Performance Analysis of Adaptive Filters Equipped with the Dual Sign Algorithm

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Abstract—**Adaptive filters equipped with the sign algorithm are attractive in many applications because of their computational simplicity. Unfortunately, their slow speed of convergence is a major limitation. The dual sign algorithm (DSA) is a means by which the convergence speed can be increased without overly degrading the steady-state performance and with a minimal amount of additional computational complexity. This paper presents a convergence analysis for adaptive filters equipped with the dual sign algorithm. Previous analyses of the dual sign algorithm were based on two assumptions: 1) the input sequence to the adaptive filter is white; 2) the behavior for the DSA is such that it switches from an adaptive filter equipped with the sign algorithm with a relatively large convergence constant to another one with a smaller convergence constant a certain amount of time after the filter is initialized. Both these restrictions are removed for Gaussian input signals in our analysis. A simulation example that shows good match between theoretical and empirical results is also presented in this paper.**

I. Introduction

STOCHASTIC gradient adaptive filters using nonlinear correlation multipliers have received a great deal of attention recently [1]-[4], [6]-[11], [13], [15]. A particularly attractive method in terms of computational complexity is the sign algorithm (SA) [3], [10], which updates the coefficient vector as

$$H(n+1) = H(n) + \mu \operatorname{sign}\{e(n)\}X(n) \tag{1}$$

where $H(n)$ is the vector of N adaptive filter coefficients at time n, $X(n)$ is the input vector to the adaptive filter, μ is a time invariant convergence constant, sign $(\cdot)$ denotes the signum function, and $e(n)$ is the error in estimating the desired response signal $d(n)$ using the input vector $X(n)$; i.e.,

$$e(n) = d(n) - H^T(n)X(n). \tag{2}$$

In (2), $(\cdot)^T$ denotes the matrix transpose of $(\cdot)$. While the sign algorithm is computationally simpler than the popular LMS algorithm, it suffers from the disadvantage that its convergence speed is too slow to be useful in many applications. The dual sign algorithm (DSA) was introduced [8], [9] to overcome this limitation with only a minimal increase in computational complexity. The coefficients are updated in the DSA using the following equation:

$$H(n+1) = H(n) + \mu r(n)X(n) \tag{3}$$

where

$$r(n) = \begin{cases} \operatorname{sign}\{e(n)\}; & |e(n)| \le \tau \\ L \operatorname{sign}\{e(n)\}; & |e(n)| > \tau. \end{cases} \tag{4}$$

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L is a constant larger than or equal to 1 and τ is a preselected threshold. The basic idea here is that the adaptive filter makes large corrections to the coefficient vector if the magnitude of the estimation error is larger than the threshold τ and smaller corrections otherwise. Consequently, one would expect the DSA to have fast convergence properties and small steady-state errors.

The objective of this paper is to present a convergence analysis of the dual sign algorithm. Previous analyses [9] of the DSA were based on two major assumptions, as follows.

1) The input signals are zero mean and white. In particular, the input correlation matrix is of the form

$$R_{xx} = E\{X(n)X^T(n)\} = \sigma_x^2 I \tag{5}$$

where I denotes the $N \times N$ identity matrix. In many practical situations, this assumption is grossly violated. In such cases, analysis using this assumption tends to show faster than true convergence, especially when the eigenvalue spread for the input autocorrelation matrix is large.

2) The DSA consists of two sign algorithms, one with convergence constant μL and the other with convergence constant μ and the DSA switches from updating the filter coefficients using the larger constant μL to updating using the convergence constant μ a certain amount of time after it is initialized. This model for the convergence behavior of the DSA reduces the problem to that for analyzing the sign algorithm, but is overly simplistic. Furthermore, the analysis using this assumption cannot be extended easily to the case when the input signal is nonstationary.

The analysis presented in this paper relaxes both the above assumptions for Gaussian input signals. The rest of the paper is organized as follows. In the next section, the DSA is analyzed for the case when the input signal is zero mean, Gaussian, and stationary. Numerical results including a simulation example that shows close match between theoretical and empirical results are given in Section III. Finally, Section IV contains the concluding remarks.

II. Convergence Analysis for the DSA

Our analysis makes use of the following assumptions.

1) The input signal pair $\{X(n), d(n)\}$ are real, jointly Gaussian, zero-mean and stationary random signals. As in many convergence analyses of this type, we will assume that the input pair $\{X(n), d(n)\}$ is independent of $\{X(k), d(k)\}$ if $n \ne k$. (Note that we are not restricting the nature of R_{xx}). $H(n)$ is then uncorrelated with $\{X(n), d(n)\}$ since $H(n)$ depends only on the input samples at time $n-1$ and before. This independence assumption is almost never true in practice, but it is a commonly used assumption in the analysis of adaptive filters and such analyses produce results that are very close to the true

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adaptive filter behavior of the system for small values of the convergence parameters.

