A FINITE INDEX SUBGROUP OF $\mathbf{B_N}(\mathcal{O}_S)$ WITH INFINITE DIMENSIONAL COHOMOLOGY

by

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ABSTRACT

Let $\mathbb{F}_p$ be the finite field with p elements, let S be a finite nonempty set of inequivalent valuations on $\mathbb{F}_p(t)$, and let $\mathcal{O}_S$ be the ring of S-integers. If $\mathbf{B_n}$ is the solvable, linear algebraic group of upper triangular matrices with determinant 1, then the solvable S-arithmetic group $\mathbf{B_n}(\mathcal{O}_S)$ has a finite-index subgroup with infinite-dimensional cohomology group in dimension $|S|$.

CONTENTS

CHAPTER 1

INTRODUCTION

We begin by recalling definitions of various finiteness properties for groups.

Definition 1 *A group G is said to be of type F_m if G acts freely on a contractible CW complex X and the m-skeleton of $G\backslash X$ is finite.*

If a group G is of type F_m, then by definition (using the cellular chain complex of X), there is a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ which is finitely generated up to dimension m. This suggests the following weakening of the finiteness condition F_m.

Definition 2 *A group G is of type FP_m (with respect to $\mathbb{Z}$) if there is a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ that is finitely generated up to dimension m.*

It is clear that if a group is F_m, then it is FP_m. In [1], Bestvina-Brady give an example to show that FP_m is strictly weaker than F_m (see example 6.3.3).

Let S be a finite nonempty set of inequivalent valuations on a global field K, and $\mathcal{O}_S$ be the ring of S-integers. Given an affine algebraic group $\mathbf{G}$ defined over K, we can realize $\mathbf{G}(K)$ as a subgroup of $\mathbf{GL_n}(K)$. An S-arithmetic group is defined by restricting the entries of $\mathbf{G}(K)$ to $\mathcal{O}_S \subseteq K$. Given a valuation v on K let K_v denote the completion of K with respect to the norm $||\cdot||_v$ induced by v.

For any field extension L/K, the L-rank of $\mathbf{G}$, denoted $\text{rank}_L\mathbf{G}$, is the dimension of a maximal L-split torus of $\mathbf{G}$. For any K-group $\mathbf{G}$ and set of places S, we define the nonnegative integer

$$k(\mathbf{G}, S) = \sum_{v \in S} \text{rank}_{K_v}\mathbf{G}. \tag{1.1}$$

This number is called the sum of the local ranks.

Bux-Köhl-Witzel recently showed that every S-arithmetic subgroup $\mathbf{G}(\mathcal{O}_S)$ of a non-commutative K-isotropic absolutely almost simple group $\mathbf{G}$ defined over a global function field K is of type $F_{k(\mathbf{G},S)-1}$ but not of type $F_{k(\mathbf{G},S)}$ [2]. Applying this theorem to $\mathbf{SL_n}(\mathbb{F}_p[t])$ shows that $\mathbf{SL_n}(\mathbb{F}_p[t])$ is of type F_{n-2} but not of type F_{n-1}.

For the case of solvable groups, the result is different in that it does not depend at all on the rank of the group. In [3], Bux shows that if $\mathbf{G}$ is a Chevalley group and $\mathbf{B} \leq \mathbf{G}$ is a Borel subgroup, then $\mathbf{B}(\mathcal{O}_S)$ is of type $F_{|S|-1}$ but not type $FP_{|S|}$.

If a finitely-generated group G is of type FP_m, then $H^m(G;R)$ is a finitely-generated R-module. However, if a group fails to be FP_m, then it is not necessarily the case that $H^m(G;R)$ is an infinitely-generated R-module. For an example of this, let H be a nontrivial perfect group (the abelenization of H is trivial). Now let

$$G = \bigoplus_{\mathbb{N}} H \tag{1.2}$$

then obviously G is not FP_1 since it is not finitely generated and FP_1 is equivalent to being finitely generated. However, $H_1(G;\mathbb{Z}) = 0$, since H_1 is the abelianization of G. So asking if $H_m(G,-)$ is finitely generated becomes an interesting question even when we know that G is not FP_m.

The group $\mathbf{SL_n}(\mathbb{Z}[t])$ is not an S-arithmetic group. However, many of the techniques used for S-arithmetic groups can be employed to gain results about finiteness properties of $\mathbf{SL_n}(\mathbb{Z}[t])$. In [4], Bux-Mohammadi-Wortman show that $\mathbf{SL_n}(\mathbb{Z}[t])$ is not FP_{n-1} and in [5], Cesa-Kelly demonstrate that certain principal congruence subgroups of $\mathbf{SL_n}(\mathbb{Z}[t])$ have infinite-dimensional cohomology in dimension $(n-1)$. Knudson has shown that $H_2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Z})$ is infinite .imensional [6]. In [7], Cobb gives a new proof of this theorem by studying the Euclidean building for $\mathbf{SL_n}(\mathbb{Q}((t^{-1})))$.

This paper brings together techniques from [5] and [3] to prove the following theorem.

Theorem 1 *Let Γ_n be the finite-index subgroup of $\mathbf{B_n}(\mathcal{O}_S)$ such that the diagonal entries all are of the form $\frac{f}{g}$ where $f, g \in \mathbb{F}_p[t]$ are monic polynomials. If $p \neq 2$, then $H^{|S|}(\Gamma_n;\mathbb{F}_p)$ is an infinite-dimensional vector space.*

In Chapter 2, we show that dimension of $H^k(\Gamma_n;\mathbb{F}_P)$ is the same as the dimension of $H^k(\Gamma_2;\mathbb{F}_p)$. Therefore, to prove Theorem 1, we will focus our attention on $\Gamma_2 \subseteq \mathbf{B_2}(\mathcal{O}_S)$. To simplify notation, in what follows, let $\Gamma = \Gamma_2$.

In Chapter 3, we show that Γ acts on X a product of trees and Y, a modified horosphere. The goal is to construct an infinite family of cocylces on $\Gamma\backslash Y$ and show that they are independent.

CHAPTER 2

AN ALGEBRAIC RETRACT

In [3], Bux shows that the finiteness length of $\mathbf{B_n}(\mathcal{O}_S)$ does not depend on the rank of $\mathbf{B_n}$ but instead only the number of places in S. This surprising result is uncovered for the group $\mathbf{B_n}$ by linking the finiteness properties of $\mathbf{B_n}$ and the finiteness properties of $\mathbf{B_2}$.

Definition 3 *A retract between two groups G and H is given by a surrjection $G \to H$ and an inclusion $H \to G$ such that the composition $H \to G \to H$ is the identity. Denote a retract of groups by $G \leftrightarrows H$.*

In Section 4 of [3], it is shown that if there is a retract $G \rightleftarrows H$, then G and H have the same finiteness length.

