$.

Also, it follows that

$$
(\text{IV}) \leq \text{const} \cdot \left(\int_D \frac{1}{|x|^{\alpha+d-2}} dx \right) \cdot |z|^2 = \text{const} \cdot |z|^2. \tag{4.32}
$$

Case 2. $\mathbf{0 < \alpha < 2, d = 1}$. This can be done using the same approach as the previous case, and the end result is that (4.18) holds.

Case 3. α= **2**. We have

$$
\begin{aligned}
1 - \phi_{\alpha,d}(z) &= \frac{1}{d} \sum_{j=1}^{d} \big(1 - \cos(z_j)\big) \\
&= \frac{2}{d} \sum_{j=1}^{d} \sin^2\left(\frac{z_j}{2}\right) \\
&= \frac{1}{2d}|z|^2 + \frac{2}{d} \sum_{j=1}^{d} \left(\sin^2\big(\frac{z_j}{2}\big) - \frac{z_j^2}{4} \right).
\end{aligned}
\tag{4.33}
$$

Because $|x - \sin(x)| \leq |x^3|/6$ for all $x \in \mathbf{R}$, when $|z| \leq 1$,

$$
\begin{aligned}
\sum_{j=1}^{d} \left| \sin^2\left(\frac{z_j}{2}\right) - \frac{z_j^2}{4} \right| &= \sum_{j=1}^{d} \left| \sin\left(\frac{z_j}{2}\right) + \frac{z_j}{2} \right| \cdot \left| \sin\left(\frac{z_j}{2}\right) - \frac{z_j}{2} \right| \\
&\leq \text{const} \cdot \sum_{j=1}^{d} |z_j|^3 \leq \text{const} \cdot |z|^3.
\end{aligned}
\tag{4.34}
$$

Now we prove assertion (2). For all $0 < \alpha \leq 2$ and $d \geq 1$,

$$
1 - \phi_\alpha(z) \geq \text{const} \cdot \sum_{\substack{\mathbf{x}=(x_1,\cdots,x_d)\in\mathbf{Z}^d \\ |x_1|+\cdots+|x_d|=1}} \big(1 - \cos(z \cdot x)\big) = \text{const} \cdot \sum_{j=1}^{d} \big(1 - \cos(z_j)\big). \tag{4.35}
$$

It can then be easily checked that assertion (2) is true. **Q.E.D.**

We are now ready for the next lemma, which is one of the keys to prove Theorem 23.

Lemma 25. *Fix $T_2 > T_1 > 0$. $\epsilon^{-d} P_t^{(\epsilon)}(x) \to p_t(x)$ uniformly for $t \in [T_1, T_2]$ and $x \in (\epsilon \mathbf{Z})^d$ as $\epsilon \downarrow 0$, where $P_t^{(\epsilon)}(x)$ is defined in (4.12), and $p_t(x)$ is defined in (1.3).*

Proof: Since $P_t^{(\epsilon)}$ is supported on $(\epsilon \mathbf{Z})^d$,

$$\left(\frac{2\pi}{\epsilon}\right)^d P_t^{(\epsilon)}(x) = \int_{[-\pi/\epsilon, \pi/\epsilon]^d} e^{-iz\cdot x} \cdot \psi_{\epsilon,\alpha,d}(z)\, dz$$
$$= \int_{[-\pi/\epsilon, \pi/\epsilon]^d} e^{-iz\cdot x} \cdot e^{-\nu \cdot C_\alpha \cdot \epsilon^{-\alpha} \cdot t(1-\phi_{\alpha,d}(\epsilon z))}\, dz. \tag{4.36}$$

By Assertion 1 of Lemma 24, there exists $0 < C < d^{-1/2} < \pi$ and $C' > 0$ such that for all $z = (z_1, \cdots, z_d)$, $|z_1| \le C, \cdots, |z_d| \le C$,

$$1 - \phi_{\alpha,d}(z) \ge C'|z|^\alpha. \tag{4.37}$$

$$(2\pi)^d \left| \frac{P_t^{(\epsilon)}(x)}{\epsilon^d} - p_t(x) \right| \le \int_{[-C/\epsilon, C/\epsilon]^d} \left| e^{-\nu t |z|^\alpha} - e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t(1-\phi_{\alpha,d}(\epsilon z))} \right| dz$$
$$+ \int_{[-\pi/\epsilon, \pi/\epsilon]^d \setminus [-C/\epsilon, C/\epsilon]^d} e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t(1-\phi_{\alpha,d}(\epsilon z))}\, dz + \int_{\mathbf{R}^d \setminus [-C/\epsilon, C/\epsilon]^d} e^{-\nu t |z|^\alpha}\, dz.$$

It is easily seen that $\int_{\mathbf{R}^d \setminus [-C/\epsilon, C/\epsilon]^d} e^{-\nu t |z|^\alpha}\, dz \to 0$ uniformly for $t \ge T_1$ as $\epsilon \downarrow 0$. By assertion 2 of Lemma 24, there exists $C'' > 0$ so that

$$\int_{[-\pi/\epsilon, \pi/\epsilon]^d \setminus [-C/\epsilon, C/\epsilon]^d} e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t(1-\phi_{\alpha,d}(\epsilon z))}\, dz \le \int_{[-\pi/\epsilon, \pi/\epsilon]^d \setminus [-C/\epsilon, C/\epsilon]^d} e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t \cdot C''}\, dz$$
$$\le e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t \cdot C''} \cdot \frac{(2\pi)^d}{\epsilon^d}, \tag{4.38}$$

which goes to 0 uniformly for $t \ge T_1$ as $\epsilon \downarrow 0$.

The last estimate is given by

$$\int_{[-C/\epsilon, C/\epsilon]^d} \left| e^{-\nu t |z|^\alpha} - e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t(1-\phi_{\alpha,d}(\epsilon z))} \right| dz$$
$$\le \int_{[-C/\epsilon, C/\epsilon]^d} e^{-\nu t |z|^\alpha} \left| 1 - e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t(1-\phi_{\alpha,d}(\epsilon z) - C_{\alpha,d}^{-1} \cdot |\epsilon z|^\alpha)} \right| dz. \tag{4.39}$$

Due to assertion (1) of Lemma 24, because $|\epsilon z| < 1$, there exists some $a, b, C > 0$ depends only on α, d so that

$$\left| \nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t(1 - \phi_{\alpha,d}(\epsilon z) - C_{\alpha,d}^{-1} \cdot |\epsilon z|^\alpha) \right| \le C \cdot t \cdot \epsilon^{-\alpha} \cdot R_{\alpha,d}(\epsilon z) \le C \cdot t \cdot \epsilon^a |z|^b, \tag{4.40}$$

which goes to 0 as $\epsilon \downarrow 0$. For all $t \leq T_2$, either

$$\left|1 - e^{-\nu \cdot C_\alpha \cdot \epsilon^{-\alpha} \cdot t(1-\phi_\alpha(\epsilon z) - C_\alpha^{-1} \cdot |\epsilon z|^\alpha)}\right| \leq \left|1 - e^{-C \cdot T_2 \cdot \epsilon^a |z|^b}\right| \tag{4.41}$$

or

$$\left|1 - e^{-\nu \cdot C_\alpha \cdot \epsilon^{-\alpha} \cdot t(1-\phi_\alpha(\epsilon z) - C_\alpha^{-1} \cdot |\epsilon z|^\alpha)}\right| \leq \left|e^{C \cdot T_2 \cdot \epsilon^a |z|^b} - 1\right| \tag{4.42}$$

holds. So the integrand in (4.39) converges to 0 pointwise as $\epsilon \downarrow 0$, uniformly in $t \leq T_2$. Besides, by (4.37), for $t \geq T_1$ we have

$$\begin{aligned}
1_{[-C/\epsilon, C/\epsilon]^d}(z) \cdot & \left|e^{-\nu t |z|^\alpha} - e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot t(1-\phi_{\alpha,d}(\epsilon z))}\right| \\
& \leq 1_{[-C/\epsilon, C/\epsilon]^d}(z) \cdot \left(e^{-\nu T_1 |z|^\alpha} + e^{-\nu \cdot C_{\alpha,d} \cdot \epsilon^{-\alpha} \cdot T_1 \cdot C' |\epsilon z|^\alpha}\right) \\
& \leq e^{-\nu T_1 |z|^\alpha} + e^{-\nu \cdot C_{\alpha,d} \cdot T_1 \cdot C' |z|^\alpha}.
\end{aligned} \tag{4.43}$$

This function is integrable on $\mathbf{R}^d$. So the dominated convergence theorem is applicable to (4.39) to show uniform convergence of (4.39) in $t \in [T_1, T_2]$ as $\epsilon \downarrow 0$. In the end, we remind the reader the convergence is uniform in $x \in \mathbf{R}^d$, because all the convergence in the proof do not depend on $x \in \mathbf{R}^d$. **Q.E.D.**

4.1 Proof of Theorem 23

We recall from (1.2) that the solution $u_t(x)$ to (SHE(1)) satisfies

$$u_t(x) := (p_t * u_0)(x) + \int_{(0,t) \times \mathbf{R}^d} p_{t-s}(y-x) \sigma\big(u_s(y)\big)\, \eta(ds\, dy),$$

where $(p_t * u_0)(x) := \int_{\mathbf{R}^d} p_t(x-y) u_0(y)\, dy$. Fix $\delta > 0$. We introduce the following random fields indexed by $x \in \mathbf{R}^d$ and $t \geq \delta$:

$$u_t^{(1,\delta)}(x) := (p_t * u_0)(x) + \int_{(0,t-\delta) \times \mathbf{R}^d} p_{t-s}(y-x) \cdot \sigma\big(u_s(y)\big)\, \eta(ds\, dy), \tag{4.44}$$

$$u_t^{(2,\epsilon,\delta)}(x) := (p_t * u_0)(x) + \int_{(0,t-\delta) \times \mathbf{R}^d} p_{t-s}(y-x) \cdot \sigma\big(u_s(\epsilon[y/\epsilon])\big)\, \eta(ds\, dy), \tag{4.45}$$

$$u_t^{(3,\epsilon,\delta)}(x) := (p_t * u_0)(x) + \int_{(0,t-\delta) \times \mathbf{R}^d} p_{t-s}(\epsilon[y/\epsilon] - \epsilon[x/\epsilon]) \cdot \sigma\big(u_s(\epsilon[y/\epsilon])\big)\, \eta(ds\, dy), \tag{4.46}$$

$$u_t^{(4,\epsilon,\delta)}(x) := (p_t * u_0)(x) + \int_{(0,t-\delta) \times \mathbf{R}^d} \frac{P_{t-s}^{(\epsilon)}(\epsilon[y/\epsilon] - \epsilon[x/\epsilon])}{\epsilon^d} \cdot \sigma\big(u_s(\epsilon[y/\epsilon])\big)\, \eta(ds\, dy), \tag{4.47}$$