2) We will approximate the mean-squared value of the estimation error $e(n)$, conditioned on the coefficient vector $H(n)$, with the unconditional mean-squared estimation error, i.e.,

$$E\{e^2(n)|H(n)\} \approx E\{e^2(n)\} = \sigma_e^2(n). \quad (6)$$

This approximation is also valid for small convergence parameters and has been successfully used in the past for analyzing the sign algorithm [10].

The following two lemmas are very useful in the analysis of the dual sign algorithm.

Lemma 1: Let X_1 and X_2 be real, jointly Gaussian and zero-mean random variables and let

$$R = \begin{bmatrix} \sigma_1^2 & r_{12} \\ r_{12} & \sigma_2^2 \end{bmatrix} \quad (7)$$

be the autocorrelation matrix of the random vector x defined as

$$x = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}. \quad (8)$$

Also, let $r(X_2)$ be a random variable defined as

$$r(X_2) = \begin{cases} \text{sign}\{X_2\} & |X_2| \le \tau \\ L\,\text{sign}\{X_2\} & |X_2| > \tau \end{cases} \quad (9)$$

where $\tau \ge 0$ and $L \ge 1$.

Then

$$E\{r(X_2)X_1\} = \sqrt{\frac{2}{\pi}}\left\{1 + (L-1)e^{-(\tau^2/\sigma_2^2)}\right\}\frac{r_{12}}{\sigma_2}. \quad (10)$$

Lemma 2: Let X_1, X_2, and X_3 be real, jointly Gaussian and zero mean random variables and let

$$R = \begin{bmatrix} \sigma_1^2 & r_{12} & r_{13} \\ r_{12} & \sigma_2^2 & r_{23} \\ r_{13} & r_{23} & \sigma_3^2 \end{bmatrix} \quad (11)$$

be the autocorrelation matrix of the random vector

$$Z = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}. \quad (12)$$

Let $r(X_3)$ be defined as in Lemma 1, with X_2 replaced by X_3. Then

$$E\{r^2(X_3)X_1X_2\} = \sqrt{\frac{2}{\pi}}e^{-(\tau^2/2\sigma_3^2)}(L^2-1)\frac{\tau}{\sigma_3^3}r_{13}r_{23} + \left\{L^2 - (L^2-1)\,\text{erf}\left(\frac{\tau}{\sigma_3}\right)\right\}r_{12} \quad (13)$$

where erf (z) is the error function defined as

$$\text{erf}(z) = \sqrt{\frac{2}{\pi}}\int_0^z e^{-(t^2/2)}\,dt. \quad (14)$$

The results of both lemmas can be verified using Price's theorem [12], [14] or by direct evaluation. The proofs are given in Appendix A.

Let

$$V(n) = H(n) - H_{\text{opt}} \quad (15)$$

denote the coefficient misalignment vector at time n, where

$$H_{\text{opt}} = R_{xx}^{-1}R_{xd} \quad (16)$$

is the optimum coefficient vector and R_{xd} is the cross-correlation vector of $X(n)$ and $d(n)$. Also, let

$$K(n) = E\{V(n)V^T(n)\} \quad (17)$$

denote a second moment matrix of the coefficient misalignment vector. Then the following recursive expressions describe the mean and mean-squared behavior of the adaptive filter coefficients equipped with the dual sign algorithm. The derivations are given in Appendix B:

$$E\{V(n+1)\} = \left[I - \frac{\mu}{\sigma_e(n)}\sqrt{\frac{2}{\pi}}\left\{1 + (L-1)e^{-[\tau^2/2\sigma_e^2(n)]}\right\}R_{xx}\right] \cdot E\{V(n)\} \quad (18)$$

and

$$\sigma_e^2(n) = \xi_{\min} + \text{tr}\,\{K(n)R_{xx}\} \quad (19)$$

where tr $\{\cdot\}$ denotes the trace of $\{\cdot\}$ and $\xi_{\min}$ is the minimum mean-squared estimation error given by

$$\xi_{\min} = E\{d^2(n)\} - H_{\text{opt}}^T R_{Xd}. \quad (20)$$

Also, the second moment $K(n)$ of the misalignment vector can be evaluated recursively as

$$K(n+1) = K(n) + \mu^2\left[\sqrt{\frac{2}{\pi}}e^{-[\tau^2/2\sigma_e^2(n)]}(L^2-1) \cdot \frac{\tau}{\sigma_e^3(n)}R_{xx}K(n)R_{xx} + \left\{L^2 - (L^2-1)\,\text{erf}\left(\frac{\tau}{\sigma_e(n)}\right)\right\}R_{xx}\right] - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)}\left\{1 + (L-1)e^{-[\tau^2/2\sigma_e^2(n)]}\right\} \cdot \{R_{xx}K(n) + K(n)R_{xx}\}. \quad (21)$$