Lemma 2 *Suppose $G \leftrightarrows H$ is a retract of groups. Then there is an injection between $H^i(H;\mathbb{F}_p)$ and $H^i(G;\mathbb{F}_p)$.*

Proof. The proposition follows since functors and cofunctors preserve the structure of the retract. ■

Lemma 3 *There is a retract $\Gamma_n \rightleftarrows \Gamma$.*

Proof.

$$\Gamma \simeq \begin{pmatrix} * & * & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \subseteq \Gamma_n$$

■

CHAPTER 3

A GOOD CHOICE OF UNIFORMIZERS

The valuations on $\mathbb{F}_p(t)$ arise in two manners. The first way to build a discrete valuation is to choose $f \in \mathbb{F}_p[t]$, an irreducible polynomial. Every element $h \in \mathbb{F}_p(t)$ can be associated to a unique integer k by writing

$$h = f^k \frac{g}{q} \tag{3.1}$$

where $g, q \in \mathbb{F}_p[t]$ and f does not divide g or q. Therefore, for each irreducible polynomial $f \in \mathbb{F}_p[t]$, we have a valuation

$$\nu_f(h) = k. \tag{3.2}$$

There is one more valuation that is not accounted for by a suitable choice of $f \in \mathbb{F}_p[t]$. Given a rational function

$$h = \frac{g}{q} \tag{3.3}$$

let

$$\nu_\infty(h) = \deg(q) - \deg(g). \tag{3.4}$$

For any discrete valuation ν_i there is an element $\pi_i \in \mathbb{F}_p(t)$ such that $\nu_i(\pi_i) = 1$. The element π_i is called a uniformizer and plays an important role in the construction for the Euclidean building corresponding to $\mathbf{SL_2}(\mathbb{F}_p(t)_{\nu_i})$.

There are many choices for π_i and if your are working with one place (considering the $|S| = 1$ case), it is not so important which element you choose to be your uniformizer. However, working in a setting with multiple places, it will be useful to make sure that we can limit the interaction between different valuations and uniformizers. The content of the Lemma 4 makes this assurance for us.

Lemma 4 *There exists $d \in \mathbb{N}$ such that for each $\nu_i \in S$, there exists $\pi_{i,S} \in \mathbb{F}_p(t)$ with the following relationship*

$$\nu_i(\pi_{j,S}) = \begin{cases} d & : i = j \\ \\ 0 & : i \neq j \end{cases} \tag{3.5}$$

Note that the elements $\pi_{i,S}$ are not uniformizers if $d \neq 1$. Before we supply a proof for Lemma 4, consider the following example.

Example 1 *For this example, fix a prime* $p = 2$ *and a set of inequivalent valuations* $S = \{\nu_t, \nu_\infty, \nu_{t+1}\}$*. The following choices of elements satisfy the lemma:*

$$\pi_{t,S} = \frac{t^2}{t^2+t+1} \tag{3.6}$$

$$\pi_{t+1,S} = \frac{(t+1)^2}{t^2+t+1} \tag{3.7}$$

$$\pi_{\infty,S} = \frac{1}{t^2+t+1} \tag{3.8}$$

Note that there is no choice of elements such that

$$\nu_i(\pi_{j,S}) = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases} \tag{3.9}$$

Proof.[Lemma 4] Given a valuation $\nu_i \in S$, let $f_i \in \mathbb{F}_p[t]$ be the monic irreducible polynomial that is associated to ν_i (in the case where $\nu_i = \nu_\infty$, let $f_i = 1/t$). We know that there is a monic irreducible polynomial associated to ν_i since we are working over a finite field. Because there are infinitely many primes in $\mathbb{F}_p[t]$, there is an irreducible polynomial $h \in \mathbb{F}_p[t]$ such that ν_h is not equivalent to any of the valuations $\nu_i \in S$. Now let

$$\pi_{i,S} = \begin{cases} \frac{f_i^{\deg(h)}}{h^{\deg(f_i)}} & : \nu_i \neq \nu_\infty \\ \frac{1}{h} & : \nu_i = \nu_\infty \end{cases} \tag{3.10}$$

From the construction, we see that

$$\nu_i(\pi_{j,S}) = \begin{cases} \deg(h) & : i = j \\ 0 & : i \neq j \end{cases} \tag{3.11}$$

The integer d in the lemma can be chosen to be the least integer d such that there is an irreducible polynomial $h \in \mathbb{F}_p[t]$ of degree d with $\nu_h \notin S$.

■

The elements $\pi_{i,S}$ constructed in the previous lemma are elements of $\mathbb{F}_p(t)$. They are not elements of $\mathcal{O}_S$. This can be witnessed by seeing that

$$\nu_h(\pi_{i,S}) < 0 \tag{3.12}$$

for each $1 \leq i \leq |S|$. The following lemma shows that there cannot be a nontrivial element $a \in \mathcal{O}_S$ such that $\nu_i(a) > 0$ for all $\nu_i \in S$.

Lemma 5 *If* $a \in \mathcal{O}_S \subseteq \mathbb{F}_p(t)$ *and* $\nu_i(a) > 0$ *for all* $\nu_i \in S$*, then* $a = 0$*.*

Proof. Assume there is nonzero element $a \in \mathcal{O}_S$ such that $\nu_i(a) > 0$ for all $\nu_i \in S$. Then, $a = \frac{g}{h}$ where $g, h \in \mathbb{F}_p[t]$ and either g or h has degree at least one. Since $a \in \mathcal{O}_S$ and either $\nu_\infty \in S$ or $\nu_\infty \notin S$, $\nu_\infty(a) \geq 0$. Therefore, $\deg(h) \geq \deg(g)$ and therefore $\deg(h) > 0$. Choose a prime polynomial p such that p divides h. Notice that $\nu_p(a) < 0$. Either $\nu_p \in S$ or $\nu_p \notin S$. If $\nu_p \in S$, then by the hypothesis of the lemma $\nu_p(a) > 0$. If $\nu_p \notin S$, then $\nu_p(a) \geq 0$ since $a \in \mathcal{O}_S$. This shows our assumption leads to a contradiction.

■

CHAPTER 4

TWO SPACES Γ ACTS ON

In this chapter wee record two spaces that Γ acts on.

4.1 A tree for each place

For each place $\nu \in S$, let $\mathbb{F}_p(t)_\nu$ be the completion of $\mathbb{F}_p(t)$ with respect to ν. Let $A_\nu = \{x \in \mathbb{F}_p(t)_\nu : \nu(x) \geq 0\}$ be the valuation ring associated to the field $\mathbb{F}_p(t)_\nu$. An A_ν-lattice of $V_\nu = \mathbb{F}_p(t)_\nu \times \mathbb{F}_p(t)_\nu$ is an A_ν-submodule of V_ν of the form $L = A_\nu e_1 \oplus A_\nu e_2$ where e_1, e_2 is the standard basis of V.