When $t \leq \delta$,

$$u_t^{(2,\epsilon,\delta)}(x) = u_t^{(3,\epsilon,\delta)}(x) = u_t^{(4,\epsilon,\delta)}(x) := (p_t * u_0)(x). \tag{4.48}$$

We then define another random field indexed by $x \in \mathbf{R}^d$ and $t \geq 0$:

$$u_t^{(5,\epsilon)}(x) := (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}^d} \frac{P_{t-s}^{(\epsilon)}(\epsilon[y/\epsilon] - \epsilon[x/\epsilon])}{\epsilon^d} \cdot \sigma\big(u_s(\epsilon[y/\epsilon])\big)\, \eta(ds\, dy). \tag{4.49}$$

Finally, we also recall from (4.11) and (2.23) that

$$\begin{aligned}
&U_t^{(\epsilon)}(\epsilon[x/\epsilon]) \\
&= \sum_{y\in(\epsilon\mathbf{Z})^d} P_t^{(\epsilon)}(y - \epsilon[x/\epsilon]) \cdot u_0(y) + \sum_{k\in\mathbf{Z}^d} \int_0^t P_{t-s}^{(\epsilon)}(\epsilon k - \epsilon[x/\epsilon]) \cdot \sigma\big(U_s(\epsilon k)\big)\, dB_s^{(\epsilon)}(\epsilon k) \\
&= \sum_{y\in(\epsilon\mathbf{Z})^d} P_t^{(\epsilon)}(y - \epsilon[x/\epsilon]) \cdot u_0(y) \\
&\quad + \sum_{k\in\mathbf{Z}^d} \int_{(0,t)\times\mathbf{R}^d} 1_{[\epsilon k, \epsilon(k+1))}(y) \frac{P_{t-s}^{(\epsilon)}(\epsilon k - \epsilon[x/\epsilon])}{\epsilon^d} \cdot \sigma\big(U_s(\epsilon k)\big)\, \eta(ds, dy) \\
&= \sum_{y\in(\epsilon\mathbf{Z})^d} P_t^{(\epsilon)}(y - \epsilon[x/\epsilon]) \cdot u_0(y) + \int_{(0,t)\times\mathbf{R}^d} \frac{P_{t-s}^{(\epsilon)}(\epsilon[y/\epsilon] - \epsilon[x/\epsilon])}{\epsilon^d} \cdot \sigma\big(U_s(\epsilon[y/\epsilon])\big)\, \eta(ds, dy).
\end{aligned}$$

In the next few sections, we would do a few approximations, then proceed to the main proof of (4.9).

4.1.1 The approximation of $u_t(x)$ by $u_t^{(1,\delta)}(x)$

By (2.15), Proposition 6, and Plancherel's Theorem, for any $x \in \mathbf{R}^d$ and $t \geq 0$,

$$\begin{aligned}
&\big\|u_t^{(1,\delta)}(x) - u_t(x)\big\|_k^2 \\
&\leq 4k \cdot \int_{(t-\delta,t)} \int_{\mathbf{R}^d\times\mathbf{R}^d} p_{t-s}(z-x) p_{t-s}(y-x) f(y-z) \big\|\sigma(u_s(y)u_s(z))\big\|_{k/2}\, dy\, dz\, ds \\
&\leq \text{const} \cdot \int_{(t-\delta,t)} \left(\int_{\mathbf{R}^d} p_{t-s}(y-x)\, dy \cdot \int_{\mathbf{R}^d} p_{t-s}(z-x)\, dz \right) ds \\
&\leq \text{const} \cdot \delta.
\end{aligned} \tag{4.50}$$

4.1.2 The approximation of $u_t^{(1,\delta)}(x)$ by $u_t^{(2,\epsilon,\delta)}(x)$

We first note that

$$\begin{aligned}
p_t(x) - p_t(x') &= (2\pi)^{-d} \int_{\mathbf{R}^d} (e^{-iz\cdot x} - e^{-iz\cdot x'}) e^{-\nu t|z|^\alpha}\, dz \\
&\leq \text{const} \cdot |x - x'| \cdot \int_{\mathbf{R}^d} |z| e^{-\nu t|z|^\alpha}\, dz,
\end{aligned} \tag{4.51}$$

and that

$$
\begin{aligned}
p_t(x) &= (2\pi)^{-d} \int_{\mathbf{R}^d} e^{-iz\cdot x} e^{-\nu t|z|^\alpha}\, dz \\
&= (2\pi)^{-d} \int_{\mathbf{R}^d} e^{-it^{-1/\alpha} y\cdot x} e^{-\nu |y|^\alpha} t^{-d/\alpha}\, dy \\
&= t^{-d/\alpha} p_1(t^{-1/\alpha} x). \qquad (4.52)
\end{aligned}
$$

By (4.51), (2.15), and Proposition 6, and Plancherel's Theorem, for any $x, x' \in \mathbf{R}^d$, $|x - x'| = \epsilon$, $t \geq 0$,

$$
\begin{aligned}
&\left\| u_t(x) - u_t(x') \right\|_k^2 \\
&\leq 4k \cdot \int_0^r \int_{\mathbf{R}^d \times \mathbf{R}^d} \left| p_{t-s}(y-x) - p_{t-s}(y-x') \right| \cdot \left| p_{t-s}(z-x) - p_{t-s}(z-x') \right| f(y-z) \\
&\quad \cdot \left\| \sigma(u_s(y))\sigma(u_s(z)) \right\|_{k/2} dy\, dz\, ds \\
&\leq \text{const} \cdot \int_0^t \left(\int_{\mathbf{R}^d} \left| p_s(y) - p_s(y-(x'-x)) \right| dy \right)^2 ds \\
&\leq \text{const} \cdot \int_0^{\epsilon'} 4\, ds + \text{const} \cdot \int_{\epsilon'}^t \int_{\{|y| \geq K\}} \left| p_s(y) - p_s(y-(x'-x)) \right| dy\, ds \\
&\qquad + \text{const} \cdot \int_{\epsilon'}^t \left(\int_{|y|<K} dy \cdot \epsilon \cdot \int_{\mathbf{R}^d} |z| e^{-\nu\epsilon'|z|^\alpha}\, dz \right) ds \\
&= \text{(I)} + \text{(II)} + \text{(III)}. \qquad (4.53)
\end{aligned}
$$

Here (I) is small when ϵ' is small. To approximate (II), by (4.52), we have

$$
\begin{aligned}
\text{(II)} &= \text{const} \cdot \int_{\epsilon'}^t \int_{\{|y| \geq K\}} s^{-d/\alpha} \left| p_1(s^{-1/\alpha} y) - p_1(s^{-1/\alpha} \cdot (y - (x'-x))) \right| dy\, ds \\
&\leq \text{const} \cdot \int_{\epsilon'}^t \left(\int_{\{|z| \geq K s^{-1/\alpha}\}} p_1(z)\, dz + \int_{\{|z| \geq (K-\epsilon)s^{-1/\alpha}\}} p_1(z)\, dz \right) ds \\
&\leq \text{const} \cdot t \cdot \left(\int_{\{|z| \geq K t^{-1/\alpha}\}} p_1(z)\, dz + \int_{\{|z| \geq (K-\epsilon)t^{-1/\alpha}\}} p_1(z)\, dz \right). \qquad (4.54)
\end{aligned}
$$

This quantity can be again made small by selecting large K, for $\epsilon' > 0$ fixed and all ϵ small enough. (III) is small by letting $\epsilon \downarrow 0$, when ϵ' and K are both fixed. These estimates imply that

$$
\lim_{|x-x'| \downarrow 0, x, x' \in \mathbf{R}^d} \sup_{t \in [0,T]} \left\| u_t(x) - u_t(x') \right\|_k^2 = 0. \qquad (4.55)
$$

By Proposition 6,

$$\begin{aligned}
&\left\|u_t^{(1,\delta)}(x) - u_t^{(2,\epsilon,\delta)}(x)\right\|_k^2 \\
&\leq 4k \cdot \int_0^{t-\delta} \int_{\mathbf{R}^d \times \mathbf{R}^d} p_{t-s}(y-x) p_{t-s}(z-x) f(y-z) \\
&\quad \cdot \left\|\sigma(u_s(y)) - \sigma(u_s(\epsilon[y/\epsilon])\right\|_k \cdot \left\|\sigma(u_s(z)) - \sigma(u_s(\epsilon[z/\epsilon])\right\|_k \, dy\, dz\, ds \\
&\leq \text{const} \cdot t \cdot \left(\sup_{s \in [\delta,t]} \sup_{y \in \mathbf{R}^d} \left\|u_s(y) - u_s(\epsilon[y/\epsilon])\right\|_k \right)^2 . \qquad (4.56)
\end{aligned}$$

Equation (4.56) implies, by (4.55), for any $T > 0$,

$$\lim_{\epsilon \downarrow 0} \sup_{t \in [0,T]} \sup_{x \in \mathbf{R}^d} \left\|u_t^{(1,\delta)}(x) - u_t^{(2,\epsilon,\delta)}(x)\right\|_k = 0. \qquad (4.57)$$

4.1.3 The approximation of $u_t^{(2,\epsilon,\delta)}(x)$ by $u_t^{(3,\epsilon,\delta)}(x)$

We recall Theorem 7.3.1 of [12] that for any $K > 0$ there exists $c = c(K) > 0$ such that

$$p_t(x) \leq \frac{c\nu t}{|x|^{\alpha+d}} \qquad \text{for all } |x| \geq (\nu t)^{1/\alpha} \cdot K, \qquad (4.58)$$

$$p_t(x) \leq c(\nu t)^{-d/\alpha} \qquad \text{for all } |x| < (\nu t)^{1/\alpha} \cdot K. \qquad (4.59)$$