Appendix C shows that the distribution functions of the coefficient misalignment vector $V(n)$ converges for any $\mu > 0$, $\tau \ge 0$, and $L \ge 1$. In particular, it implies that the $E\{V(n)\}$ and $K(n)$ also converge to limiting values. From (18), it is easy to see that since $E\{V(n)\}$ converges it must converge to the zero vector. It is shown in Appendix B that the steady state values of $K(n)$ and $\sigma_e^2(n)$ are given by

$$K(\infty) = \mu\sqrt{\frac{\pi}{2}}\sigma_e(\infty)\left[2\left\{1 + (L-1)e^{-[\tau^2/2\sigma_e^2(\infty)]}\right\}I - \mu e^{-[\tau^2/2\sigma_e^2(\infty)]}\frac{\tau}{\sigma_e^2(\infty)}(L^2-1)R_{xx}\right]^{-1} \cdot \left\{L^2 - (L^2-1)\,\text{erf}\left(\frac{\tau}{\sigma_e(\infty)}\right)\right\}I \quad (22)$$

where $\sigma_e^2(\infty)$ and $K(\infty)$ are the limiting values of the mean-squared estimation error and $K(n)$, respectively. $\sigma_e(\infty)$ is the solution of the nonlinear equation

$$\sigma_e^2(\infty) = \xi_{\min} + \text{tr}\left\{R_{xx}K(\infty)\right\} \tag{23}$$

$$= \xi_{\min} + \mu\sqrt{\frac{\pi}{2}}\,\sigma_e(\infty)\sum_{i=1}^{N}\frac{\left\{L^2 - (L^2-1)\,\text{erf}\left(\frac{\tau}{\sigma_e(\infty)}\right)\right\}\lambda_i}{2\left\{1+(L-1)e^{-[\tau^2/2\sigma_e^2(\infty)]}\right\} - \mu e^{-[\tau^2/2\sigma_e^2(\infty)]}\frac{\tau}{\sigma_e^2(\infty)}(L^2-1)\lambda_i}. \tag{24}$$

If we assume that $\tau/\sigma(\infty) >> 1$, we can simplify the above equation as

$$\sigma_e^2(\infty) = \xi_{\min} + \mu\left(\sum_{i=1}^{N}\lambda_i\right)\sqrt{\frac{\pi}{2}}\frac{\sigma_e(\infty)}{2}. \tag{25}$$

This result is the same as that for the sign algorithm with convergence parameter μ. Before using this result, one must make sure that this approximation is valid. When the approximation is valid, we can obtain a closed form expression for the root mean-squared value of the steady-state estimation error by solving for $\sigma_e(\infty)$ in (25) and retaining the positive root. This will give

$$\sigma_e(\infty) = \frac{\alpha + \sqrt{\alpha^2 + 4\xi_{\min}}}{2} \tag{26}$$

where

$$\alpha = \frac{\mu}{2}\left(\sum_{i=1}^{N}\lambda_i\right)\sqrt{\frac{\pi}{2}}. \tag{27}$$

For many choices of the parameters L and τ, the approximations will not be valid. (See, for example, the numerical results in the next section.) In such cases, one must solve for $\sigma_e(\infty)$ numerically from (24). In all the numerical results presented in the next section, the steady-state values were evaluated in this way. One rule of thumb is that one can use (26) if $\tau >> \sqrt{\xi_{\min}}$.

III. Numerical Results

One advantage that the dual sign algorithm has over its single step counterpart is that it is to some extent possible to control the speed of convergence and excess steady-state estimation error separately from each other. Given a specific operating environment, the performance of the sign algorithm is controlled by the choice of a single parameter, the convergence constant μ. However, the performance of the DSA depends on the choice of μ, L, and τ. Consequently, it is often possible to achieve given performance levels by a judicious choice of these parameters. Since there are no closed form expressions available for the speed of convergence and steady-state mean-squared estimation error of the algorithms, the selection of the parameters must be done by numerically evaluating the steady-state quantities and the time the algorithm takes to attain specific performance levels using the equations given in Section II.

The rest of this section deals with an example problem that demonstrates how to design the DSA algorithm to achieve the given performance levels. The validity of the derivations in the last section is also verified in this section by using some Monte Carlo simulations. The example problem considered here is that of identifying a linear system with impulse response

$$H_{\text{opt}}^T = [0.2, 0.4, 0.6, 0.8, 1.0, 0.8, 0.6, 0.4, 0.2]. \tag{28}$$

$x(n)$, the input to the system, which is used as the input signal to the adaptive filter is a zero-mean and Gaussian signal obtained as the output of an autoregressive filter with the transfer function

$$A(z) = \frac{0.4}{1 - 1.79z^{-1} + 1.9425z^{-2} - 1.27z^{-3} + 0.5z^{-4}} \tag{29}$$

when its input is a zero-mean, Gaussian and white pseudorandom sequence with unit variance. The eigenvalue spread of the input autocorrelation matrix is more than 650. The desired response signal of the adaptive filter (denoted by $d(n)$) was obtained by adding uncorrelated, zero mean and white noise with variance 0.1024 to the output of the system to be identified when its input was $x(n)$.