To build the Euclidean building for $\mathbf{SL_2}(\mathbb{F}_p(t)_\nu)$, take for vertices homethety classes of $A_\nu-$lattices (two lattices L and L' are homethetic if $\lambda L = L'$ for some $\lambda \in \mathbb{F}_p(t)_\nu$). Choose an element $\omega_\nu \in A_\nu$ such that $\nu(\omega_\nu) = 1$. There is an edge between two lattice classes Λ, Λ' if there are representative lattices L, L' such that

$$\omega_\nu L < L' < L. \tag{4.1}$$

Let X_ν denote the tree that is constructed in this way. The tree X_ν is a regular tree where each vertex has valence equal to the cardinality of the residue field $A_\nu/\omega_\nu A_\nu$. The tree can be realized as a union of lines each isometric to $\mathbb{R}$. These lines are called apartments. Fix a basis e_1, e_2 for V_ν. The standard apartment is the orbit of the two edges

$$\{\omega_\nu^0 e_1, e_2\} - - - - \{\omega_\nu^1 e_1, e_2\} - - - - \{\omega_\nu^2 e_1, e_2\} \tag{4.2}$$

under the action of diagonal matrices. The vertices of the standard apartment have representative latices of the form $\{\omega_\nu^k e_1, e_2\}$. The stabilizer of the vertex $\{\omega_\nu^k e_1, e_2\}$ contains matrices of the form

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

where $\nu(f) \geq k$.

Consult Chapter 2 of Serre's book Trees [8] or Brown's book Buildings [9] for more details.

4.2 A product of trees - X

Since Γ embeds diagonally into $\prod_{\nu\in S}\mathbf{SL_2}(\mathbb{F}_p(t)_\nu)$ and each $\mathbf{SL_2}(\mathbb{F}_p(t)_\nu)$ acts on X_ν, the group Γ acts on $X=\prod_{\nu\in S}X_\nu$. The product X is a Euclidean building with apartments isometric to $\mathbb{R}^{|S|}$. The standard apartment in X is the product of the standard apartments from each of the factors:

$$\mathcal{A}_S=\prod_{\nu\in S}\mathcal{A}_\nu. \tag{4.4}$$

Therefore, all of the vertices in $\mathcal{A}_S$ can be described as an $|S|$-tuple, where $(a_1,a_2,\ldots,a_{|S|})\in\mathbb{Z}^{|S|}$ describes the point associated to the following point in the product

$$\prod_{\nu_i\in S}\{\omega_{\nu_i}^{a_i}e_1,e_2\}\in X. \tag{4.5}$$

Lemma 6 *Let $\mathcal{O}_S^*$ denote the units of $\mathcal{O}_S$. For every $1\le i,j\le|S|$, the element $\frac{\pi_{i,S}^{deg(f_j)}}{\pi_{j,S}^{deg(f_i)}}$ is a quotient of monic polynomials in $\mathcal{O}_S^*$.*

Proof. The units in $\mathcal{O}_S$ are exactly the elements a such that a and a^{-1} are both in the ring of S-integers. We will show that $\frac{\pi_{i,S}^{\deg(f_j)}}{\pi_{j,S}^{\deg(f_i)}}\in\mathcal{O}_S^*$ by showing that $\frac{\pi_{i,S}^{\deg(f_j)}}{\pi_{j,S}^{\deg(f_i)}}\in\mathcal{O}_S$ and not making use of the fact that $i<j$ or $j<i$.

$$\frac{\pi_{i,S}^{\deg(f_j)}}{\pi_{j,S}^{\deg(f_i)}}=\frac{f_i^{\deg(f_j)\deg(h)}}{h^{\deg(f_i)\deg(f_j)}}\frac{h^{\deg(f_i)\deg(f_j)}}{f_j^{\deg(f_i)\deg(h)}} \tag{4.6}$$

$$=\frac{f_i^{\deg(f_j)\deg(h)}}{f_j^{\deg(f_i)\deg(h)}} \tag{4.7}$$

To show that this is an S-integer, we will show that $\nu(\frac{f_i^{\deg(f_j)\deg(h)}}{f_j^{\deg(f_i)\deg(h)}})\ge 0$ for all $\nu\notin\{\nu_{f_i},\nu_{f_j}\}$. The only possible ν to present a challenge is showing that $\nu_\infty(\frac{f_i^{\deg(f_j)\deg(h)}}{f_j^{\deg(f_i)\deg(h)}})\ge 0$. However, since the denominator and numerator have the same degree $\nu_\infty(\frac{f_i^{\deg(f_j)\deg(h)}}{f_j^{\deg(f_i)\deg(h)}})=0$. This shows that $\frac{\pi_{i,S}^{\deg(f_j)}}{\pi_{j,S}^{\deg(f_i)}}\in\mathcal{O}_S^*$.

■

Lemma 7 *The convex hull of $\Gamma\cdot(0,0,\ldots,0)\cap\mathcal{A}_S$ contains a $(|S|-1)-$ dimensional flat. Using the coordinates described above, the convex hull of $\Gamma\cdot(0,0,\ldots,0)\cap\mathcal{A}_S$ is the span of the vectors*

$$v_1 = (deg(f_1), -deg(f_2), 0, \ldots, 0), \tag{4.8}$$

$$v_2 = (0, deg(f_2), -deg(f_3), 0, \ldots, 0), \tag{4.9}$$

$$v_3 = (0, 0, deg(f_3), -deg(f_4), 0, \ldots, 0), \tag{4.10}$$

$$\vdots \tag{4.11}$$

$$v_{|S|-1} = (0, \ldots, deg(f_{|S|-1}), -deg(f_{|S|})). \tag{4.12}$$

Furthermore, $\Gamma \cdot (0, 0, \ldots, 0) \cap \mathcal{A}_S$ *is quasi-isometric to this* $(|S|-1)$*-dimensional flat.*

Proof. The first remark is that $\mathcal{O}_S^*$ contains a copy of $\mathbb{Z}^{|S|-1}$ as a finite index subgroup. This follows by an application of Dirichlet's unit theorem (see Theorem 5.12 [10]). Such a subgroup containing only $\frac{f}{g}$ with $f, g \in \mathbb{F}_p[t]$ and f, g monic polynomials is constructed in Lemma 6. This demonstrates that the orbit of $(0, 0, \ldots, 0)$ under the orbit of diagonal elements in Γ is quasi-isometric to an $(|S|-1)$-dimensional flat.