Equations (4.58) and (4.59) together imply that for all $\delta \leq s \leq t$,

$$p_s(x) \leq c(\nu\delta)^{-d/\alpha} 1_{\{z:|z| \leq (\nu t)^{1/\alpha}\}}(x) + \frac{c\nu t}{|x|^{\alpha+d}} 1_{\{z:|z| \geq (\nu\delta)^{1/\alpha}\}}(x). \qquad (4.60)$$

Therefore, for all $\delta \leq s \leq t$, $|y - x| \leq \frac{1}{3}(\nu\delta)^{1/\alpha}$,

$$p_s(y) \leq c(\nu\delta)^{-d/\alpha} 1_{\{z:|z| \leq \frac{3}{2}(\nu t)^{1/\alpha}\}}(x) + \frac{c\nu t}{(|x| - \frac{1}{3}(\nu\delta)^{1/\alpha})^{\alpha+d}} 1_{\{z:|z| \geq \frac{2}{3}(\nu\delta)^{1/\alpha}\}}(x). \qquad (4.61)$$

By (2.15) and Proposition 6,

$$\begin{aligned}
&\left\|u_t^{(2,\epsilon,\delta)}(x) - u_t^{(3,\epsilon,\delta)}(x)\right\|_k^2 \\
&\leq \int_0^{t-\delta} \int_{\mathbf{R}^d \times \mathbf{R}^d} \left|p_{t-s}(y-x) - p_{t-s}(\epsilon[y/\epsilon] - \epsilon[x/\epsilon])\right| \cdot \left|p_{t-s}(z-x) - p_{t-s}(\epsilon[z/\epsilon] - \epsilon[x/\epsilon])\right| \\
&\quad \cdot f(y-z) \left\|\sigma(u_s(\epsilon[y/\epsilon]))\sigma(u_s(\epsilon[z/\epsilon]))\right\|_{k/2} \, dy\, dz\, ds \cdot (4k) \\
&\leq \text{const} \cdot \int_\delta^t \left(\int_{\mathbf{R}^d} \left|p_s(y) - p_s(\epsilon[y/\epsilon] + x - \epsilon[x/\epsilon])\right| dy \right)^2 ds. \qquad (4.62)
\end{aligned}$$

Now, by (4.61), for all $\epsilon < \frac{1}{3}(\nu\delta)^{1/\alpha}$, $\delta \leq s \leq t$, we have

$$\begin{aligned}
&p_s(\epsilon[y/\epsilon] + x - \epsilon[x/\epsilon]) \\
&\leq c(\nu\delta)^{-d/\alpha} 1_{\{z:|z| \leq \frac{3}{2}(\nu t)^{1/\alpha}\}}(y) + \frac{c\nu t}{(|x| - \frac{1}{3}(\nu\delta)^{1/\alpha})^{\alpha+d}} 1_{\{z:|z| \geq \frac{2}{3}(\nu\delta)^{1/\alpha}\}}(y). \qquad (4.63)
\end{aligned}$$

Therefore, due to (4.63) and (4.51), we can apply dominated convergence theorem to obtain

$$\lim_{\epsilon\downarrow 0}\sup_{t\in[0,T]}\sup_{x\in\mathbf{R}^d}\left\|u_t^{(2,\epsilon,\delta)}(x)-u_t^{(3,\epsilon,\delta)}(x)\right\|_k=0. \tag{4.64}$$

4.1.4 The approximation of $u_t^{(3,\epsilon,\delta)}(x)$ by $u_t^{(4,\epsilon,\delta)}(x)$

By (2.15) and Proposition 6,

$$\begin{aligned}
&\left\|u_t^{(3,\epsilon,\delta)}(x)-u_t^{(4,\epsilon,\delta)}(x)\right\|_k^2\\
&\le 4k\cdot\int_0^{t-\delta}\int_{\mathbf{R}^d\times\mathbf{R}^d}\left|p_{t-s}(\epsilon[y/\epsilon]-\epsilon[x/\epsilon])-\frac{P_{t-s}^{(\epsilon)}(\epsilon[y/\epsilon]-\epsilon[x/\epsilon])}{\epsilon^d}\right|\cdot f(y-z)\\
&\cdot\left|p_{t-s}(\epsilon[z/\epsilon]-\epsilon[x/\epsilon])-\frac{P_{t-s}^{(\epsilon)}(\epsilon[z/\epsilon]-\epsilon[x/\epsilon])}{\epsilon^d}\right|\cdot\|\sigma(u_s(\epsilon[y/\epsilon]))\sigma(u_s(\epsilon[z/\epsilon]))\|_{k/2}\,dy\,dz\,ds\\
&\le \text{const}\cdot\int_\delta^t\left(\int_{\mathbf{R}^d}\left|p_s(\epsilon[y/\epsilon]-\epsilon[x/\epsilon])-\frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon]-\epsilon[x/\epsilon])}{\epsilon^d}\right|dy\right)^2 ds.\\
&= \text{const}\cdot\int_\delta^t\left(\int_{\mathbf{R}^d}\left|p_s(\epsilon[y/\epsilon])-\frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon])}{\epsilon^d}\right|dy\right)^2 ds.
\end{aligned} \tag{4.65}$$

By (4.12), for $\delta\le s\le t$,

$$\begin{aligned}
&\int_{\mathbf{R}^d\setminus[-M,M]^d}\frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon])}{\epsilon^d}\,dz\le\sum_{\substack{\mathbf{y}=(y_1,\cdots,y_d)\in(\epsilon\mathbf{Z})^d\\ |y_i|>\epsilon([\frac{M}{\epsilon}]-1)\text{ for all }1\le i\le d}}P_s^{(\epsilon)}(y)\\
&\le\sum_{\substack{\mathbf{y}=(y_1,\cdots,y_d)\in\epsilon\mathbf{Z}^d\\ |y_i|\ge[\frac{M}{\epsilon}]\text{ for all }1\le i\le d}}P_s^{(1)}(y)\\
&=\sum_{\substack{\mathbf{y}=(y_1,\cdots,y_d)\in\epsilon\mathbf{Z}^d\\ |y_i|\ge[\frac{M}{\epsilon}]\text{ for all }1\le i\le d}}\sum_{n=0}^{\infty}(\widetilde{P}_1)^n_{0,x}\frac{e^{-\nu\cdot C_\alpha\cdot s}(\nu\cdot C_\alpha\cdot s)^n}{n!}\\
&=\sum_{\substack{\mathbf{y}=(y_1,\cdots,y_d)\in\epsilon\mathbf{Z}^d\\ |y_i|\ge[\frac{M}{\epsilon}]\text{ for all }1\le i\le d}}\sum_{n=0}^{N}(\widetilde{P}_1)^n_{0,x}\frac{e^{-\nu\cdot C_\alpha\cdot s}(\nu\cdot C_\alpha\cdot s)^n}{n!}+\sum_{n=N+1}^{\infty}\frac{e^{-\nu\cdot C_\alpha\cdot s}(\nu\cdot C_\alpha\cdot s)^n}{n!},
\end{aligned} \tag{4.66}$$

where $\widetilde{P}_1:=\left(p_{i,j}^{(1,\alpha,d)}\right)_{i,j\in\mathbf{Z}^d}$. We may select N large so that

$$\sum_{n=N+1}^{\infty}\frac{e^{-\nu\cdot C_\alpha\cdot s}(\nu\cdot C_\alpha\cdot s)^n}{n!} \tag{4.67}$$

is small for all $\delta\le s\le t$, by Markov's inequality for Poisson random variables. We could then make (4.66) uniformly small for all ϵ small enough and $\delta\le s\le t$, by selecting large M.

By (4.61), for all $\epsilon < \frac{1}{3}(\nu\delta)^{1/\alpha}$, $\delta \le s \le t$, we have

$$p_s(\epsilon[y/\epsilon]) \le c(\nu\delta)^{-d/\alpha} 1_{\{z:|z|\le \frac{3}{2}(\nu t)^{1/\alpha}\}}(y) + \frac{c\nu t}{(|x| - \frac{1}{3}(\nu\delta)^{1/\alpha})^{\alpha+d}} 1_{\{z:|z|\ge \frac{2}{3}(\nu\delta)^{1/\alpha}\}}(y). \quad (4.68)$$

We also note that

$$\int_{\mathbf{R}^d} \frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon])}{\epsilon^d}\, dz = 1. \quad (4.69)$$

Therefore, (4.68), (4.69), and the fact that (4.66) is small as M gets larger, uniformly for all $\epsilon < \frac{1}{3}(\nu\delta)^{1/\alpha}$ and $\delta \le s \le t$, imply

$$\lim_{M\to\infty} \sup_{\epsilon < \frac{1}{3}(\nu\delta)^{1/\alpha}} \int_\delta^t \left(\left(\int_{\mathbf{R}^d} \left| p_s(\epsilon[y/\epsilon]) - \frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon])}{\epsilon^d} \right| dy \right)^2 \right.$$
$$\left. - \left(\int_{[-M,M]^d} \left| p_s(\epsilon[y/\epsilon]) - \frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon])}{\epsilon^d} \right| dy \right)^2 \right) ds = 0. \quad (4.70)$$

By Lemma 25, for any fixed $M > 0$,

$$\lim_{\epsilon\downarrow 0} \int_\delta^t \left(\int_{[-M,M]^d} \left| p_s(\epsilon[y/\epsilon]) - \frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon])}{\epsilon^d} \right| dy \right)^2 ds = 0. \quad (4.71)$$

Therefore, (4.65), (4.70), and (4.71) together imply

$$\lim_{\epsilon\downarrow 0} \sup_{t\in[0,T]} \sup_{x\in\mathbf{R}^d} \left\| u_t^{(3,\epsilon,\delta)}(x) - u_t^{(4,\epsilon,\delta)}(x) \right\|_k = 0. \quad (4.72)$$