The dependence of the convergence time as well as the steady-state mean-squared estimation error on the parameters μ, L, and τ is shown in Table I for the DSA. In this table, the steady-state mean-squared errors were evaluated by numerically solving for $\sigma_e(\infty)$ from (26). Also, the two different convergence times in number of samples were evaluated as the time taken for the mean-square estimation error to go below twice and 1.1 times the steady-state mean-squared error.

We can observe several things from Table I. The steady-state error power increases with increasing values of L and decreases with increasing values of τ, when the other parameters are fixed. Moreover, for several choices of the parameters, assuming that the steady-state behavior of the DSA is similar to that of the sign algorithm with convergence constant μ will give misleading results. Choice of τ and L is a compromise between the convergence speed and steady-state errors—a large L will increase convergence speed but increase the steady-state error power whereas a large τ will slow down convergence but reduce the steady-state mean-squared error. From Table I we can see that a choice of $\mu = 2^{-10}$, $L = 16$, and $\tau = 1$ will give a reasonable compromise between the convergence speed and steady-state mean-squared error. The results of simulation examples using these parameters are plotted in Fig. 1. The results presented are ensemble averages of 250 independent runs using 4000 samples each.

The measure of convergence displayed in Fig. 1 is the sum of the mean-squared values of each coefficient misalignment sequence (tr $\{K(n)\}$) normalized by $H_{\text{opt}}^T H_{\text{opt}}$. Comparing the theoretical and empirical curves, we find that the two curves show very good match in spite of the fairly large eigenvalue spread for the input autocorrelation matrix in this example. Also plotted in Fig. 1 are the theoretical performance measures of the sign algorithm when the choices of the convergence param-

TABLE I
STEADY-STATE MEAN-SQUARED ESTIMATION ERROR AND CONVERGENCE TIMES FOR THE DSA

μ	L	τ	mse	R_1*	R_2**
2^{-8}	32	0.0	0.7859	13	30
2^{-8}	32	1.0	0.3780	24	44
2^{-8}	32	2.0	0.1104	239	529
2^{-8}	16	0.0	0.3209	34	58
2^{-8}	16	0.5	0.2016	44	70
2^{-8}	16	1.0	0.1169	98	310
2^{-8}	16	2.0	0.1104	278	583
2^{-8}	8	0.0	0.1854	72	110
2^{-8}	8	0.5	0.1337	89	153
2^{-8}	8	1.0	0.1114	153	403
2^{-8}	8	2.0	0.1104	336	645
2^{-8}	1	0.0	0.1104	567	877
2^{-10}	64	0.0	0.3209	34	58
2^{-10}	64	1.0	0.1473	84	257
2^{-10}	64	2.0	0.1043	754	1985
2^{-10}	32	0.0	0.1854	72	110
2^{-10}	32	1.0	0.1080	197	929
2^{-10}	32	2.0	0.1043	904	2166
2^{-10}	16	0.0	0.1382	143	217
2^{-10}	16	0.5	0.1158	176	337
2^{-10}	16	1.0	0.1051	345	1262
2^{-10}	1	0.0	0.1043	2259	3536
2^{-12}	64	0.0	0.1382	143	217
2^{-12}	64	0.5	0.1169	177	344
2^{-12}	64	1.0	0.1056	395	2503
2^{-12}	32	0.0	0.1190	285	435
2^{-12}	32	0.5	0.1088	351	716
2^{-12}	32	1.0	0.1036	740	3706
2^{-12}	16	0.0	0.1104	567	877
2^{-12}	16	1.0	0.1030	1341	5057
2^{-12}	1	0.0	0.1029	9028	14173

*R_1: Time to converge in number of samples to 2 times the steady-state mean-squared error.

**R_2: Time to converge to 1.1 times the steady-state mean-squared error.

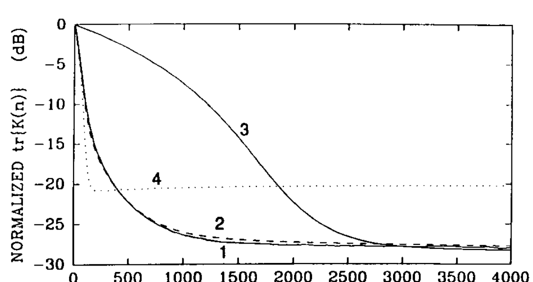

Fig. 1. Curves comparing the performance of the dual sign algorithm with that of the sign algorithm. 1) DSA simulation; $\mu = 2^{-10}$, $L = 16$, $\tau = 1$, 2) DSA theory, 3) SA theory; $\mu = 2^{-10}$, and 4) SA theory; $\mu = 2^{-6}$.

eters are μ and μL, respectively. We can see that the DSA has the fast convergence properties close to that of the sign algorithm with a large convergence constant and also the steady-state properties comparable to that of the sign algorithm with a small convergence constant.