From Lemma 6, we know that

$$\begin{pmatrix} \frac{\pi_{i,S}^{\deg(f_{i+1})}}{\pi_{i+1,S}^{\deg(f_i)}} & 0 \\ 0 & \frac{\pi_{i+1,S}^{\deg(f_i)}}{\pi_{i,S}^{\deg(f_{i+1})}} \end{pmatrix} \in \mathbf{B_2}(\mathcal{O}_S). \tag{4.13}$$

This shows that

$$2d \cdot v_i = \begin{pmatrix} \frac{\pi_{i,S}^{\deg(f_{i+1})}}{\pi_{i+1,S}^{\deg(f_i)}} & 0 \\ 0 & \frac{\pi_{i+1,S}^{\deg(f_i)}}{\pi_{i,S}^{\deg(f_{i+1})}} \end{pmatrix} (0, 0, \ldots, 0) \tag{4.14}$$

is in the convex hull of the orbit and therefore, the convex hull of the orbit contains $\text{span}(v_1, v_2, \ldots, v_{|S|-1})$.

■

Let $\mathcal{A}_{\mathcal{O}}$ denote the $(|S|-1)$-dimensional flat described in Lemma 7.

Lemma 8 *The sequence of points* $x_m = \{(-m, -m, ..., -m)\}_{m \in \mathbb{N}}$ *in* $\mathcal{A}_S$ *is unbounded in the quotient* $\mathbf{SL_2}(\mathcal{O}_S) \backslash X$*.*

Proof. The proof is modeled after a result of Bux-Wortman (see [11] Lemma 2.2).

The group $G = \prod_{\nu \in S} \mathbf{SL_2}(\mathbb{F}_p(t))$ acts on X component wise. The valuations $\nu_i \in S$ define a metric on G such that the point stabilizers are bounded subgroups. To show that

x_m is unbounded in $\mathbf{SL_2}(\mathcal{O}_S)\backslash X$, it suffices to prove that the preimage of x_m is unbounded under the canonical projection

$$\mathbf{SL_2}(\mathcal{O}_S)\backslash G \to \mathbf{SL_2}(\mathcal{O}_S)\backslash X. \tag{4.15}$$

Let $D_i \in \mathbf{SL_2}(\mathbb{F}_p(t))$ be the diagonal matrix with entries $\pi_{i,S}$ and $\pi_{i,S}^{-1}$ for $1 \leq i \leq |S|$. Now take $D = (D_1, D_2, \ldots, D_{|S|}) \in G$ and observe that

$$D^{-m} \cdot (0, 0, 0, \ldots, 0) = (-2dm, -2dm, -2dm, \ldots, -2dm). \tag{4.16}$$

If $\mathbf{SL_2}(\mathcal{O}_S)D^{-m}$ were bounded in $\mathbf{SL_2}(\mathcal{O}_S)\backslash G$ then there would exist a global constant $C \in \mathbb{Z}$ such that for any $n \in \mathbb{N}$, there exists a matrix

$$M_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathbf{SL_2}(\mathcal{O}_S) \tag{4.17}$$

such that the values of the entries of $M_n D_i^n$ under ν_i are bounded below by C. This would imply that for each $\nu_i \in S$,

$$C \leq \nu_i(a_n \left(\frac{1}{\pi_{i,S}}\right)^n) = \nu_i(a_n) - n \cdot d \tag{4.18}$$

therefore, $\nu_i(a_n) \geq 1$ whenever $n \cdot d \geq 1 - C$ which by Lemma 5 implies that $a_n = 0$. However, the same argument also shows that $c_n = 0$, but this implies that $M_{1-C} \notin \mathbf{SL_2}(\mathcal{O}_S)$.

■

Lemma 8 also shows that the sequence of points x_m is unbounded in the quotient $\Gamma\backslash X$.

4.3 A space with a free Γ action - Y

Notice that the action of Γ on X is not free. The Γ point stabilizers are finite groups, and there is no bound on the order of the point stabilizers. This section contains a construction of a complex, Y_S, which is $|S|$-connected and has a Γ-action. There will be a map from Y_S to the building X.

Let $c : [0, \infty) \to X$ be the unit speed geodesic ray based at x_0 that passes through x_m for all $m \in \mathbb{N}$. Define $\beta_c(x) = \lim_{\tau \to \infty} (\tau - d(x, c(\tau)))$. This is called the *Busemann function* associated to c. The function is well studied and provides a notion of height in the building X. Given $x \in [0, \infty)$, the inverse image $\beta_c^{-1}(x)$ is called a horosphere and the inverse image of $\beta_c^{-1}[x, \infty)$ is called a horoball. The ray c represents a point in the visual boundary of X and is fixed by $\prod_{\nu \in S} \mathbf{B_2}(\mathbb{F}_p(t)_\nu)$. Furthermore, $\mathbf{B_2}(\mathcal{O}_S)$ fixes every horosphere based at c.

Let Y_0 be a horosphere associated to c. In [12], Bux shows that Y_0 is $(|S|-2)$-connected. Our goal is to build an $|S|$-connected space, Y, containing Y_0 such that Γ acts freely outside of Y_0, and a map $\psi : Y \to X$ that extends the inclusion $Y_0 \subseteq Y$ and that is Γ equivariant.

If Y_0 is not $(|S|-1)$-connected, there is some map of an $|S|-2$ dimensional sphere $f: S^{|S|-2} \to Y_0$ whose image is not contractible in Y_0. Using the inclusion map $\psi : Y_0 \to X$ and the fact that X is $|S|$-connected, there is a $(|S|-1)$-disk, $\Delta^{|S|-1} \subseteq X$ such that $\partial\Delta^{|S|-1} = f(S^{|S|-1})$

Let

$$Y_1' = Y_0 \bigsqcup_{\gamma\in\Gamma} \gamma\Delta^{|S|-1}/\sim \tag{4.19}$$

where the boundary of the disk $\gamma\Delta^{|S|-1}$ is identified with its image $\gamma f(\partial\Delta^{|S|-1})$ in Y_0. The inclusion map from Y_0 to X can be extended to ψ_1' by mapping the disk $\gamma(\Delta^{|S|-1}) \subseteq Y_1'$ to $\gamma\Delta^{|S|-1} \subseteq X$. Continue this process till you have constrcuted an $(|S|-1)$-connected space Y_1. Along with Y_1, we get a map Γ- equivariant $\psi_1 : Y_1 \to X$.

To obtain a space Y which is $|S|$-connected, begin by choosing some $f : S^{|S|-1} \to Y_1$ with a noncontractible image in Y_1. For an arbitrary $|S|$-disk, $\Delta^{|S|}$, let

$$Y_2' = Y_1 \bigsqcup_{\gamma\in\Gamma} \gamma\Delta^{|S|}/\sim \tag{4.20}$$

where the boundary of $\gamma\Delta^{|S|}$ is identified with the sphere $\gamma f(\partial\Delta^{|S|})$ in Y_1. Repeat this process until the resulting space is $|S|$-connected, and call this space Y. Note that the major difference in this step of the construction and the previous step is that there is no induced cellular map from Y (which is $|S|+1$-dimensional), to the building X (which is $|S|$-dimensional). However, ψ can be extended to a map from Y to X by mapping each $(|S|+1)$-cell continuously. The map is not unique, but this will not be a problem.