4.1.5 The approximation of $u_t^{(4,\epsilon,\delta)}(x)$ by $u_t^{(5,\epsilon)}(x)$

By (2.15) and Proposition 6, when $t \ge \delta$,

$$\begin{aligned}
&\left\| u_t^{(4,\epsilon,\delta)}(x) - u_t^{(5,\epsilon)}(x) \right\|_k^2 \\
&\le 4k \cdot \int_{t-\delta}^t \int_{\mathbf{R}^d\times\mathbf{R}^d} \frac{P_{t-s}^{(\epsilon)}(\epsilon[y/\epsilon] - \epsilon[x/\epsilon])}{\epsilon^d} \cdot \frac{P_{t-s}^{(\epsilon)}(\epsilon[z/\epsilon] - \epsilon[x/\epsilon])}{\epsilon^d} \cdot f(y-z) \\
&\quad \cdot \left\| \sigma(u_s(\epsilon[y/\epsilon]))\sigma(u_s(\epsilon[z/\epsilon])) \right\|_{k/2} dy\, dz\, ds \\
&\le \text{const} \cdot \int_0^\delta \int_{\mathbf{R}^d\times\mathbf{R}^d} \frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon])}{\epsilon^d} \cdot \frac{P_s^{(\epsilon)}(\epsilon[z/\epsilon])}{\epsilon^d}\, dy\, dz\, ds \\
&\le \text{const} \cdot \delta. \qquad (4.73)
\end{aligned}$$

When $t < \delta$,

$$\begin{aligned}
&\left\| u_t^{(4,\epsilon,\delta)}(x) - u_t^{(5,\epsilon)}(x) \right\|_k^2 \\
&\le \text{const} \cdot \int_0^t \int_{\mathbf{R}^d\times\mathbf{R}^d} \frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon])}{\epsilon^d} \cdot \frac{P_s^{(\epsilon)}(\epsilon[z/\epsilon])}{\epsilon^d}\, dy\, dz\, ds \\
&\le \text{const} \cdot \delta. \qquad (4.74)
\end{aligned}$$

4.1.6 The approximation of $\int_{\mathbf{R}^d} p_t(x-y)u_0(y)\,dy$

We note that

$$
\begin{aligned}
&\left|\sum_{y\in(\epsilon\mathbf{Z})^d} P_t^{(\epsilon)}(\epsilon[x/\epsilon]-y)\cdot u_0(y) - \int_{\mathbf{R}^d} p_t(x-y)u_0(y)\,dy\right| \\
&= \left|\int_{\mathbf{R}^d} \frac{P_t^{(\epsilon)}(\epsilon[x/\epsilon]-\epsilon[y/\epsilon])}{\epsilon^d}\cdot u_0(\epsilon[y/\epsilon])\,dy - \int_{\mathbf{R}^d} p_t(x-y)\cdot u_0(y)\,dy\right| \\
&\le \int_{\mathbf{R}^d} \left|\frac{P_t^{(\epsilon)}(\epsilon[x/\epsilon]-\epsilon[y/\epsilon])}{\epsilon^d} - p_t(y-x)\right|\cdot u_0(\epsilon[y/\epsilon])\,dy \\
&\quad + \int_{\mathbf{R}^d} p_t(x-y)\cdot\Big|u_0(y)-u_0(\epsilon[y/\epsilon])\Big|\,dy \\
&\le \text{const}\cdot \int_{\mathbf{R}^d} \left|\frac{P_t^{(\epsilon)}(\epsilon[x/\epsilon]-\epsilon[y/\epsilon])}{\epsilon^d} - p_t(\epsilon[y/\epsilon]-\epsilon[x/\epsilon])\right|\,dy \\
&\quad + \int_{\mathbf{R}^d} p_t(x-y)\cdot|u_0(y)-u_0(\epsilon[y/\epsilon])|\,dy \\
&\quad + \text{const}\cdot \int_{\mathbf{R}^d} \Big|p_t(\epsilon[x/\epsilon]-\epsilon[y/\epsilon]) - p_t(x-y)\Big|\,dy \\
&= \text{(I)}+\text{(II)}+\text{(III)}.
\end{aligned} \tag{4.75}
$$

Using the same method as we used to bound (4.65), it can be shown that for any $T_2 > T_1 > 0$,

$$\lim_{\epsilon\downarrow 0}\sup_{T_1\le t\le T_2}\big|\text{(I)}\big| = 0. \tag{4.76}$$

Due to (4.60), for any $T_2 > T_1 > 0$, $T_1 \le t \le T_2$,

$$
\begin{aligned}
\big|\text{(II)}\big| \le \int_{\mathbf{R}^d} &\left(c(\nu T_1)^{-d/\alpha}1_{\{z:|z|\le(\nu T_2)^{1/\alpha}\}}(x-y) + \frac{c\nu T_2}{|x-y|^{\alpha+d}}1_{\{z:|z|\ge(\nu T_1)^{1/\alpha}\}}(x-y)\right) \\
&\cdot\Big|u_0(y)-u_0(\epsilon[y/\epsilon])\Big|\,dy.
\end{aligned} \tag{4.77}
$$

Therefore, by the continuity of u_0 and the dominated convergence theorem,

$$\lim_{\epsilon\downarrow 0}\sup_{T_1\le t\le T_2}\big|\text{(II)}\big| = 0. \tag{4.78}$$

Our estimate of (III) follows the same approach as was used to bound (4.64). Thus,

$$\lim_{\epsilon\downarrow 0}\sup_{T_1\le t\le T_2}\big|\text{(III)}\big| = 0. \tag{4.79}$$

By (4.76), (4.78), and (4.79), we have proved

$$\lim_{\epsilon\downarrow 0}\sup_{T_1\le t\le T_2}\left|\sum_{y\in(\epsilon\mathbf{Z})^d} P_t^{(\epsilon)}(\epsilon[x/\epsilon]-y)\cdot u_0(y) - \int_{\mathbf{R}^d} p_t(x-y)u_0(y)\,dy\right| = 0. \tag{4.80}$$

We also point out here if, in particular, $u_0 \equiv c$, then

$$\sum_{y\in(\epsilon\mathbf{Z})^d} P_t^{(\epsilon)}(\epsilon[x/\epsilon]-y)\cdot u_0(y) = \int_{\mathbf{R}^d} p_t(x-y)u_0(y)\,dy = 1. \tag{4.81}$$

4.1.7 Proof of Theorem 23, final step

First we let

$$\begin{aligned} v_t^{(\epsilon,\delta)}(x) := {}& \left\|u_t(x)-u_t^{(1,\delta)}(x)\right\|_k + \left\|u_t^{(1,\delta)}(x)-u_t^{(2,\epsilon,\delta)}(x)\right\|_k + \left\|u_t^{(2,\epsilon,\delta)}(x)-u_t^{(3,\epsilon,\delta)}(x)\right\|_k \\ &+ \left\|u_t^{(3,\epsilon,\delta)}(x)-u_t^{(4,\epsilon,\delta)}(x)\right\|_k + \left\|u_t^{(4,\epsilon,\delta)}(x)-u_t^{(5,\epsilon)}(x)\right\|_k \\ &+ \left|\sum_{y\in(\epsilon\mathbf{Z})^d} P_t^{(\epsilon)}(\epsilon[x/\epsilon]-y)\cdot u_0(y) - \int_{\mathbf{R}^d} p_t(x-y)u_0(y)\,dy\right|. \end{aligned} \tag{4.82}$$

By (4.82) and Proposition 6,

$$\begin{aligned} &\left\|u_t(x)-U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k^2 \\ \le{}& 2\left(v_t^{(\epsilon,\delta)}(x)\right)^2 \\ &+ 2\left\|\int_{(0,t)\times\mathbf{R}^d} \frac{P_{t-s}^{(\epsilon)}(\epsilon[y/\epsilon]-\epsilon[x/\epsilon])}{\epsilon^d}\cdot\Big(\sigma\big(u_s(\epsilon[y/\epsilon])\big)-\sigma\big(U_s(\epsilon[y/\epsilon])\big)\Big)\,\eta(ds\,dy)\right\|_k^2 \\ \le{}& 2\left(v_t^{(\epsilon,\delta)}(x)\right)^2 + \text{const}\cdot\int_0^t\int_{\mathbf{R}^d\times\mathbf{R}^d} \frac{P_s^{(\epsilon)}(\epsilon[y/\epsilon]-\epsilon[x/\epsilon])}{\epsilon^d}\cdot\frac{P_s^{(\epsilon)}(\epsilon[z/\epsilon]-\epsilon[x/\epsilon])}{\epsilon^d}\cdot f(y-z) \\ &\cdot\left\|\Big(\sigma\big(u_s(\epsilon[y/\epsilon])\big)-\sigma\big(U_s(\epsilon[y/\epsilon])\big)\Big)\cdot\Big(\sigma\big(u_s(\epsilon[z/\epsilon])\big)-\sigma\big(U_s(\epsilon[z/\epsilon])\big)\Big)\right\|_{k/2} dy\,dz \\ \le{}& 2\left(v_t^{(\epsilon,\delta)}(x)\right)^2 + \text{const}\cdot\int_{\delta'}^t\left(\sup_{z\in\mathbf{R}^d}\left\|u_s(\epsilon[z/\epsilon]-U_s(\epsilon[z/\epsilon])\right\|_k\right)^2 ds + \text{const}\cdot\delta'. \end{aligned} \tag{4.83}$$

Now we replace x with $\epsilon[x/\epsilon]$ in (4.83), and then take supremum over $x\in\mathbf{R}^d$, to see that

$$\begin{aligned} &\left(\sup_{x\in\mathbf{R}^d}\left\|u_t(\epsilon[x/\epsilon])-U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k\right)^2 = \sup_{x\in\mathbf{R}^d}\left\|u_t(\epsilon[x/\epsilon])-U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k^2 \\ \le{}& 2\left(\sup_{x\in\mathbf{R}^d} v_t^{(\epsilon,\delta)}(\epsilon[x/\epsilon])\right)^2 + \text{const}\cdot\int_{\delta'}^t\left(\sup_{z\in\mathbf{R}^d}\left\|u_s(\epsilon[z/\epsilon]-U_s(\epsilon[z/\epsilon])\right\|_k\right)^2 ds + \text{const}\cdot\delta'. \end{aligned} \tag{4.84}$$