Fig. 2 displays the theoretical and empirical performance measures of the DSA when $\mu = 2^{-12}$, $L = 32$, and $\tau = 0.5$. With these parameters we get somewhat faster convergence but slightly higher steady-state mean-squared error than before. Once again, we can see that the theoretical and empirical curves show very good match. The theoretical values of the mean-squared estimation error have been compared with empirical results for several other values of the triple (μ, L, τ). Except for the cases where μL is very large and τ is relatively small, the

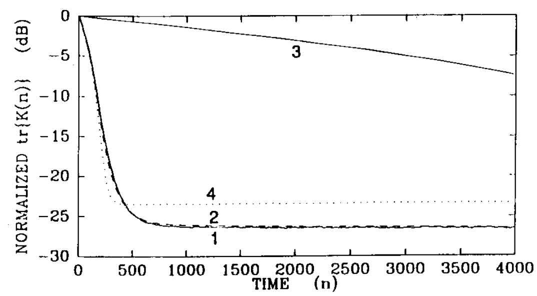

Fig. 2. Curves comparing the performance of the dual sign algorithm with that of the sign algorithm. 1) DSA simulation; $\mu = 2^{-12}$, $L = 32$, $\tau = 0.5$, 2) DSA theory, 3) SA theory; $\mu = 2^{-12}$, and 4) SA theory; $\mu = 2^{-7}$.

theory presented in this paper does predict the performance of the DSA well.

IV. CONCLUDING REMARKS

This paper presented a convergence analysis for stochastic gradient adaptive filters equipped with the dual sign algorithm. Expressions for the mean and mean-squared values of the coefficient misalignment vector were derived under the assumption that the input signal is Gaussian. The main differences between the current analysis and previous analyses are: 1) the present analysis is valid for arbitrary autocorrelation matrices, and 2) the analysis of this paper does not approximate the behavior for the DSA as switching between two sign algorithms a certain amount of time after the DSA is initialized. A simulation example comparing the analytical results with empirical ones was presented and the two curves showed excellent agreement with each other.

The advantages of the dual sign algorithm over the sign algorithm are essentially two. Proper choice of the parameters of the DSA will result in fast convergence and small steady-state errors. Since there are three parameters that control the performance of the DSA, the designer has somewhat better flexibility in the choice of these parameters than sign algorithm when specified performance level must be achieved. With the sign algorithm, the design methodology involves a compromise between the speed of convergence and steady-state behavior of the filter. With the DSA, it is certainly a smaller compromise, since it is to some extent possible to better control the speed of convergence and steady-state performance by selecting μ, L, and τ appropriately. The above advantages of the DSA are possible with very little additional computational complexity over the sign algorithm, and consequently, the author believes that the DSA is a very good choice in practical applications.

APPENDIX A
PROOF OF LEMMAS

Lemma 1: By Price's theorem [12], [14]

$$\frac{\partial}{\partial r_{12}} E\{r(X_2)X_1\}$$

$$= E\left\{\frac{\partial}{\partial X_2} r(X_2) \frac{\partial}{\partial X_1} X_1\right\} \tag{A.1}$$

$$= E\{2\delta(X_2) + (L-1)[\delta(X_2+\tau) + \delta(X_2-\tau)]\} \tag{A.2}$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sigma_2}\left\{1 + (L-1)e^{-(\tau^2/2\sigma_2^2)}\right\}. \tag{A.3}$$

In (A.2), $\delta(\cdot)$ denotes the unit impulse function. It immediately follows that

$$E\{r(X_2)X_1\} = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma_2}\{1 + (L-1)e^{-(\tau^2/2\sigma_2^2)}\}r_{12} + C_1 \tag{A.4}$$

where C_1 is the constant of integration. C_1 can be found to be zero by noting that if $r_{12} = 0$, $E\{r(X_2)X_1\}$ must be zero.

Lemma 2: Following the same procedure as before

$$\frac{\partial}{\partial r_{12}} E\{r^2(X_3)X_1X_2\} = E\{r^2(X_3)\} \tag{A.5}$$

$$= L^2 - (L^2 - 1)\,\mathrm{erf}\left(\frac{\tau}{\sigma_3}\right) \tag{A.6}$$

where erf (z) is defined as in (14). Integrating (A.6) with respect to r_{12}, we get

$$E\{r^2(X_3)X_1X_2\} = \left(L^2 - (L^2-1)\,\mathrm{erf}\left(\frac{\tau}{\sigma_3}\right)\right) r_{12} + C_2 \tag{A.7}$$

where C_2 is the constant of integration and must be evaluated. We evaluate C_2 by setting $r_{12} = 0$ and solving for C_2 in (A.7). With this constraint,

$$\frac{\partial}{\partial r_{13}} E\{r^2(X_3)X_1X_2\}$$
$$= E\{X_2[(L^2-1)\{\delta(X_3 - \tau) - \delta(X_3 + \tau)\}]\}. \tag{A.8}$$

Some straightforward calculation will show that the above equation can be simplified to

$$C_2 = E\{r^2(X_3)X_1X_2\}\big|_{r_{12}=0}$$
$$= \sqrt{\frac{2}{\pi}}e^{-(\tau^2/2\sigma_3^2)}(L^2-1)\frac{\tau}{\sigma_3^3}r_{13}r_{23}. \tag{A.9}$$

The result of Lemma 2 follows when the value of C_2 in (A.9) is substituted in (A.7).

