Deformed Hubbard operator, bosonization, and phase diagram of the one-dimensional t-J model

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(Received 18 June 2002; published 23 October 2002)

We present an analytic study of the phase diagram of the one-dimensional t-J model and a couple of its cousins. To deal with the interactions induced by the no double occupancy constraints, we introduce a deformation of the Hubbard operators. When the deformation parameter Δ is small, the induced interactions are softened, accessible by perturbation theory. We combine bosonization with renormalization group techniques to map out the phase diagram of the system. We argue that when $\Delta \rightarrow 1$, there is no essential change in the phase diagram. A comparison with the existing results in the literature obtained by other methods justifies our deformation approach.

DOI: 10.1103/PhysRevB.66.155111 PACS number(s): 71.10.Hf, 71.30.+h, 74.20.Mn

I. INTRODUCTION

A. Historical review

Since the discoveries of quantum Hall effects and high-T_c oxides in 1980s, strongly correlated systems have been of great interest both theoretically and experimentally. As far as the high-T_c problem is concerned, the t-J model is believed to be the appropriate starting model Hamiltonian, because it captures the essence of the interplay between charge and spin degrees of freedom in superconducting Cu oxides.[1,2] Although high-T_c cuprates are (at least) two-dimensional systems, it is very interesting to study its one-dimensional (1D) counterpart. As argued by Anderson,[1] two-dimensional strongly correlated systems may share some properties of 1D systems. In addition, the physical understanding of the 1D systems is also extremely helpful for the study of the ladder systems, which have been realized experimentally and have attracted a lot of attention in recent years.[3–5]

In 1D systems, the phase space of the particle scattering is highly restricted. The occurrence of a single scattering event will spread quickly among all other particles, which invalidates the concept of individual excitations. Consequently, we are often confronted with correlated collective excitations. On the other hand, in some cases we can benefit from such a phase space restriction. That is, the many-particle scattering matrix could be nicely decomposed into the product of two-particle ones which satisfy the so-called Yang-Baxter integrable conditions.[6] This property provides us with the possibility to exactly solve some 1D models, e.g., the Hubbard model,[7] the Heisenberg model,[8,9] and the supersymmetric t-J model[10] The exact solutions in turn provide us with powerful guidelines to develop and to justify certain approximate schemes for other problems.

In some sense, the 1D t-J model could be viewed as a descendant of the Hubbard model in large on-site repulsion limit. That is, the strong coupling limit of the Hubbard model can be mapped into the weak coupling limit of the t-J model. Naively, one may speculate that the integrability of the 1D Hubbard model would be inherited by the t-J model in the whole parameter space. Unfortunately, this speculation is not correct: The t-J model is only integrable at two special points in parameter space. The reason for this difference is that, in contrast to other 1D integrable models, the Hilbert space of the t-J model is highly constrained: Double occupancy of any site is completely excluded. Furthermore, the integrable points of the t-J model are located in the strong coupling regime ($|J|=2t$ in our convention below), not in the weak coupling limit. Therefore, the integrability of the supersymmetric t-J model is not simply inherited from the Hubbard model. Rather it is better to be viewed as a separate miracle of the interacting 1D many-particle system. Since the 1D t-J model cannot be exactly solved at a generic point in parameter space, the analytical studies of the t-J model have been a painstaking task even in the 1D case.