Here all the constants in (4.84) are independent of our choice of $t\in[0,T]$. In view of (2.15) and (3.14), we could now apply Gronwall's inequality to (4.84) (see Appendix A.2) to see for all $\delta'\le t\le T$,

$$
\begin{aligned}
&\left(\sup_{x\in\mathbf{R}^d}\left\|u_t(\epsilon[x/\epsilon]) - U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k\right)^2\\
&\le \left[2\sup_{t\in[\delta',T]}\left(\sup_{x\in\mathbf{R}^d} v_t^{(\epsilon,\delta)}(\epsilon[x/\epsilon])\right)^2 + \mathrm{const}\cdot\delta'\right]\cdot e^{\mathrm{const}\cdot(t-\delta')}\\
&\le \left[2\sup_{t\in[\delta',T]}\left(\sup_{x\in\mathbf{R}^d} v_t^{(\epsilon,\delta)}(\epsilon[x/\epsilon])\right)^2 + \mathrm{const}\cdot\delta'\right]\cdot e^{\mathrm{const}\cdot T}. \qquad (4.85)
\end{aligned}
$$

By (4.50), (4.56), (4.64), (4.72), (4.74), and (4.80),

$$
\limsup_{\epsilon\downarrow 0}\sup_{t\in[\delta',T]}\left(\sup_{x\in\mathbf{R}^d} v_t^{(\epsilon,\delta)}(\epsilon[x/\epsilon])\right)^2 \le \mathrm{const}\cdot\delta^2. \qquad (4.86)
$$

It follows from (4.86) that for all $T_1 \le t \le T_2$,

$$
\begin{aligned}
&\limsup_{\epsilon\downarrow 0}\sup_{t\in[T_1,T_2]}\left(\sup_{x\in\mathbf{R}^d}\left\|u_t(\epsilon[x/\epsilon]) - U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k\right)^2\\
&\le \inf_{0\le\delta'\le T_1}\limsup_{\epsilon\downarrow 0}\sup_{t\in[\delta',T_2]}\left(\sup_{x\in\mathbf{R}^d}\left\|u_t(\epsilon[x/\epsilon]) - U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k\right)^2\\
&\le \inf_{0\le\delta'\le T_1}\limsup_{\epsilon\downarrow 0}\left(2\sup_{t\in[\delta',T_2]}\left(\sup_{x\in\mathbf{R}^d} v_t^{(\epsilon,\delta)}(\epsilon[x/\epsilon])\right)^2 + \mathrm{const}\cdot\delta'\right)\cdot e^{\mathrm{const}\cdot T_2}\\
&\le \inf_{0\le\delta'\le T_1}\limsup_{\epsilon\downarrow 0}\left(\mathrm{const}\cdot\delta^2 + \mathrm{const}\cdot\delta'\right)\cdot e^{\mathrm{const}\cdot T_2}\\
&\le \mathrm{const}\cdot\delta^2. \qquad (4.87)
\end{aligned}
$$

The first assertion of Theorem 23 is proved because $\delta > 0$ can be arbitrarily chosen. If, in particular, $u_0 \equiv c$ for some constant $c \ge 0$, then (4.83), (4.84), (4.85), (4.86) becomes

$$
\left\|u_t(x) - U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k^2 \le 2\left(v_t^{(\epsilon,\delta)}(x)\right)^2 + \mathrm{const}\cdot\int_0^t\left(\sup_{z\in\mathbf{R}^d}\left\|u_s(\epsilon[z/\epsilon] - U_s(\epsilon[z/\epsilon])\right\|_k\right)^2 ds \qquad (4.88)
$$

$$
\begin{aligned}
&\left(\sup_{x\in\mathbf{R}^d}\left\|u_t(\epsilon[x/\epsilon]) - U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k\right)^2\\
&\le 2\left(\sup_{x\in\mathbf{R}^d} v_t^{(\epsilon,\delta)}(\epsilon[x/\epsilon])\right)^2 + \mathrm{const}\cdot\int_0^t\left(\sup_{z\in\mathbf{R}^d}\left\|u_s(\epsilon[z/\epsilon] - U_s(\epsilon[z/\epsilon])\right\|_k\right)^2 ds \qquad (4.89)
\end{aligned}
$$

For all $0 \le t \le T$,

$$
\left(\sup_{x\in\mathbf{R}^d}\left\|u_t(\epsilon[x/\epsilon]) - U_t^{(\epsilon)}(\epsilon[x/\epsilon])\right\|_k\right)^2 \le \left(2\sup_{t\in[0,T]}\left(\sup_{x\in\mathbf{R}^d} v_t^{(\epsilon,\delta)}(\epsilon[x/\epsilon])\right)^2\right)\cdot e^{\mathrm{const}\cdot T}. \qquad (4.90)
$$

For all $T > 0$,

$$\limsup_{\epsilon\downarrow 0} \sup_{t\in[0,T]} \left(\sup_{x\in\mathbf{R}^d} \left\| u_t(\epsilon[x/\epsilon]) - U_t^{(\epsilon)}(\epsilon[x/\epsilon]) \right\|_k \right)^2 \le \text{const}\cdot\delta^2. \tag{4.91}$$

Thus the second assertion of Theorem 23 is proved, by letting $\delta \to 0$ in (4.91).

4.2 Proof of Theorem 2

Let $u_t(x)$ and $u_t(x)$ satisfy the assumptions in Theorem 2. We could find solutions $U_t^{(\epsilon)}(x)$ and $U_t^{(\epsilon)}(x)$ to (SDE(ϵ)) accordingly, such that (4.9) holds, by Theorem 23.

By Theorem 1 applied to $U_t^{(\epsilon)}(x)$ and $V_t^{(\epsilon)}(x)$, we have for any $x_1, x_2, \cdots, x_n \in (\epsilon\mathbf{Z})^d$, $t_1, t_2 \cdots t_n \ge 0$, and $k_1, \cdots, k_n \in [0, \infty)$,

$$E\left[U_{t_1}^{(\epsilon)}(x_1)^{k_1}\cdots U_{t_n}^{(\epsilon)}(x_n)^{k_n}\right] \le E\left[V_{t_1}^{(\epsilon)}(x_1)^{k_1}\cdots V_{t_n}^{(\epsilon)}(x_n)^{k_n}\right]. \tag{4.92}$$

By Lemma 22 and Theorem 23, Theorem 2 is proved. Note that the assumptions (3.89), (3.90), (3.91), and (3.92) are all satisfied, due to (2.15) and Theorem 23. Theorem 23 also implies the convergence in probability for $U_t^{(\epsilon)}(\epsilon[x/\epsilon)$ to $u_t(x)$.

CHAPTER 5

$L^K(P)$ APPROXIMATION FROM SHE(1) TO SHE(2)

Throughout this chapter, SHE(1) and SHE(2) denote the stochastic heat equations defined in Chapter 1 on page 2. The goal of this section is to prove Theorem 3. Before we proceed to its proof, we first prove the following lemma.

Lemma 26. *Let $p_t(x)$ be defined as in (1.3), and $f_\beta := |z|^{-\beta}$, $\beta < d$. Then we have*

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} p_t(y)p_t(z)f_\beta(y-z)\,dy\,dz = \text{const}\cdot t^{-\beta/\alpha}. \tag{5.1}$$

Proof: We first show that

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} p_1(y)p_1(z)f_\beta(y-z)\,dy\,dz < \infty. \tag{5.2}$$

By (4.58), (4.59), we have

$$p_1(x) \le \frac{c}{|x|^{\alpha+d}}1_{\mathbf{R}^d\setminus B(0;1)}(x) + c\cdot 1_{B(0;1)}(x), \tag{5.3}$$

for some constant $c > 0$ depending on ν only. (5.2) then follows from the following three approximations.

Approximation 1.

$$\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} 1_{B(0;1)}(y)1_{B(0;1)}(z)f_\beta(y-z)\,dy\,dz \le \int_{B(0;1)}\left(\int_{B(0;1)} f_\beta(z)\,dz\right)dy < \infty. \tag{5.4}$$

Approximation 2.

$$\begin{aligned}
&\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} 1_{B(0;1)}(y)1_{\mathbf{R}^d\setminus B(0;1)}(z)\frac{1}{|z|^{\alpha+d}}f_\beta(y-z)\,dy\,dz \\
&\quad= \int_{\mathbf{R}^d\setminus B(0;1)}\frac{1}{|z|^{\alpha+d}}\left(\int_{\mathbf{R}^d} 1_{B(0;1)}(y+z)f_\beta(y)\,dy\right)dz \\
&\quad\le \text{const}\cdot\int_{\mathbf{R}^d\setminus B(0;1)}\frac{1}{|z|^{\alpha+d}}\left(1_{B(0;2)}(z) + \frac{1}{|z|^\beta}1_{\mathbf{R}^d\setminus B(0;2)}(z)\right)dz \\
&\quad\le \text{const} + \text{const}\cdot\int_{\mathbf{R}^d\setminus B(0;2)}\frac{1}{|z|^{\alpha+d+\beta}}\,dz < \infty.
\end{aligned} \tag{5.5}$$

Approximation 3. First we note that by Hölder's inequality, for all $z \in \mathbf{R}^d$,

$$\begin{aligned} g(z) &:= \int_{\mathbf{R}^d} 1_{\mathbf{R}^d\setminus B(0;1)}(y-z) 1_{\mathbf{R}^d\setminus B(0;1)}(y) \frac{1}{|y-z|^{\alpha+d}} \frac{1}{|y|^{\alpha+d}}\, dy \\ &\leq \int_{\mathbf{R}^d\setminus B(0;1)} \frac{1}{|y|^{2(\alpha+d)}}\, dy < \infty. \end{aligned} \tag{5.6}$$