Appendix B
Derivation of (18), (21), and (22)

From (3) and (15)

$$V(n+1) = V(n) + \mu r(n)X(n). \tag{B.1}$$

Since $X(n)$ and $d(n)$ are zero-mean and jointly Gaussian random processes, the error signal $e(n)$ is also zero mean and Gaussian, conditioned on the misalignment vector $V(n)$. Taking the conditional expected value of both sides, we get

$$E\{V(n+1)|V(n)\} = V(n) + \mu E\{r(n)X(n)|V(n)\}. \tag{B.2}$$

Applying Lemma 1 to the second term on the right-hand side of (B.2), using the approximation in (6) and then taking the expectation of both sides of (B.2) again, the following results:

$$E\{V(n+1)\}$$
$$= E\{V(n)\} + \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)}\{1 + (L-1)e^{-(\tau^2/2\sigma_e^2(n))}\}$$
$$\cdot E\{X(n)e(n)\}. \tag{B.3}$$

Equation (18) follows easily since

$$E\{X(n)e(n)\} = -R_{xx}E\{V(n)\}. \tag{B.4}$$

Postmultiplying both sides of (B.1) with their respective transposes and taking the statistical expectations,

$$K(n+1) = K(n) + \mu^2 E\{r^2(n)X(n)X^T(n)\}$$
$$+ \mu E\{r(n)V(n)X^T(n)\}$$
$$+ E\{r(n)X(n)V^T(n)\}. \tag{B.5}$$

The second, third, and fourth terms on the right-hand side can be evaluated by first taking the conditional (on $V(n)$) expectation of each quantity using the lemmas, then applying the approximation of (6) and then taking the expectation again. Thus

$$E\{r^2(n)X(n)X^T(n)\}$$
$$= E\{E\{r^2(n)X(n)X^T(n)|V(n)\}\} \tag{B.6}$$

$$\approx E\left\{\sqrt{\frac{2}{\pi}}e^{-[\tau^2/2\sigma_e^2(n)]}(L^2-1)\frac{\tau}{\sigma_e^3(n)}R_{xx}V(n)V^T(n)R_{xx}\right\}$$
$$+ \left\{L^2 - (L^2-1)\,\mathrm{erf}\left(\frac{\tau}{\sigma_e(n)}\right)\right\}R_{xx} \tag{B.7}$$

$$= \sqrt{\frac{2}{\pi}}e^{-[\tau/2\sigma_e^2(n)]}(L^2-1)\frac{\tau}{\sigma_e^3(n)}R_{xx}K(n)R_{xx}$$
$$+ \left\{L^2 - (L^2-1)\,\mathrm{erf}\left(\frac{\tau}{\sigma_e(n)}\right)\right\}R_{xx}. \tag{B.8}$$

Also,

$$E\{V(n)X^T(n)r(n)\}$$
$$= E\{E\{V(n)X^T(n)r(n)|V(n)\}\} \tag{B.9}$$

$$= E\left\{-V(n)\sqrt{\frac{2}{\pi}}\{1 + (L-1)e^{-[\tau^2/2\sigma_e^2(n)]}\}\right.$$
$$\left.\cdot\frac{1}{\sigma_e(n)}V^T(n)R_{xx}\right\} \tag{B.10}$$

$$= -\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)}\{1 + (L-1)e^{-[\tau^2/2\sigma_e^2(n)]}\}K(n)R_{xx}. \tag{B.11}$$

Similarly,

$$E\{r(n)X(n)V^T(n)\}$$
$$= -\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(n)}\{1 + (L-1)e^{-[\tau^2/2\sigma_e^2(n)]}\}R_{xx}K(n). \tag{B.12}$$

Equation (21) results from substituting (B.8), (B.11) and (B.12) in (B.5).

It is shown in Appendix C that $V(n)$ converges in distribution. Consequently, $K(n)$ converges, and we can easily show that the following equality holds in the steady state:

$$\mu^2\left[\sqrt{\frac{2}{\pi}}e^{-[\tau^2/2\sigma_e^2(n)]}(L^2-1)\frac{\tau}{\sigma_e^3(\infty)}R_{xx}K(n)R_{xx} + \left\{L^2-(L^2-1)\,\mathrm{erf}\left(\frac{\tau}{\sigma_e(\infty)}\right)\right\}R_{xx}\right] - \mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(\infty)}\left\{1+(L-1)e^{-[\tau^2/2\sigma_e^2(\infty)]}\right\} \cdot \left\{R_{xx}K(\infty)+K(\infty)R_{xx}\right\} = \phi \tag{B.13}$$