Let U be the subgroup of $\prod_{\nu\in S} \mathbf{B_2}(\mathbb{F}_p(t)_\nu)$ with matrices of the form

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}. \tag{4.21}$$

Let U_m be the subgroup of U that fixes x_m. The group U is isomorphic to the field $\mathbb{F}_p(t)_\nu)$ and U_m is a subspace of this vector space. Therefore, we can write $U = U_m \times U^m$. Let $g_m : X_S \to U^m \backslash X$ be the quotient map. Notice that $X_S = U\mathcal{A}_S$.

Let $c : [0,\infty) \to X$ be the unit speed geodesic ray based at x_0 that passes through x_m for all $m \in \mathbb{N}$. Define $\beta_c(x) = \lim_{\tau\to\infty}(\tau - d(x, c(\tau)))$. This is called the *Busemann function* associated to c. The function is well studied and provides a notion of height in the building X. Given $x \in [0,\infty)$, the inverse image $\beta_c^{-1}(x)$ is called a horosphere and the inverse image of $\beta_c^{-1}[x,\infty)$ is called a horoball. The ray c represents a point in the visual boundary of X_S and is fixed by $\prod_{\nu\in S} \mathbf{B_2}(\mathbb{F}_p(t)^\nu)$. Furthermore, $\mathbf{B_2}(\mathcal{O}_S)$ fixes every horosphere based at c.

Lemma 9 *The Γ orbit of x_0 has bounded height with respect to the Busemann function β_c.*

Proof. See Theorem 6.2 in [13].

■

Lemma 10 *There exists an N such that for $m > N$, given any chain $\sigma \subseteq X_{S,\Gamma}$ with $(\partial\sigma)^0 \subseteq \Gamma x$ then $g_m(\psi(\sigma)) \cap Lk(x_m)$ is supported on $Lk(x_m)^{\downarrow}$.*

Proof. To begin, we choose N such that for $m > N$, $\beta_c(x_m) > \beta_c(\Gamma x_0)$. Assume otherwise. Then there is a chamber $C_1 \subseteq \mathrm{supp}(g_m(\psi(\sigma)) \cap \mathrm{Lk}(x_m))$ such that $C_1 \not\subseteq \mathrm{Lk}(x_m)^{\downarrow}$. This means that there is a face F_1 of C_1 such that for every $x \in F_1$

$$\beta_c(x) \geq \beta_c(x_m). \tag{4.22}$$

Because of Lemma 9, this means that $F_1 \not\subseteq \partial(\psi(\sigma))$ and therefore, there is another chamber C_2 such that $C_1 \cap C_2 = F_1$ and $C_2 \subseteq (g_m(\psi(\sigma))$.

Let $\mathcal{A}_1$ be an apartment that contains C_1 and contains the point at infinity fixed by U. Every chamber $C' \subseteq X$ for which $C' \cap C_1 = F_1$ is either in $\mathcal{A}_1$ or is equal to uC_1 for some $u \in U$. We can write $u = u^* u_*$ for some $u^* \in U^m$ and $u_* \in U_m$. But U_m fixes C_1 so $uC_1 = u^* C_1$ and $g_m(u^* C_1) = C_1$. Since $C_2 \neq C_1$, it must be the case that $C_2 \subseteq \mathcal{A}_1$.

The above shows that there is only one C' in the image of g_m such that $C' \cap C_1 = F_1$ and that there is a face F_2 of C_2 such that for every $x \in F_2$

$$\beta_c(x) \geq \beta_c(x_m). \tag{4.23}$$

This process can be repeated indefinitely. However, this would imply that the support of $g_m(\psi(\sigma))$ contains infinitely many cells, which is absurd.

■

CHAPTER 5

LOCAL PROPERTIES OF X

In this section we define local properties.

Definition 4 *Given a polysimplicial complex C and a vertex $x \in C$ the link of x denoted $Lk(x)$ is a subcomplex of C consisting of the polytopes τ that are disjoint from x and such that both x and τ are faces of some higher-dimensional simplex in C.*

In this section, for each $x \in X$, we will construct a local cocylce $\phi \in H^{|S|-1}(\mathrm{Lk}(x); \mathbb{F}_p)$. The cocycle will be extended to a global cocylce of $\Gamma \backslash Y$ by making use of the map ψ and an averaging technique. As in Section 3.1, let $A_\nu = \{x \in \mathbb{F}_p(t) : \nu(x) \geq 0\}$ be the valuation ring associated to ν. The quotient $\mathbb{F}_\nu = A_\nu / \omega_\nu A_\nu$ is a finite field called the residue field.

For a vertex $x \in X_\nu$, the link of x can be understood several ways. Consistent with the general theory of Euclidean buildings, you can see $\mathrm{Lk}(x)$ as the spherical building for $\mathbf{SL_2}(\mathbb{F}_\nu)$. However, in this special case, you can see the link of x as $\mathbb{P}^1(\mathbb{F}_\nu)$, the projective line over the field $\mathbb{F}_\nu$. The stabilizer of x in Γ acts on $\mathrm{Lk}(x)$. The action fixes the point $[1 : 0]$ that corresponds to infinity in $\mathbb{P}^1(\mathbb{F}_\nu)$.

Definition 5 *The join of two complexes C_1 and C_2 denoted $C_1 \star C_2$ is*

$$C_1 \ \times \ C_2 \times \ [0,1] \ / \ \sim, \tag{5.1}$$

where $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in C_1$ and $y, y' \in C_2$

The link of a vertex $(x, y) \in C_1 \times C_2$ is the join $\mathrm{Lk}(x) \star \mathrm{Lk}(y)$. This shows that if you have a vertex $x \in X$, then $\mathrm{Lk}(x)$ is the join of $|S|$ spherical buildings one for each $\mathbf{SL_2}(\mathbb{F}_\nu)$.

The join of $\mathbb{P}^1(\mathbb{F}_{\nu_1})$ and $\mathbb{P}^1(\mathbb{F}_{\nu_2})$ is a complete bipartite graph. The edges in the graph correspond to elements in $\mathbb{P}^1(\mathbb{F}_{\nu_1}) \times \mathbb{P}^1(\mathbb{F}_{\nu_2})$. In general, given $x \in X$, the link of x is a simplicial complex that is analogous to a complete bipartite graph. The analogy is made precise by the following lemma.

Lemma 11 *Given a vertex $x \in X$, the link of x is a simplicial complex that can be described by the following:*

1. *The vertices of the Lk(x) can be enumerated by elements of $\sqcup_{\nu \in S}\mathbb{P}^1(\mathbb{F}_\nu)$. In the disjoint union, each $\mathbb{P}^1(\mathbb{F}_\nu)$ is considered distinct and therefore, the vertices are partitioned into $|S|$ different sets.*
2. *The edges of Lk(x) correspond to choosing two vertices from different sets in the disjoint union $\sqcup_{\nu \in S}\mathbb{P}^1(\mathbb{F}_\nu)$*
3. *Lk(x) is a flag complex.*

Proof. The vertices of $C_1 \star C_2$ correspond to the disjoint union $C_1^0 \sqcup C_2^0$. Therefore by induction, given $x \in X$, the vertices exactly correspond to elements of $\mu \in \sqcup_{\nu \in S}\mathbb{P}^1(\mathbb{F}_\nu)$.