To illustrate the points more clearly, let us take a close look at the t-J model. The model delineates the behavior of hard core fermions on a discrete lattice, and the dynamics is given by the model Hamiltonian

$$H_{tJ}=-t\mathcal{P}\sum_{j,\sigma}(c_{j,\sigma}^{\dagger}c_{j+1,\sigma}+\text{H.c.})\mathcal{P}+J\sum_{j}\mathbf{S}_j\cdot\mathbf{S}_{j+1}. \quad (1)$$

Here $\mathcal{P}$ is the projection operator that prohibits double occupancy of any site, σ and $\bar{\sigma}\equiv-\sigma$ the spin orientations (with $\sigma=1$ for $\uparrow$, and -1 for $\downarrow$); t is the hopping amplitude and J the antiferromagnetic ($J>0$) or ferromagnetic ($J<0$) coupling. Due to the aforementioned constraints, at each site the states $|a\rangle$ can only be one of the following three possible states: with $a=\uparrow,\downarrow$, and $a=0$ (empty). This Hilbert space is neither fermionic nor bosonic. One can check that the projection operators $\chi^{ab}=|a\rangle\langle b|$ close, under commutation and anticommutation, to form a semi-simple supersymmetric Lie algebra, the $Spl(1,2)$ given by the relations[11]

$$\{\chi_i^{ab},\chi_j^{cd}\}_{\pm}=\delta_{ij}(\chi_i^{ad}\delta^{bc}\pm\chi_i^{bc}\delta^{ad}), \quad (2)$$

where $\chi_i^{\sigma 0}$ and $\chi_i^{0\sigma}$ are fermionic operators that, respectively, create and annihilate a single electron. The bosonic operator $\chi^{\sigma\sigma'}$ are identified as the generators of the group SU(2). Using these operators, the t-J model can be neatly written as

$$H_{tJ}=-t\sum_{j,\sigma=\uparrow,\downarrow}(\chi_j^{0\sigma}\chi_{j+1}^{\sigma 0}+\text{H.c.})+J\sum_{j,\sigma,\sigma'}\chi_j^{\sigma\sigma'}\chi_{j+1}^{\sigma'\sigma}, \quad (3)$$

in terms of the bilinears in the generators of $Spl(1,2)$. But the price we have to pay is to introduce both fermionic and bosonic operators simultaneously. As a consequence, it is difficult to make a simple, controlled approximation in this representation. To overcome the difficulties associated with the no-double-occupancy constraints, the slave boson and slave fermion method,[12,13] and, more recently, the supersymmetric Hubbard operator method[14] have been invented to treat the t-J model, with the hope of the mean field ground state being relevant to the high-T_c problem. However, after more than one decade effort, it seems that a reliable ground state is still elusive.

B. Deformed Hubbard operators

The seminal work by Jordan and Wigner[15] and, later, by Lieb, Schultz, and Mattis[16] provided an alternative idea to handle the above hybridized situation in statistics: That is, the spin operators are uniformly expressed in terms of fermionic operators, though the spin systems are neither bosonic nor fermionic ones. In the same spirit, one can also rewrite the t-J model in terms of fermions exclusively.

In addition to rewriting the magnetic interactions using the fermionic realization of the spin operators,

$$\vec{S}_j=\frac{1}{2}c^{\dagger}_{j\alpha}\vec{\sigma}_{\alpha\beta}c_{j\beta}, \tag{4}$$

one may also introduce the Hubbard operators[17]

$$\bar{c}^{\dagger}_{j\sigma}=c^{\dagger}_{j\sigma}(1-n_{j\bar{\sigma}}), \tag{5}$$

and rewrite the hopping terms in terms of them. These operators also realize the constraints that exclude double occupancy on each lattice site. In this way, one obtains a formulation of the t-J model completely in terms of fermionic operators.

However, one immediately sees that the old hopping terms will induce extra four- and six-fermion interactions. These interactions are "hard" ones, in the sense that their strengths are exactly the same as the hopping amplitude t. This fact defies the attempts to treat the additional four- and six-fermion terms perturbatively. Therefore, at first glance, it seems silly to adopt this strategy to solve the t-J model unless techniques can be invented to make the induced interactions tractable.