where ϕ denotes an $N \times N$ matrix whose every element is zero, and $K(\infty)$ and $\sigma_e(\infty)$ are the steady-state values of $K(n)$ and $\sigma_e(n)$. Equation (22) results from solving for $K(\infty)$ from (B.13). In order to do this, we transform the above equation into the "primed coordinate system." Let Q be an orthonormal matrix such that

$$Q^T R_{xx} Q = \Lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_N) \tag{B.14}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_N$ are the eigenvalues of R_{xx} and let

$$K'(\infty) = Q^T K(\infty) Q. \tag{B.15}$$

It is relative easy to show that $K'(\infty)$ is also a diagonal matrix. Premultiplying and postmultiplying both sides of (B.13) will give

$$\mu^2\left[\sqrt{\frac{2}{\pi}}e^{-[\tau^2/2\sigma_e^2(\infty)]}(L^2-1)\frac{\tau}{\sigma_e^3(\infty)}\Lambda^2K'(\infty) + \left\{L^2-(L^2-1)\,\mathrm{erf}\left(\frac{\tau}{\sigma_e(\infty)}\right)\right\}\Lambda\right] - 2\mu\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_e(\infty)}\left\{1+(L-1)e^{-[\tau^2/2\sigma_e^2(\infty)]}\right\} \cdot \Lambda K'(\infty) = \phi. \tag{B.16}$$

Solving for $K(\infty)$ from (B.16) and then premultiplying and postmultiplying by Q and Q^T, respectively, will result in (22).

Appendix C
Proof of the Convergence of $V(n)$ in Distribution

The proof of convergence follows the one used in [7] very closely and therefore only sketches of the proof will be given here. To prove convergence, we first establish a very loose upper bound for the long-term time average (LTTA) of the mean absolute estimation error (MAEE). One of the consequences of the independence assumption is that $V(n)$ is a Markov process. With the help of the upper bound on the LTTA of the MAEE, we will show that $V(n)$ has a stationary distribution. Once this is established, it follows from Doob's theorem [5], [7] that the sequence of probability distribution functions of $V(n)$ converges to the above stationary distribution.

From (3) and (15)

$$V(n+1) = V(n) + \mu|r(n)|\,\mathrm{sign}\,\{e(n)\}X(n). \tag{C.1}$$

Taking the squared norm of both sides will give

$$\|V(n+1)\|^2 = \|V(n)\|^2 + \mu^2|r(n)|^2\|X(n)\|^2 + 2\mu|r(n)|\,\mathrm{sign}\,\{e(n)\}V^T(n)X(n). \tag{C.2}$$

Note that

$$V^T(n)X(n) = e_{\min}(n) - e(n) \tag{C.3}$$

where $e_{\min}(n)$ is the optimal estimation error given by

$$e_{\min}(n) = d(n) - H_{\mathrm{opt}}^T X(n). \tag{C.4}$$

Substituting this in (C.2) and realizing that

$$e_{\min}(n)\,\mathrm{sign}\,\{e(n)\} \le |e_{\min}(n)| \tag{C.5}$$

and that $|r(n)| \le L$, we get the following inequality:

$$\|V(n+1)\|^2 \le \|V(n)\|^2 + \mu^2L^2\|X(n)\|^2 + 2L\mu|e_{\min}(n)| - 2\mu|e(n)|. \tag{C.6}$$

Iterating (C.4) n times will give an expression of $\|V(n+1)\|^2$ in terms of $\|V(1)\|^2$ as

$$\|V(n+1)\|^2 \le \|V(1)\|^2 + \mu^2L^2\sum_{i=1}^{n}\|X(i)\|^2 + 2\mu L\sum_{i=1}^{n}|e_{\min}(i)| - 2\mu\sum_{i=1}^{n}|e(i)|. \tag{C.7}$$

Now, take the statistical expectations of both sides of (C.7). Also, recognize that $\|V(1)\|^2$ is bounded in all practical situations and that $\|V(n+1)\|^2 \ge 0$. It is now straightforward to show that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}E\{|e(i)|\} \le LE\{|e_{\min}(n)|\} + \frac{\mu L^2}{2}E\{\|X(n)\|^2\}. \tag{C.8}$$

In deriving the above result we have also made use of the fact that $X(n)$ and $e_{\min}(n)$ are stationary processes. Obviously, the above bound on the long-term time average of the mean absolute estimation error sequence is a very loose one, but this will suffice for our purpose.