Given two polysimplicial complexes C_1 and C_2, the edges in $C_1 \star C_2$ are edges from C_1, edges from C_2, and edges between vertices in C_1 and vertices in C_2. Because

$$\mathrm{Lk}(x) = (\ldots(\mathbb{P}^1(\mathbb{F}_{\nu_3}) \star (\mathbb{P}^1(\mathbb{F}_{\nu_2}) \star \mathbb{P}^1(\mathbb{F}_{\nu_1}))\ldots), \tag{5.2}$$

given any two vertices $y_1, y_2 \in \mathrm{Lk}(x)$ that come from different elements of the partition, there is an edge between y_1 and y_2.

The fact that Lk(x) is a flag complex is deduced from the well-known fact that Lk(x) is a spherical building. ■

The previous lemma gives an understanding of Lk(x) that is important in defining a cocyle $\phi \in H^{|S|-1}(\mathrm{Lk}(x); \mathbb{F}_p)$.

In each place, there is a distinguished point at infinity $[1:0]$. It is set apart from the rest of the vertices in Lk(x) because it is fixed under the statbilizer of x. There is a distinguished chamber $\mathcal{C}_\infty \subseteq \mathrm{Lk}(x)$ where each vertex of $\mathcal{C}_\infty$ corresponds to a different point $[1:0]$ in one of the partition sets $\mathbb{P}^1(\mathbb{F}_\nu)$. This allows us to define the following set of "downward facing" chambers

$$\mathrm{Lk}(x)^{\downarrow} = \bigcup_{\mathcal{C} \cap \mathcal{C}_\infty = \emptyset} \mathcal{C}. \tag{5.3}$$

Let P_m be the Γ stabilizer of the point $x_m = (-m, -m, -m, \ldots, -m) \in \mathcal{A}$. The stabilizer of x_m consists of matrices of the form

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \in \Gamma \tag{5.4}$$

with $\nu_i(f) \geq -m$ for $1 \leq i \leq |S|$. The diagonal entries are equal to 1 since we have chosen Γ to have diagonal entries of the from $\frac{f}{g}$ where f and g are monic polynomials. Because

P_m acts on X and fixes x_m, P_m acts on $\mathrm{Lk}(x_m)$. Notice that for every $g \in P_m$, g pointwise fixes the chamber $\mathcal{C}_\infty$ and therefore, there is also a P_m action on $\mathrm{Lk}(x_m)^\downarrow$.

Lemma 12 *There is a cocycle $\phi \in H^{|S|-1}(Lk(x_m); \mathbb{F}_p)$ that is P_m invariant on cycles that are supported on $Lk(x_m)^\downarrow$.*

Proof. The Solomon-Tits theorem informs us that a rank $(|S|-1)$ spherical building has the homotopy type of a connect sum of $(|S|-1)$-spheres. Furthermore, given a chamber $\mathcal{C}$, there is a basis for homology given by all the apartments that contain $\mathcal{C}$. A convenient index for this basis is representing any apartment $\mathcal{A}$ that contains $\mathcal{C}$ by the unique chamber in $\mathcal{A}$ that is opposite $\mathcal{C}$.

The fact that $\mathrm{Lk}(x_m)$ is a spherical building is well known. Since $\mathrm{Lk}(x_m)^\downarrow$ is the join of a set of finite points, it is also a spherical building. The basis we will use for $\mathrm{Lk}(x_m)$ will be given by choosing $\mathcal{C}_\infty$. Any chamber given by the point $(a_1, a_2, ..., a_{|S|})$ with each $a_i \neq [1:0]$ is opposite C_∞. Let the basis element that corresponds to the chamber given by the points $(a_1, a_2, \ldots, a_{|S|})$ be denoted by $\mathcal{C}_{a_1, a_2, \ldots, a_n}$. In this way, any cycle $\sigma \in H_{|S|-1}(\mathrm{Lk}(x_m), \mathbb{F}_p)$ can be written

$$\sigma_m = \sum c_i \mathcal{C}_{a_{1,i}, a_{2,i}, \ldots, a_{|S|,i}}. \tag{5.5}$$

The field $\mathbb{F}_\nu$ is isomorphic to $\mathbb{F}_p[t]/f$ for some irreducible monic polynomial f. So elements of $\mathbb{F}_\nu$ can be uniquely expressed as polynomials with degree less than $\deg(f)$. For an element $a_i \in \mathbb{F}_{\nu_i}$, define $\widetilde{a_{\nu_i}}$ to be the degree 0 term of a_i. Now define a cocycle ϕ_m such that

$$\phi_m(\sigma_m) = \sum c_i \widetilde{a_{1,i}} \widetilde{a_{2,i}} \ldots \widetilde{a_{|S|,i}} \text{ where } c_i \in \mathbb{F}_p \tag{5.6}$$

We can choose a basis for homology for $\mathrm{Lk}(x_m)^\downarrow$ by choosing the chamber with vertices $(0, 0, \ldots, 0)$ in $\mathrm{Lk}(x_m)^\downarrow$. An apartment in the basis for homology is given by selecting a chamber opposite $(0, 0, \ldots, 0)$. Any chamber opposite $(0, 0, \ldots, 0)$ in $\mathrm{Lk}(x_m)^\downarrow$ has vertices $(a_1, a_2, \ldots, a_{|S|})$ with $a_i \neq 0$ and $a_i \neq \infty$ for all $0 \leq i \leq |S|$.

A combinatorial approach to labeling each chamber in the apartment that contains $(0, 0, \ldots, 0)$ and $(a_1, a_2, \ldots, a_n)$ is to look at the product

$$(a_1 - 0)(a_2 - 0)(a_3 - 0) \ldots (a_{|S|} - 0). \tag{5.7}$$

This product is the sum $2^{|S|}$ terms. Each term in the product is a string of length $|S|$ of a_is and 0s and corresponds to a chamber. The sign of each term will give an orientation to each chamber such that the sum of the chambers is the apartment.