In the present paper we propose a technique that allows us to deal with these induced four- and six-fermion interactions. The key point is to use the idea of "adiabatic continuity" *to soften* the above-mentioned interactions induced by the no-double-occupancy constraints. That is, we propose to introduce the *deformed* Hubbard operators

$$\bar{c}^{\dagger}_{j\sigma}=c^{\dagger}_{j\sigma}(1-\Delta n_{j\bar{\sigma}}), \tag{6}$$

with a deformation parameter $0<\Delta\leqslant 1$. When Δ approaches unity, we recover the genuine Hubbard operators [Eq. (5)]. For $0<\Delta<1$, there is a nonzero probability to allow leakage into states with double occupancy. With these deformed Hubbard operators [Eq. (6)] replacing the genuine Hubbard operators [Eq. (5)] in the hopping terms, we obtain a deformation of the original t-J model. The deformed model has the advantage that, for small Δ, the induced four- and six-fermion interactions are no longer "hard." This is because these interactions have strengths proportional to the deformation parameter Δ and, therefore, are tractable in the sense of perturbation theory when Δ is small.

Though small values of Δ may not be "physical," after extracting possible structures in the phase diagram for small Δ, we analytically continue our results back to $\Delta=1$. The fundamental assumption underlying this continuation is the *adiabatic continuity*, namely, that when the Hamiltonian of the model is adiabatically changed with Δ varying from a small positive value to unity, there is no essential, qualitative change in the phase diagram of the model, though various phase boundaries in parameter space may undergo a continuous deformation. Historically, our idea of considering a deformed model is parallel to the ideas that underlie the replica method in treating disordered system, or the large N-expansion in field theory. Actually, even in the field of 1D exactly solvable models one can find a precedent: Yang and Yang[9] proposed the XXZ model as a deformation of the XXX model, i.e., the spin-$\frac{1}{2}$ 1D Heisenberg model, and used it to justify the Bethe ansatz method in the latter by first studying the large anisotropic limit and then continuing back to the isotropic limit. In this paper we will first discuss some simpler cases and give arguments to justify the adiabatic continuity assumption together with our deformed Hubbard operators.

Of course, practically the justification may depend on how we treat the deformed model, which is a fully fermionized model containing four- and six-fermion interactions. In the present paper, we are going to combine the bosonization method and perturbative renormalization group (RG) techniques to deal with the deformed t-J model. That is, we first bosonize the deformed model, and then use the RG flows to map out the phase diagram of the bosonized model. We will argue that the phase diagram obtained in this way does not change in an essential way, when the deformation parameter Δ varies from a small positive value to unity.

For convenience, we will start with a simplified model. That is, we will first consider a model in which the magnetic spin-spin interactions are Ising-like, i.e., of the form $J_z\Sigma_j S^z_j S^z_{j+1}$. This model, together with the usual hopping term, we call the t-J_z model. Then, with a bit more complication, we modify the isotropic magnetic interactions in Eq. (1) to anisotropic XXZ-type interactions:

$$H=-t\mathcal{P}\sum_{j,\sigma}(c^{\dagger}_{j\sigma}c_{j+1,\sigma}+\text{H.c.})\mathcal{P}+J_{\perp}\times\sum_j(S^x_jS^x_{j+1}+S^y_jS^y_{j+1})+J_z\sum_j S^z_jS^z_{j+1}. \tag{7}$$

This model we call the t-$J_{\perp}$-J_z model. The phase diagram of the SU(2) invariant t-J model can be obtained in the double limit with $J_{\perp}\to J_z$ (the isotropic limit) and with the deformation parameter $\Delta\to 1$ (the physical limit with no double occupancy).