As noted earlier, $V(n)$ is a Markov process. Let S denote a Borel measurable subset of an N-dimensional Euclidean space and let

$$\mathfrak{F}_n\{S\} = P\{V(n)\epsilon S\} \tag{C.9}$$

where $P\{\cdot\}$ is the probability of the event $\{\cdot\}$. Also let

$$\mathcal{G}_n\{S\} = \frac{1}{n}\sum_{k=1}^{n}\mathfrak{F}_k\{S\}. \tag{C.10}$$

Obviously $\mathcal{G}_n\{S\}$ is also a probability distribution function. Let us define a cost function ψ as

$$\psi(V(n), V(n-1), \cdots, V(1)) = E\{|e(n)|\,|\,V(n), \cdots, V(1)\} = E\{|e_{\min}(n) - V^T(n)X(n)|\,|\,V(n), \cdots, V(1)\}. \tag{C.11}$$

Since $V(n)$ is a Markov process, it follows that

$$\psi(V(n), V(n-1), \cdots, V(1)) = E\{|e(n)|\,|\,V(n)\} = \psi(V(n)). \tag{C.12}$$

Also, note that

$$E\{\psi(V(n))\} = E\{|e(n)|\} \tag{C.13}$$

Substituting (C.13) in (C.8) will give

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} E\left\{\psi(V(k))\right\} \leq LE\left\{\left|e_{\min}(n)\right|\right\} + \frac{\mu L^2}{2} E\left\{\|X(n)\|^2\right\}. \quad \text{(C.14)}$$

This implies that there exists a constant t such that for any finite and nonzero value of μ,

$$\frac{1}{n}\sum_{k=1}^{n} E\left\{\psi(V(k))\right\} \leq t \quad \forall n. \quad \text{(C.15)}$$

Since $\psi(V(n)) \geq 0$ for all n, we can use the Chebyshev inequality to get

$$P\left\{\psi(V(n)) \geq M\right\} \leq \frac{1}{M} E\left\{\psi(V(n))\right\} \quad \text{(C.16)}$$

for any positive M. Combining (C.16) and (C.15) gives

$$\frac{1}{n}\sum_{k=1}^{n} P\left\{\psi(V(k)) \geq M\right\} \leq \frac{t}{M}. \quad \text{(C.17)}$$

Since $\{e_{\min}(n), X(n)\}$ is jointly Gaussian distributed (implying that the distribution function is always positive) $\psi(V(n))$ becomes unbounded as $V(n)$ becomes unbounded, and it follows that there exists some positive R (that depends on M) such that

$$P\left\{\|V(n)\| > R\right\} \leq P\left\{\psi(V(n)) \geq M\right\}. \quad \text{(C.18)}$$

Combining (C.15) and (C.18), we have the following result:

$$\frac{1}{n}\sum_{k=1}^{n} P\left\{\|V(k)\| > R\right\} \leq \frac{t}{M}. \quad \text{(C.19)}$$

The left-hand side of (C.19) is nothing but $\mathcal{G}_n\{\|V(n)\| > R\}$, implying that

$$\mathcal{G}_n\left\{\|V(n)\| > R\right\} \leq \frac{t}{M}. \quad \text{(C.20)}$$

The above inequality implies that the sequence of distributions $\mathcal{G}_n$ is stochastically bounded. This in turn implies that there exists a convergent subsequence of distributions $\mathcal{G}_{n_i}\{S\}$ such that

$$\lim_{i\to\infty} \mathcal{G}_{n_i}\{S\} = \mathcal{G}\{S\} \quad \text{(C.21)}$$

where $\mathcal{G}\{S\}$ denotes a limiting distribution.

Now, since $V(n)$ belongs to a Markov process, there exists a linear transformation operator $\mathcal{H}$ ($\mathcal{H}$ is also time invariant since it depends only on $X(n)$ and $e_{\min}(n)$, both of which are stationary) such that

$$\mathcal{F}_{n+1}\{S\} = \mathcal{H}\left\{\mathcal{F}_n\{S\}\right\}. \quad \text{(C.22)}$$

Note that

$$\mathcal{H}\left\{\mathcal{G}_{n_i}\{S\}\right\} = \frac{1}{n_i}\sum_{k=1}^{n_i} \mathcal{H}\left\{\mathcal{F}_k\{S\}\right\} \quad \text{(C.23a)}$$

$$= \frac{1}{n_i}\sum_{k=1}^{n_i} \mathcal{F}_{k+1}\{S\} \quad \text{(C.23b)}$$

$$= \mathcal{G}_{n_i}\{S\} + \frac{1}{n_i}\left[\mathcal{F}_{n_i+1}\{S\} - \mathcal{F}_1\{S\}\right]. \quad \text{(C.23c)}$$

Taking the limit as $i \to \infty$, this becomes

$$\lim_{i\to\infty} \mathcal{H}\left\{\mathcal{G}_{n_i}\{S\}\right\} = \mathcal{G}\{S\}. \quad \text{(C.24)}$$

It is straightforward to employ the arguments used in [7] to show that

$$\mathcal{H}\left\{\mathcal{G}\{S\}\right\} = \mathcal{G}\{S\}. \quad \text{(C.25)}$$

This implies that $\mathcal{G}$ is a stationary distribution of the Markov process $V(n)$. It is also easy to show (again we omit details here since the reasoning is essentially the same as that in [7]) that $V(n)$ satisfies all the hypotheses of Doob's theorem [5], [7]. Then, by Doob's theorem, the successive distributions $\mathcal{F}_n$ converge to $\mathcal{G}$ (in the weak sense) for any initial misalignment vector $V(1)$.

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