We note that $g(z)$ is a radial function. For $|z| > 3$,

$$\begin{aligned} & g(z) = g(|z|e_1) \\ &= \int_{\mathbf{R}^d} 1_{\mathbf{R}^d\setminus B(0;1)}(y' - |z|e_1) 1_{\mathbf{R}^d\setminus B(0;1)}(y') \frac{1}{|y'-|z|e_1|^{\alpha+d}} \frac{1}{|y'|^{\alpha+d}}\, dy' \\ &= \text{const} \cdot |z|^d \cdot \int_{\mathbf{R}^d} 1_{\mathbf{R}^d\setminus B(0;1)} \left(\frac{|z|}{3} y - |z|e_1 \right) 1_{\mathbf{R}^d\setminus B(0;1)} \left(\frac{|z|}{3} y \right) \frac{1}{|\frac{|z|}{3}y - |z|e_1|^{\alpha+d}} \frac{1}{|z|^{\alpha+d}|y|^{\alpha+d}} \right) dy \\ &= \text{const} \cdot |z|^{-2\alpha-d} \cdot \int_{\mathbf{R}^d} 1_{\mathbf{R}^d\setminus B(0;3/|z|)}(y-3e_1) 1_{\mathbf{R}^d\setminus B(0;3/|z|)}(y) \frac{1}{|y-3e_1|^{\alpha+d}|y|^{\alpha+d}}\, dy \\ &= \text{const} \cdot |z|^{-2\alpha-d} \cdot \Bigg(\int_{\mathbf{R}^d} 1_{\mathbf{R}^d\setminus B(0;1)}(y-3e_1) 1_{\mathbf{R}^d\setminus B(0;1)}(y) \frac{1}{|y-3e_1|^{\alpha+d}|y|^{\alpha+d}}\, dy \\ &\quad + \int_{\frac{3}{|z|}<|y-3e_1|<1} \frac{1}{|y-3e_1|^{\alpha+d}|y|^{\alpha+d}}\, dy + \int_{\frac{3}{|z|}<|y|<1} \frac{1}{|y-3e_1|^{\alpha+d}|y|^{\alpha+d}}\, dy \Bigg) \\ &= \text{const} \cdot |z|^{-2\alpha-d} \cdot \left(g(3e_1) + \int_{\frac{3}{|z|}<|y'|<1} \frac{1}{|y'|^{\alpha+d}}\, dy + \int_{\frac{3}{|z|}<|y|<1} \frac{1}{|y|^{\alpha+d}}\, dy \right) \\ &\leq \text{const} \cdot |z|^{-2\alpha-d} \cdot |z|^{\alpha} = \text{const} \cdot |z|^{-\alpha-d}. \end{aligned} \tag{5.7}$$

Therefore, by (5.6) and (5.7),

$$\begin{aligned} & \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} 1_{\mathbf{R}^d\setminus B(0;1)}(z) 1_{\mathbf{R}^d\setminus B(0;1)}(y) \frac{1}{|z|^{\alpha+d}} \frac{1}{|y|^{\alpha+d}} f_\beta(y-z)\, dy\, dz \\ &= \text{const} \cdot \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} 1_{\mathbf{R}^d\setminus B(0;1)}(z+y) 1_{\mathbf{R}^d\setminus B(0;1)}(y) \frac{1}{|z+y|^{\alpha+d}} \frac{1}{|y|^{\alpha+d}}\, dy \right) f_\beta(z)\, dz \\ &= \text{const} \cdot \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} 1_{\mathbf{R}^d\setminus B(0;1)}(y) 1_{\mathbf{R}^d\setminus B(0;1)}(y-z) \frac{1}{|y|^{\alpha+d}} \frac{1}{|y-z|^{\alpha+d}}\, dy \right) f_\beta(z)\, dz \\ &\leq \text{const} \cdot \int_{\mathbf{R}^d} \left(1_{B(0;3)}(z) + \frac{1}{|z|^{\alpha+d}} 1_{\mathbf{R}^d\setminus B(0;3)}(z) \right) f_\beta(z)\, dz \\ &\leq \text{const} \cdot \left(\int_{B(0;3)} \frac{1}{|z|^\beta}\, dz + \int_{\mathbf{R}^d\setminus B(0;3)} \frac{1}{|z|^{\alpha+d+\beta}} \right) < \infty. \end{aligned} \tag{5.8}$$

Hence (5.2) is proved. Now by (4.52), the scaling property for $p_t(x)$, we have

$$\begin{aligned} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p_s(y) p_s(z) f_\beta(y-z)\, dy\, dz &= t^{-2d/\alpha} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p_1(t^{-1/\alpha}y) p_1(t^{-1/\alpha}y) f_\beta(y-z)\, dy\, dz \\ &= t^{-2d/\alpha} t^{2d/\alpha} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p_1(y') p_1(z') f_\beta(t^{1/\alpha}y' - t^{1/\alpha}z')\, dy'\, dz' \\ &= t^{-\beta/\alpha} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} p_1(y') p_1(z') f_\beta(y'-z')\, dy'\, dz'. \end{aligned} \tag{5.9}$$

This completes the proof. **Q.E.D.**

5.1 Proof of Theorem 3

Let $u_t^\delta(x)$ solve the (SHE(1)) with spatially homogeneous noise η_β^δ, and its covariance kernel is given by

$$f_\beta^\delta(x) := \left(h_\beta^\delta * h_\beta^\delta\right)(x), \tag{5.10}$$

where

$$h_\beta^\delta(x) := \frac{C_2}{\delta + |x|^{(d+\beta)/2}}, \delta > 0. \tag{5.11}$$

Here C_2 is the same as the one in Theorem 3. First we would like to show f_β^δ satisfies all the assumptions in (SHE(1)), as in the following theorem.

Theorem 27. *f_β^δ is a bounded, continuous, symmetric positive definite function.*

Proof: We prove the several assertions of the theorem separately.

1. Proof of boundedness:

For all $x \in \mathbf{R}^d$, by Hölder's inequality,

$$\begin{aligned} f_\beta^\delta(x) &= \int_{\mathbf{R}^d} \frac{1}{\delta + |x-y|^{(d+\beta)/2}} \cdot \frac{1}{\delta + |y|^{(d+\beta)/2}}\, dy \\ &\leq \left(\int_{\mathbf{R}^d} \frac{1}{(\delta + |x-y|^{(d+\beta)/2})^2}\, dy\right)^{1/2} \cdot \left(\int_{\mathbf{R}^d} \frac{1}{(\delta + |y|^{(d+\beta)/2})^2}\, dy\right)^{1/2} \\ &= \int_{\mathbf{R}^d} \frac{1}{(\delta + |y|^{(d+\beta)/2})^2}\, dy \leq C < \infty. \end{aligned} \tag{5.12}$$

2. Proof of symmetry:

$$\begin{aligned} f_\beta^\delta(x) &= \int_{\mathbf{R}^d} \frac{1}{\delta + |x-y|^{(d+\beta)/2}} \cdot \frac{1}{\delta + |y|^{(d+\beta)/2}}\, dy \\ &= \int_{\mathbf{R}^d} \frac{1}{\delta + |-x+y|^{(d+\beta)/2}} \cdot \frac{1}{\delta + |-y|^{(d+\beta)/2}}\, dy \\ &= \int_{\mathbf{R}^d} \frac{1}{\delta + |-x-y'|^{(d+\beta)/2}} \cdot \frac{1}{\delta + |y'|^{(d+\beta)/2}}\, dy' \\ &= f_\beta^\delta(-x). \end{aligned} \tag{5.13}$$

3. Proof of continuity:

Let $x_n \to x$, and there exists $C > 0$ such that $|x_n| \le C$ for all $n \in \mathbf{N}$. Because

$$\frac{1}{\delta + |x-y|^{(d+\beta)/2}} \cdot \frac{1}{\delta + |y|^{(d+\beta)/2}}\, dy \le \frac{1}{\delta + \max(|y| - C, 0)^{(d+\beta)/2}} \cdot \frac{1}{\delta + |y|^{(d+\beta)/2}}\, dy, \tag{5.14}$$

where the right-hand side of (5.14) is integrable on $\mathbf{R}^d$, so by the dominated convergence theorem, we have $f^\delta_\beta(x_n) \to f^\delta_\beta(x)$ as $n \to \infty$.

4. Proof of positive-definiteness:

The proof of this part is included in Appendix A.1. **Q.E.D.**

Now we recall Section 2.2, given a space–time white noise ξ defined on $(0,\infty) \times \mathbf{R}^d$, we may construct η_β, η^δ_β by

$$\eta_\beta(\phi) := \int_{(0,\infty)\times\mathbf{R}^d} \left(\int_{\mathbf{R}^d} \phi(s,y) h_\beta(y-x)\, dy \right) \xi(ds,dx), \tag{5.15}$$

$$\eta^\delta_\beta(\psi) := \int_{(0,\infty)\times\mathbf{R}^d} \left(\int_{\mathbf{R}^d} \phi(s,y) h^\delta_\beta(y-x)\, dy \right) \xi(ds,dx). \tag{5.16}$$

for all ϕ, ψ such that

$$(t,x) \mapsto \widetilde{\phi}(t,x) = \int_{\mathbf{R}^d} \phi(t,y) h_\beta(y-x)\, dy \in L^2((0,\infty) \times \mathbf{R}^d), \tag{5.17}$$

$$(t,x) \mapsto \widetilde{\psi}(t,x) = \int_{\mathbf{R}^d} \psi(t,y) h^\delta_\beta(y-x)\, dy \in L^2((0,\infty) \times \mathbf{R}^d). \tag{5.18}$$

Note that all the results we have shown in Section 2.2 are for h_β, but they remain true if we replaced h_β with h^δ_β. From the mild forms of $u_t(x)$ and $u^\delta_t(x)$, we have

$$u_t(x) = \int_{(0,t)\times\mathbf{R}^d} p_t(x-y) u_0(y)\, dy + \int_{(0,t)\times\mathbf{R}^d} p_{t-s}(x-y) \sigma(u_s(y)) \eta_\beta(dy,ds), \tag{5.19}$$

$$u^\delta_t(x) = \int_{(0,t)\times\mathbf{R}^d} p_t(x-y) u_0(y)\, dy + \int_{(0,t)\times\mathbf{R}^d} p_{t-s}(x-y) \sigma(u^\delta_s(y)) \eta^\delta_\beta(dy,ds). \tag{5.20}$$

By (2.21),

$$\begin{aligned} u_t(x) &= \int_{(0,t)\times\mathbf{R}^d} p_t(x-y) u_0(y)\, dy \\ &\quad + \int_{(0,t)\times\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p_{t-s}(x-z) \sigma(u_s(z)) h_\beta(z-y)\, dz \right) \xi(dy,ds), \\ u^\delta_t(x) &= \int_{(0,t)\times\mathbf{R}^d} p_t(x-y) u_0(y)\, dy \\ &\quad + \int_{(0,t)\times\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p_{t-s}(x-z) \sigma(u^\delta_s(z)) h^\delta_\beta(z-y)\, dz \right) \xi(dy,ds). \end{aligned} \tag{5.21}$$