The paper is organized as follows: In Sec. II, we discuss the phase structure of the extremely anisotropic limit of the t-$J_\perp$-J_z model, namely, the t-J_z model. The convention of our bosonization scheme is also presented in detail in this section. Then discussions of the phase diagram for the 1D t-$J_\perp$-J_z model are presented in Sec. III. In Sec. IV, we compare our results with other work. The discussions and conclusions are summarized in Sec. V.[18]

II. AN EXTREMELY ANISOTROPIC LIMIT: THE t-J_z MODEL

A. Model

The 1D t-J_z model represents a strongly anisotropic limit of the SU(2) t-J model, in which only has the Ising part of the magnetic interactions been included. Without hopping, this simplification is significant for understanding purely magnetic interactions. However, with hopping the model is more interesting in that it has incorporated the interplay between hopping and the exchange interactions, which makes the physics of the model highly nontrivial. Therefore, the model has recently attracted a lot of interests.[19] It is known from the numerical studies that the low-energy physics in both the t-J_z and t-J models[20] shares some common features even in two dimensions. In the real world, the possible origin of exchange anisotropy is the spin-orbital coupling.[21] In the extremely anisotropic limit, the Hamiltonian (for the t-J_z model) reads

$$\begin{aligned}\mathcal{H}_{tJ_z}&=-t\sum_{j\sigma}(\bar{c}^\dagger_{j\sigma}\bar{c}_{j+1,\sigma}+\text{H.c.})+J_z\sum_j S^z_jS^z_{j+1}\\&=H_0(t)+U(J_z).\end{aligned}\tag{8}$$

Following Eq. (4), we use the representation of S^z_j given by

$$S^z_j=\frac{1}{2}(n_{j\uparrow}-n_{j\downarrow}).\tag{9}$$

Note the appearance of Hubbard operators [Eq. (5)] in the hopping terms. It is the presence of the second term in Eq. (5) that realizes the no double occupancy constraints. As a consequence, the term $H_0(t)$ is no longer a simple hopping of fermions: More interaction terms with four or six fermions are induced, with strengths of the same order of magnitude as the hopping amplitude t. How to deal with these interaction terms is an important issue.

To reduce the strengths of the interaction terms induced by the no-double-occupancy constraints, we propose to deform model Hamiltonian (8) by replacing the Hubbard operators with deformed Hubbard operators [Eq. (6)], resulting in

$$H_0(t)=H_h+H_1+H_2+H_3,\tag{10}$$

The Hamiltonians $H_i(i=h,1,2,3)$, in terms of the genuine fermion operators $c_{j\sigma}$ and $c^\dagger_{j\sigma}$, are given by

$$H_h=-t\sum_{j\sigma}(c^\dagger_{j\sigma}c_{j+1,\sigma}+\text{H.c.}),\tag{11}$$

which represents the genuine hopping term, and

$$H_1=t\Delta\sum_{j\sigma}(c^\dagger_{j\sigma}c_{j+1,\sigma}n_{j+1,\bar{\sigma}}+\text{H.c.}),\tag{12}$$

$$H_2=t\Delta\sum_{j\sigma}(c^\dagger_{j\sigma}c_{j+1,\sigma}n_{j\bar{\sigma}}+\text{H.c.}),\tag{13}$$

$$H_3=-t\Delta^2\sum_{j\sigma}(c^\dagger_{j\sigma}c_{j+1,\sigma}n_{j\bar{\sigma}}n_{j+1,\bar{\sigma}}+\text{H.c.}).\tag{14}$$

Here H_1 and H_2 are the induced four fermion repulsive interaction to prevent double occupancy of the same lattice site, and the H_3 term is attractive, representing the effects from the six fermion interactions that compensate to the excessive repulsion in H_1 and H_2. It is easy to see that now in the deformed model, all the induced terms $H_{1/2}$ and H_3 are proportional to either the deformation parameter Δ in Eq. (6) or to its square. So if Δ is small, the induced interactions are "softened," becoming tractable in perturbation theory. In the limit of $\Delta\to 1$, the total effects of the three terms precisely prevent double occupancy for each lattice site.