This combinatorial approach makes evaluating ϕ_m (up to sign) on the apartment $\mathcal{A}_{0,a}$ that contains $(0, 0, \dots, 0)$ and $(a_1, a_2, \dots, a_{|S|})$ straightforward

$$\phi_m(\mathcal{A}_{0,a}) = (\widetilde{a_1} - 0)(\widetilde{a_2} - 0)(\widetilde{a_3} - 0) \dots (\widetilde{a_{|S|}} - 0). \tag{5.8}$$

The P_m action on $\mathrm{Lk}(x_m)$ fixes all the vertices that correspond to $[1:0]$. Therefore, the action stabilizes $\mathrm{Lk}(x)^{\downarrow}$. Specifically, for any $u \in P_m$, there is a $(u_1, u_2, \dots, u_{|S|}) \in \prod_{\nu \in S} \mathbb{F}_p$ such that

$$u \cdot (a_1, a_2, \dots a_{|S|}) = (a_1 + u_1, a_2 + u_2, \dots, a_{|S|} + u_{|S|}). \tag{5.9}$$

Let $\mathcal{A}_{0,a}$ be the apartment that contains opposite chambers $(a_1, a_2, \dots, a_{|S|})$ and $(0, 0, \dots, 0)$. Then $u\mathcal{A}_{b,a}$ contains the chambers $(a_1 + u_1, a_2 + u_2, \dots, a_{|S|} + u_{|S|})$ and $(u_1, u_2, \dots, u_{|S|})$. Therefore,

$$\begin{aligned} \phi_m(u\mathcal{A}_{b,a}) &= ((a_1 + u_1) - (u_1)) \dots ((a_{|S|} + u_{|S|}) - (+u_{|S|})) && (5.10)\\ &= (a_1 - 0)(a_2 - 0)(a_3 - 0) \dots (a_{|S|} - 0) && (5.11)\\ &= \phi_m(\mathcal{A}_{0,a}). && (5.12) \end{aligned}$$

Because $\mathrm{Lk}(x_m)$ is $(|S| - 1)$ dimensional, ϕ is a top dimensional cochain and therefore represents an element of cohomology.

■

Remark 1 *Lemma 12 is what requires us to pass from* $\mathbf{B_n}(\mathcal{O}_S)$ *to* Γ_n*. In* $\mathbf{B_2}(\mathcal{O}_S)$*, the point stabilizers include diagonal matrices that do not leave* ϕ *invariant. However, if* $p = 3$*, then we could work with* $\mathbf{B_2}(\mathcal{O}_S)$ *since the only additional matrix in the stabilizer of a point is the diagonal matrix with a 2 in both entries. This diagonal matrix acts trivially on the* $Lk(x_m)$*.*

CHAPTER 6

PROOF OF THE MAIN RESULT

In this chapter, we prove the main result.

6.1 A family of cocycles on $\Gamma\backslash Y$

For every $m \in \mathbb{N}$, define

$$\Phi_m : C_{|S|+1}(\Gamma\backslash Y) \to \mathbb{F}_p \tag{6.1}$$

as follows: given an $(|S|+1)$-cell ΓB in $\Gamma\backslash Y$, let

$$\Phi_m(\Gamma B) = \sum_{\gamma P_m \in \Gamma/P_m} \phi_m\left(g_m \psi(\gamma^{-1}B) \cap \mathrm{Lk}(x_m)\right). \tag{6.2}$$

Lemma 13 *The map Φ_m is well defined. In particular, it is independent of choices of coset representatives γP_m and representative γB for an $(|S|+1)$-cell in $\Gamma\backslash Y$.*

Proof. First we check that replacing γ with γp_γ (changing the coset representatives) does not change the value of Φ_m:

$$\begin{aligned}
&\sum_{(\gamma p_\gamma)P_m \in \Gamma/P_m} \phi_m\left(g_m \psi((\gamma p_\gamma)^{-1}B) \cap \mathrm{Lk}(x_m)\right) && (6.3)\\
&= \sum_{(\gamma p_\gamma)P_m \in \Gamma/P_m} \phi_m\left(g_m \psi(p_\gamma^{-1}\gamma^{-1}B) \cap \mathrm{Lk}(x_m)\right) && (6.4)\\
&= \sum_{\gamma P_m \in \Gamma/P_m} \phi_m\left(p_\gamma^{-1} g_m \psi(\gamma^{-1}B) \cap \mathrm{Lk}(x_m)\right) && (6.5)\\
&= \sum_{\gamma P_m \in \Gamma/P_m} \phi_m\left(g_m \psi(\gamma^{-1}B) \cap \mathrm{Lk}(x_m)\right) && (6.6)\\
&= \Phi_m(\Gamma B) && (6.7)
\end{aligned}$$

Next we check that choosing a different lift of ΓB does not change the value of $\Phi_m(\Gamma B)$. If $y \in \Gamma$, then

$$\begin{aligned}
\Phi_m(\Gamma yB) &= \sum_{\gamma P_m \in \Gamma/P_m} \phi_m\left(g_m\psi(\gamma^{-1}yB) \cap \mathrm{Lk}(x_m)\right) && (6.8)\\
&= \sum_{\gamma P_m \in \Gamma/P_m} \phi_m\left(g_m\psi((y^{-1}\gamma)^{-1}B)) \cap \mathrm{Lk}(x_m)\right) && (6.9)\\
&= \sum_{y\gamma P_m \in \Gamma/P_m} \phi_m\left(g_m\psi((y^{-1}y\gamma)^{-1}B)) \cap \mathrm{Lk}(x_m)\right) && (6.10)\\
&= \sum_{y\gamma P_m \in \Gamma/P_m} \phi_m\left(g_m\psi(\gamma^{-1}B) \cap \mathrm{Lk}(x_m))\right) && (6.11)\\
&= \sum_{\gamma P_m \in \Gamma/P_m} \phi_m\left(g_m\psi(\gamma^{-1}B) \cap \mathrm{Lk}(x_m))\right) && (6.12)\\
&= \Phi_m(\Gamma B) && (6.13)
\end{aligned}$$

■

Lemma 14 *The chain map Φ_m is a representative for a cohomology class in $H^{|S|}(Y;\mathbb{F}_p)$.*

Proof. In order to show that Φ_m is a cocycle, we will demonstrate that it is trivial on boundaries of $|S|+1$-disks, and thus is in the kernel of the coboundary map.

Let $\Gamma B^{|S|+1}$ be an $(|S|+1)$-cell in $\Gamma\backslash Y$, corresponding to the $(|S|+1)$-cell $B^{|S|+1}$ in Y. Then $\Gamma(\partial B^{|S|+1})$ is an $|S|$-sphere in $\Gamma\backslash Y$ and $\partial B^{|S|+1}$ is an $|S|$-sphere in Y. Since the product of trees X contains no nontrivial $|S|$-spheres, the image of $B^{|S|}$ under the map $\psi : Y \to X$ is trivial. Thus,

$$\Phi_m(\Gamma(\partial B^{|S|+1})) = \sum_{gP_m \in \Gamma/P_m} \phi_m\left(\psi(\Gamma^{-1}\partial B^{|S|+1}) \cap \mathrm{Lk}(x_m)\right) \qquad = 0 \qquad (6.14)$$

■

Lemma 15 *The cohomology class that Φ_m represents is nontrivial.*

Proof. To prove this lemma, we will construct an explicit cycle σ_m such that $\Phi_m(\sigma_m) \neq 0$.