Therefore, by BDG inequality for space–time stochastic integral against space–time white noise (see Proposition 4.4 of [11]), Minkowski's integral inequality, and (2.15),

$$
\begin{aligned}
&\|u_t(x)-u_t^\delta(x)\|_k^2\\
&\quad\leq \int_{(0,t)\times\mathbf{R}^d} \left\| \int_{\mathbf{R}^d} p_{t-s}(x-z)\big(\sigma(u_s(z))h_\beta(z-y)-\sigma(u_s^\delta(z))h_\beta^\delta(z-y)\big)\,dz\right\|_k^2 dy\,ds\\
&\quad\leq 2\int_{(0,t)\times\mathbf{R}^d} \left\| \int_{\mathbf{R}^d} p_{t-s}(x-z)\big(\sigma(u_s(z))h_\beta(z-y)-\sigma(u_s(z))h_\beta^\delta(z-y)\big)\,dz\right\|_k^2 dy\,ds\\
&\qquad\quad + 2\int_{(0,t)\times\mathbf{R}^d} \left\| \int_{\mathbf{R}^d} p_{t-s}(x-z)\big(\sigma(u_s(z))h_\beta^\delta(z-y)-\sigma(u_s^\delta(z))h_\beta^\delta(z-y)\big)\,dz\right\|_k^2 dy\,ds\\
&\quad\leq 2\int_{(0,t)\times\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p_{t-s}(x-z)\|\sigma(u_s(z))\|_k\cdot\left|h_\beta(z-y)-h_\beta^\delta(z-y)\right|\,dz\right)^2 dy\,ds\\
&\qquad\quad + 2\int_{(0,t)\times\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p_{t-s}(x-z)\|\sigma(u_s(z))-\sigma(u_s^\delta(z))\|_k\cdot h_\beta^\delta(z-y)\,dz\right)^2 dy\,ds\\
&\quad\leq \text{const}\cdot\int_{(0,t)\times\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p_{t-s}(x-z)\cdot\left|h_\beta(z-y)-h_\beta^\delta(z-y)\right|\,dz\right)^2 dy\,ds\\
&\qquad\quad + \text{const}\cdot\int_{(0,t)\times\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p_{t-s}(x-z)\|u_s(z)-u_s^\delta(z)\|_k\cdot h_\beta^\delta(z-y)\,dz\right)^2 dy\,ds\\
&\quad=(\mathrm{I})+(\mathrm{II}).
\end{aligned}
\tag{5.22}
$$

By Lemma 26,

$$
\begin{aligned}
&\int_{(0,t)\times\mathbf{R}^d}\left(\int_{\mathbf{R}^d} p_{t-s}(x-z)h_\beta(z-y)\,dz\right)^2 dy\,ds\\
&=\int_{(0,t)\times\mathbf{R}^d}\left(\int_{\mathbf{R}^d\times\mathbf{R}^d} p_s(x-z_1)p_s(x-z_2)h_\beta(z_1-y)h_\beta(z_2-y)\,dz_1\,dz_2\right) dy\,ds\\
&=\int_{(0,t)\times\mathbf{R}^d\times\mathbf{R}^d} p_s(x-z_1)p_s(x-z_2)\left(\int_{\mathbf{R}^d} h_\beta(z_1-y)h_\beta(z_2-y)\,dy\right) dz_1\,dz_2\,ds\\
&=\int_0^t\int_{\mathbf{R}^d\times\mathbf{R}^d} p_s(x-z_1)p_s(x-z_2)f_\beta(z_1-z_2)\,dz_1\,dz_2\,ds\\
&=\int_0^t\int_{\mathbf{R}^d\times\mathbf{R}^d} p_s(z_1)p_s(z_2)f_\beta(z_1-z_2)\,dz_1\,dz_2\,ds\\
&=\text{const}\cdot\int_0^t s^{-\beta/\alpha}\,ds<\infty.
\end{aligned}
\tag{5.23}
$$

Therefore, (I) $\downarrow 0$ as $\delta\downarrow 0$, by the dominated convergence theorem. By Lemma 26, BDG inequality for space–time white noise, and the inequality $h_\beta^\delta\leq h_\beta$, we have

$$
\begin{aligned}
(\mathrm{II}) &\leq \text{const}\cdot\int_0^t \sup_{x'\in\mathbf{R}^d}\|u_s(x')-u_s^\delta(x')\|_k\left(\int_{\mathbf{R}^d\times\mathbf{R}^d} p_{t-s}(x-y)p_{t-s}(x-z)f_\beta(y-z)\,dy\,dz\right) ds\\
&\leq \text{const}\cdot\int_0^t \sup_{x'\in\mathbf{R}^d}\|u_s(x')-u_s^\delta(x')\|_k\cdot s^{-\beta/\alpha}\,ds.
\end{aligned}
\tag{5.24}
$$

As a result,

$$\sup_{x\in\mathbf{R}^d} \|u_t(x) - u_t^\delta(x)\|_k \leq (\mathrm{I}) + \mathrm{const}\cdot \int_0^t \sup_{x\in\mathbf{R}^d} \|u_s(x) - u_s^\delta(x)\|_k \cdot s^{-\beta/\alpha}\, ds. \tag{5.25}$$

By Gronwall's inequality (see Appendix A.2),

$$\sup_{x\in\mathbf{R}^d} \|u_t(x) - u_t^\delta(x)\|_k \leq (\mathrm{I}) \cdot \exp\Big(\mathrm{const}\cdot \int_0^t s^{-\beta/\alpha}\, ds\Big). \tag{5.26}$$

Therefore, we let $\delta \downarrow 0$ to complete the proof. **Q.E.D.**

5.2 Proof of Theorem 4

Let $u_t(x)$ and $u_t(x)$ satisfy the assumptions in Theorem 4. We could find solutions $u_t^\delta(x)$ and $v_t^\delta(x)$ to (SHE(1)), such that (1.16) holds, by Theorem 3.

By Theorem 2 applied to $u_t^\delta(x)$ and $v_t^\delta(x)$, we have for any

$$\begin{aligned}
&x_1,\, x_2, \cdots, x_n \in \mathbf{R}^d,\\
&t_1,\, t_2 \cdots t_n \geq 0,\\
&k_1, \cdots, k_n \in [0,\infty),\\
&E\left[u_{t_1}^\delta(x_1)^{k_1}\cdots u_{t_n}^\delta(x_n)^{k_n}\right] \leq E\left[v_{t_1}^\delta(x_1)^{k_1}\cdots v_{t_n}^\delta(x_n)^{k_n}\right]. \qquad (5.27)
\end{aligned}$$

Theorem 4 then follows from Theorem 2 and Lemma 22. Again, the assumptions (3.89), (3.90), (3.91), and (3.92) are all satisfied, due to (2.15) and Theorem 3. Theorem 3 also implies the convergence in probability for $u_t^\delta(x)$ to $u_t(x)$.

APPENDIX A

FOURIER TRANSFORM

Recall that the **Schwartz space** $\mathscr{S}(\mathbf{R}^d)$ is the collection of all functions $f : \mathbf{R}^d \to \mathbf{C}$ such that

$$\sup_{x \in \mathbf{R}^d} |x^\alpha D^\beta f(x)| < \infty \tag{A.1}$$

for all multi-indices α, β.

The space $\mathscr{S}(\mathbf{R}^d)$ is a metric space. We may construct a metric topology for $\mathscr{S}(\mathbf{R}^d)$ as follows (see also [17]). First we define metrics $\rho_{\alpha,\beta}(f,g) := \sup_{x\in\mathbf{R}^d} |x^\alpha D^\beta (f-g)(x)|$ on $\mathscr{S}(\mathbf{R}^d)$ for every multi-index α, β, and then enumerate these metrics as $\rho_1, \rho_2, \cdots$. Then, we could define a new metric on $\mathscr{S}(\mathbf{R}^d)$:

$$\rho(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(f,g)}{1+\rho_n(f,g)}. \tag{A.2}$$

We define the Fourier transform $\mathcal{F}$ on $\mathscr{S}(\mathbf{R}^d)$ by

$$\mathcal{F}[f](\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} f(x) e^{-ix\cdot\xi}\, dx \qquad \forall \xi \in \mathbf{R}^d. \tag{A.3}$$

It is known that $\mathcal{F}[f] \in \mathscr{S}(\mathbf{R}^d)$ (see [16], [17]).

The space of **tempered distributions**, namely the dual space $\mathscr{S}'(\mathbf{R}^d)$ of Schwartz space, is the space of all continuous linear functionals $\mathbf{R}^d \to \mathbf{C}$.

Given $f \in S'(\mathbf{R}^d)$, the Fourier transform of f is defined by (see [17])

$$\langle \mathcal{F}[f], \phi \rangle = \langle f, \mathcal{F}[\phi] \rangle, \tag{A.4}$$

for all $\phi \in \mathscr{S}(\mathbf{R}^d)$. When f is a function on $\mathbf{R}^d$, $\langle f, \phi \rangle := \int_{\mathbf{R}^d} f(x)\phi(x)\, dx$. Note that (A.4) extends the definition for the Fourier transform on $\mathscr{S}(\mathbf{R}^d)$.

Here we would like to prove two results. The first one is a well-known identity (see, e.g., section 3.3 of [7]), and the second one is part of the proof of Theorem 27.

Theorem 28. *Let* $f(x) = |x|^{-\beta}$, $0 < \beta < d$, $x \in \mathbf{R}^d$. *Then* $\mathcal{F}[f](\xi) = \mathrm{const} \cdot |\xi|^{-d+\beta}$.

Proof: We first note that for $k > 0$,

$$\begin{aligned}\Gamma(k) &= \int_0^\infty s^{k-1}e^{-s}\,ds \\ &= \int_0^\infty (t|\xi|^2)^{k-1}e^{-t|\xi|^2}\cdot|\xi|^2\,dt \\ &= \int_0^\infty |\xi|^{2k}\cdot t^{k-1}e^{-t|\xi|^2}\,dt\end{aligned} \tag{A.5}$$

Letting $k = \beta/2 > 0$ in (A.5), we have

$$|\xi|^{-\beta} = \Gamma(\frac{\beta}{2})^{-1}\cdot\int_0^\infty t^{\frac{\beta}{2}-1}e^{-t|\xi|^2}\,dt.$$

Now given any $f \in S'(\mathbf{R}^d)$, for any $\phi \in \mathscr{S}(\mathbf{R}^d)$, we have

$$\begin{aligned}&\Big\langle \frac{1}{|\xi|^\beta}, \mathcal{F}[\phi]\Big\rangle \\ =& \int_{\mathbf{R}^d}\Gamma(\frac{\beta}{2})^{-1}\cdot\Big(\int_0^\infty t^{\frac{\beta}{2}-1}e^{-t|\xi|^2}\,dt\Big)\cdot\mathcal{F}[\phi](\xi)\,d\xi \\ =& \Gamma(\frac{\beta}{2})^{-1}\cdot\int_0^\infty t^{\frac{\beta}{2}-1}\Big(\int_{\mathbf{R}^d} e^{-t|\xi|^2}\cdot\mathcal{F}[\phi](\xi)\,d\xi\Big)\,dt \\ =& \Gamma(\frac{\beta}{2})^{-1}\cdot\int_0^\infty t^{\frac{\beta}{2}-1}\Big(\int_{\mathbf{R}^d} e^{-t|\xi|^2}\cdot\Big(\frac{1}{(2\pi)^{d/2}}\cdot\int_{\mathbf{R}^d}\phi(x)e^{-ix\cdot\xi}\,dx\Big)\,d\xi\Big)\,dt \\ =& \Gamma(\frac{\beta}{2})^{-1}\cdot\int_0^\infty t^{\frac{\beta}{2}-1}\Big(\int_{\mathbf{R}^d}\phi(x)\cdot\Big(\frac{1}{(2\pi)^{d/2}}\cdot\int_{\mathbf{R}^d} e^{-t|\xi|^2}e^{-ix\cdot\xi}\,d\xi\Big)\,dx\Big)\,dt \\ =& \Gamma(\frac{\beta}{2})^{-1}\cdot\int_0^\infty t^{\frac{\beta}{2}-1}\Big(\int_{\mathbf{R}^d}\phi(x)\cdot\Big(\frac{1}{(2t)^{d/2}}e^{-|x|^2/4t}\Big)\,dx\Big)\,dt \\ =& \Gamma(\frac{\beta}{2})^{-1}\cdot\int_{\mathbf{R}^d}\phi(x)\cdot\Big(\int_0^\infty t^{\frac{\beta}{2}-1}\cdot\frac{1}{(2t)^{d/2}}e^{-|x|^2/4t}\,dt\Big)\,dx,\end{aligned} \tag{A.6}$$

where we have used Fubini–Tonelli's theorem three times. Apply the substitution $s = |x|^2/4t$, (A.6) equals

$$\begin{aligned}&\Gamma(\frac{\beta}{2})^{-1}\cdot\frac{1}{2^{d/2}}\cdot\int_{\mathbf{R}^d}\phi(x)\cdot\Big(\int_0^\infty\Big(\frac{|x|^2}{4s}\Big)^{\frac{\beta}{2}-\frac{d}{2}-1}\cdot e^{-s}\cdot\frac{|x|^2}{4}\cdot\frac{1}{s^2}\,ds\Big)\,dx \\ =&\Gamma(\frac{\beta}{2})^{-1}\cdot 2^{d/2-\beta}\cdot\int_{\mathbf{R}^d}\phi(x)\cdot\frac{1}{|x|^{d-\beta}}\Big(\int_0^\infty s^{\frac{d-\beta}{2}-1}\cdot e^{-s}\,ds\Big)\,dx \\ =&\Gamma(\frac{\beta}{2})^{-1}\cdot 2^{d/2-\beta}\cdot\Gamma(\frac{d-\beta}{2})\cdot\int_{\mathbf{R}^d}\phi(x)\cdot\frac{1}{|x|^{d-\beta}}\,dx \\ =&\Big\langle\phi, 2^{d/2-\beta}\cdot\frac{\Gamma(\frac{d-\beta}{2})}{\Gamma(\frac{\beta}{2})}\cdot\frac{1}{|x|^{d-\beta}}\Big\rangle.\end{aligned} \tag{A.7}$$

The calculations above show

$$\mathcal{F}[f](x) = 2^{d/2-\beta}\cdot\frac{\Gamma(\frac{d-\beta}{2})}{\Gamma(\frac{\beta}{2})}\cdot\frac{1}{|x|^{d-\beta}}. \tag{A.8}$$

Q.E.D.

Theorem 29. *Let* $f_\beta^\delta(x) = (h_\beta^\delta * h_\beta^\delta)(x)$*, where* $h_\beta^\delta(x) := \left(\delta + |x|^{(d+\beta)/2}\right)^{-1}$*,* $0 < \beta < d$*,* $x \in \mathbf{R}^d$*. Then* f_β^δ *is a positive definite function.*

Proof: Let $h_{\beta,N}^\delta(x) = \frac{1}{\delta+|x|^{(d+\beta)/2}} \cdot 1_{[-N,N]^d}(x)$, and $f_{\beta,N}^\delta(x) := h_{\beta,N}^\delta * h_{\beta,N}^\delta(x)$. We have

$$\mathcal{F}[f_{\beta,N}^\delta](\xi) = \left|\mathcal{F}[h_{\beta,N}^\delta](\xi)\right|^2, \tag{A.9}$$

and thus for any $\phi \in \mathscr{S}(\mathbf{R}^d)$,

$$\int_{\mathbf{R}^d} \mathcal{F}[f_{\beta,N}^\delta](\xi)\phi(\xi)\,d\xi = \int_{\mathbf{R}^d} \left|\mathcal{F}[h_{\beta,N}^\delta](\xi)\right|^2 \phi(\xi)\,d\xi. \tag{A.10}$$

Because $h_{\beta,N}^\delta \to h_\beta^\delta$ in $L^2(\mathbf{R}^d)$, $\mathcal{F}[h_{\beta,N}^\delta] \to \mathcal{F}[h_\beta^\delta]$ in $L^2(\mathbf{R}^d)$ (see, for example, [17]). Therefore, along with the fact that ϕ is bounded, we have

$$\lim_{N\to\infty} \int_{\mathbf{R}^d} \left|\mathcal{F}[h_{\beta,N}^\delta](\xi)\right|^2 \phi(\xi)\,d\xi = \int_{\mathbf{R}^d} \left|\mathcal{F}[h_\delta](\xi)\right|^2 \phi(\xi)\,d\xi. \tag{A.11}$$

Besides, by the monotone convergence theorem, $f_{\beta,N}^\delta(x) \uparrow f_\beta^\delta(x)$ as $N \to \infty$ for every $\xi \in \mathbf{R}^d$. This shows

$$\begin{aligned}
\lim_{N\to\infty} \Big| \int_{\mathbf{R}^d} \mathcal{F}[f_{\beta,N}^\delta](\xi)\phi(\xi)\,d\xi - \int_{\mathbf{R}^d} \mathcal{F}[f_\delta](\xi)\phi(\xi)\,d\xi \Big| \\
&= \lim_{N\to\infty} \Big| \int_{\mathbf{R}^d} f_{\beta,N}^\delta(x)\mathcal{F}[\phi](x)\,dx - \int_{\mathbf{R}^d} f_\beta^\delta(x)\mathcal{F}[\phi](x)\,dx \Big| \\
&\le \lim_{N\to\infty} \int_{\mathbf{R}^d} \left| f_{\beta,N}^\delta - f_\beta^\delta(x)\right| \cdot \left|\mathcal{F}[\phi](x)\right|\,dx \\
&= 0,
\end{aligned} \tag{A.12}$$

by the dominated convergence theorem. (A.10), (A.11), and (A.12) together show the Fourier transform of f_β^δ is given by the finite positive measure $\left|\mathcal{F}[h_\beta^\delta](\xi)\right|^2 d\xi$ (the meausre is finite because $\mathcal{F}[h_\beta^\delta] \in L^2(\mathbf{R}^d)$). Therefore, f_β^δ is positive definite by Bochner's theorem (see, for example, Theorem 2.7 of [18]). **Q.E.D.**

APPENDIX B

GRONWALL'S INEQUALITY FOR MEASURABLE FUNCTIONS

Gronwall's inequality is a famous one with many variations; the one we need for this thesis is proved below. We also refer the reader to Appendix 5 of [4].

Theorem 30. *Let $u : [a,b] \to \mathbf{R}$ be a bounded Lebesgue measurable function, $g : [a,b] \to \mathbf{R}$ be a nondecreasing function, and $\beta : [a,b] \to \mathbf{R}$ be a nonnegative Lebesgue measurable function. Assume that for all $t \in [a,b]$ we have*

$$u(t) \leq g(t) + \int_a^t \beta(s)u(s)\,ds, \tag{B.1}$$

then for all $t \in [a,b]$,

$$u(t) \leq g(t)e^{\int_a^t \beta(s)\,ds}. \tag{B.2}$$

Proof: Let $v(z) = \int_a^z \beta(s)u(s)\,ds \cdot e^{-\int_a^z \beta(s)\,ds}$, which is differentiable a.e. $z \in [a,b]$, and its derivative, if exists, is given by

$$v'(z) = \beta(z)u(z) \cdot e^{-\int_a^z \beta(s)\,ds} - \beta(z) \cdot \int_a^z \beta(s)u(s)\,ds \cdot e^{-\int_a^z \beta(s)\,ds}. \tag{B.3}$$

Therefore, for a.e. $z \in [a,b]$,

$$v'(z) \leq \beta(z)g(z)e^{-\int_a^z \beta(s)\,ds}. \tag{B.4}$$

Because v is absolute continuous on $[a,b]$, for all $t \in [a,b]$ we have

$$\begin{aligned} v(t) = v(a) + \int_a^t v'(z)\,dz &= \int_a^t v'(s)\,dz \\ &\leq \int_a^t \beta(z)g(z)e^{-\int_a^z \beta(s)\,ds}\,dz. \end{aligned} \tag{B.5}$$

Therefore, for all $t \in [a, b]$,

$$\begin{aligned} u(t) &\leq g(t) + \int_0^t \beta(s) u(s)\, ds \\ &= g(t) + v(t) e^{\int_a^t \beta(s)\, ds} \\ &\leq g(t) + \int_a^t \beta(z) g(z) e^{\int_z^t \beta(s)\, ds}\, dz \\ &\leq g(t) + g(t) \int_a^t \beta(z) e^{\int_z^t \beta(s)\, ds}\, dz \\ &= g(t) + g(t) \big(-1 + e^{\int_a^t \beta(s)\, ds} \big) \\ &= g(t) e^{\int_a^t \beta(s)\, ds}. \end{aligned} \tag{B.6}$$

Q.E.D.

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