Let δ_m be the $|S|$-simplex in $\mathcal{A}$ that is spanned by the following vectors:

$$v_{1,m} = (dm \cdot \deg(f_1), -dm \cdot \deg(f_2), 0, \ldots, 0), \tag{6.15}$$

$$v_{2,m} = (0, dm \cdot \deg(f_2), -dm \cdot \deg(f_3), 0, \ldots, 0), \tag{6.16}$$

$$v_{3,m} = (0, 0, dm \cdot \deg(f_3), -dm \cdot \deg(f_4), 0, \ldots, 0), \tag{6.17}$$

$$\vdots \tag{6.18}$$

$$v_{|S|-1,m} = (0, \ldots, dm \cdot \deg(f_{|S|-1}), -dm \cdot \deg(f_{|S|})), \tag{6.19}$$

$$v_{|S|,m} = (-dm \cdot \deg(f_1), 0, \ldots, 0, dm \cdot \deg(f_{|S|})) \tag{6.20}$$

$$v_{|S|+1,m} = (-dm, -dm, \ldots, -dm, -dm) \tag{6.21}$$

Note that the face spanned by $v_{1,m}, \ldots, v_{|S|,m}$ is contained within $\mathcal{A}_{\mathcal{O}}$. The technique to construct σ_m will be to use the action of unipotent elements in Γ to create a cycle with boundary contained in $\Gamma\mathcal{A}_{\mathcal{O}}$

For $k \leq |S|$, let F_k be the face of δ_m that is spanned by v_i for $1 \leq i \leq |S|+1$ and $i \neq k$. Let

$$f_{k,m} = \prod_{i=1}^{|S|} \frac{\pi_{i,S}^{m\deg(f_k)}}{\pi_{k,S}^{m\deg(f_i)}}. \tag{6.22}$$

We have that

$$\nu_i(f_{k,m}) = \left\{ \begin{array}{rl} md\deg(f_i) & : i \neq k \\ -md\sum_{i\neq k}\deg(f_j) & : i = k \end{array} \right. \tag{6.23}$$

The matrix

$$u_k = \begin{pmatrix} 1 & f_k \\ 0 & 1 \end{pmatrix} \in \Gamma \tag{6.24}$$

fixes the face F_k and does not fix any other face of δ_m since for an individual factor u_k fixes $\{\omega_\nu e_1, e_2\}$ only if $\nu(f) \geq k$. The group

$$U = \langle\ u_1,\ u_2,\ \ldots\ ,\ u_{|S|}\ \rangle \tag{6.25}$$

is abelian.

Let $\mathcal{P}\{u_1, ...u_{|S|}\}$ be the power set of the set $\{u_1, u_2, \ldots, u_{|S|}\}$. The chain

$$\sigma_m = \sum_{\phi\in\mathcal{P}\{u_1,...u_{|S|}\}} (-1)^{|\phi|}\mathbf{M}(\phi)\delta_m \tag{6.26}$$

is homeomorphic to a $|S|$-cell ($\mathbf{M}$ is the multiplication map in the group). We can calculate the boundary of σ_m

$$\partial(\sigma_m) = \sum_{\phi\in\mathcal{P}\{u_1,\ldots,u_{|S|}\}} \sum_{i=1}^{|S|+1} (-1)^{|\phi|}\mathbf{M}(\phi)F_i \tag{6.27}$$

$$= \sum_{\phi\in\mathcal{P}\{u_1,\ldots,u_{|S|}\}} (-1)^{|\phi|}F_{|S|+1} \tag{6.28}$$

$$\subseteq \Gamma\mathcal{A}_{\mathcal{O}} \tag{6.29}$$

By how Y was constructed, there is a cycle $\widetilde{\sigma_m} \subseteq Y$ such that $\psi(\widetilde{\sigma_m}) = \sigma_m$. In Chapter 3, the labels of the vertices in $\mathrm{Lk}(x_m)$ were chosen with only one constraint. The vertex labeled ∞ was fixed by the stabilizer of x_m. So we are free to choose the other labels. Choose the labels such that the value of $\phi_m(\delta_m \cap \mathrm{Lk}(x_m)) = 1$ and therefore, $\Phi_m(\sigma_m) = |\mathcal{P}\{u_1, ..., u_{|S|}\}| = 2^m$ because ψ is Γ-invariant on $\mathrm{Lk}(x_m)^{\downarrow}$. ■

By how Y was constructed there is a cycle $\widetilde{\sigma_m} \subseteq Y$ such that $\psi(\widetilde{\sigma_m}) = \sigma_m$. In Chapter 3 the labels of the vertices in $\mathrm{Lk}(x_m)$ were chosen with only one constraint. The vertex labeled ∞ was fixed by the stabilizer of x_m. So we are free to choose the other labe

Remark 2 *Lemma 15 is the reason we need* $p \neq 2$. *Here the cycle we build evaluates to a power of* 2 *under* Φ_m. *For* $p \neq 2$, $\Phi_m(\sigma_m) = 2^m \neq 0$.

The following proves Theorem 1.

Theorem 16 $H^{|S|}(Y; \mathbb{F}_p)$ *is infinite dimensional*

Proof. To show that $H^{|S|}(Y; \mathbb{F}_p)$ is infinite dimensional, we will show that $\Phi_k(\widetilde{\sigma_m}) = 0$ whenever $k > m$. We begin by showing that this will suffice. Choose any $N \in \mathbb{N}$. It must be the case that Φ_N and Φ_{N-1} are independent since $\Phi_N(\sigma_{N-1}) \neq 0 = \Phi_{N-1}(\sigma_{N-1})$. By induction, we can show that $\{\Phi_N, \Phi_{N-1}, ..., \Phi_1\}$ is independent. Assume that we have shown that $\{\Phi_N, \Phi_{N-1}, ..., \Phi_k\}$ is independent, then we know that $\{\Phi_N, \Phi_{N-1}, ..., \Phi_{k-1}\}$ is an independent set because

$$\Phi_{k-1}(\sigma_{k-1}) \neq 0 = \sum_{i>k-1}^{N} a_i \Phi_i(\sigma_{k-1}) \tag{6.30}$$

Now we prove that $\Phi_k(\widetilde{\sigma_m}) = 0$ whenever $k > m$.

$$\begin{aligned} \Phi_k(\Gamma\widetilde{\sigma_m}) &= \sum_{\gamma P_m \in \Gamma / P_m} \phi_k\left(\psi(\gamma^{-1}\widetilde{\sigma_m}) \cap \mathrm{Lk}(x_k)\right) & (6.31) \\ &= \phi_k(\sigma_m \cap \mathrm{Lk}(x_k)) & (6.32) \end{aligned}$$

To show that this evaluates to 0, observe that there is no chamber in $F_m \cap \mathrm{Lk}(x_k)$ for $k > m$. ■

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