By using Eq. (9), the exchange term $U(J_z)$ is given by

$$U(J_z)=\frac{J_z}{4}\sum_j(n_{j\uparrow}-n_{j\downarrow})(n_{j+1,\uparrow}-n_{j+1,\downarrow}).\tag{15}$$

In this way, we rewrite the t-J_z model in terms of fermion creation and annihilation operators exclusively. To look for the low energy effective Hamiltonian, we perform the standard procedure to bosonize the t-J_z model in Sec. II B.

B. Bosonization

The hopping term is easily diagonalized by Fourier transform; the energy spectrum is given by

$$\varepsilon(k)=-2t\cos(ka),\tag{16}$$

where a is lattice spacing. In the ground state, all the states with a momentum lower than the Fermi momentum k_F are filled. For a generic filling factor $\nu=N/M$, with N the particle number and M the number of lattice sites, the Fermi momentum is

$$k_Fa=\frac{\pi}{2}\nu.\tag{17}$$

To obtain the low energy effective action for the excitations, we only need to focus on momenta close to $\pm k_F$, and linearize the spectrum as

$$\varepsilon(\pm k_F+q)=\pm v_Fq-2t\cos(k_Fa),\tag{18}$$

where the Fermi velocity is given by $v_F=2ta\sin(k_Fa)$. The second term is a constant and can be shifted away by redefining the energy zero point. We will drop it throughout the rest of the paper.

In one dimension, the definition of exchange statistics is ambiguous, since the no-double-occupancy condition excludes the possibility to physically exchange spatial position of two particles. This makes the statistics of fermionic particles lose its absolute meaning and make an alternative description in terms of bosons possible. This situation is quite different from that of the three dimensional case, where the exchange statistics of particles has an absolute meaning. In two dimensions, the definition of particle statistics only marginally makes sense and we can transmute the statistics arbitrarily by attaching the Chern-Simons flux to particles (the composite of a particle and a flux is dubbed an anyon[22,23]). The statistics transmutation procedure in one dimension is called bosonization; it has been widely used in exploring the physics in one-dimensional systems.[24–26]

In practice, it is convenient to discuss the bosonization in real space. To do so, we expand the lattice fermion in terms of continuum fields,

$$c_{j\sigma}=\sqrt{a}[\psi_{R\sigma}(x)e^{ik_Fx}+\psi_{L\sigma}(x)e^{-ik_Fx}], \tag{19}$$

$$c_{j\sigma}^{\dagger}=\sqrt{a}[\psi_{R\sigma}^{\dagger}(x)e^{-ik_Fx}+\psi_{L\sigma}^{\dagger}(x)e^{ik_Fx}], \tag{20}$$

where $x=ja$ is used. After linearization and dropping fast varying terms, we obtain the low energy effective Hamiltonian for the H_h term as

$$\begin{aligned}H_h&=\int dx\mathcal{H}_h(x)\\&=-v_F\sum_{\sigma}\int dx[\psi_{R\sigma}^{\dagger}i\partial_x\psi_{R\sigma}-\psi_{L\sigma}^{\dagger}i\partial_x\psi_{L\sigma}],\end{aligned} \tag{21}$$

describing a one-dimensional relativistic Dirac particle in continuum.

In the following, we use the bosonization rule to bosonize the t-J_z model,

$$\psi_{P\sigma}(x)=\frac{\eta_{P\sigma}}{\sqrt{2\pi a}}e^{i\phi_{P\sigma}(x)}, \tag{22}$$

where $P=R/L=+/-$, and $\eta_{P\sigma}$ are the Klein factors to maintain the anticommuting relations between particles on different sites. One may realize the Klein factors as Majarona fermions which satisfy the following anticommuting relations:

$$\{\eta_r,\eta_s\}=2\delta_{rs}. \tag{23}$$

The introduction of $1/\sqrt{2\pi a}$ in Eq. (22) maintains the correct dimension for the field $\psi_{P\sigma}(x)$ which has dimension $[\text{length}]^{-1/2}$ [see Eq. (19)]. The ϕ fields are angular variables and thus dimensionless. To obtain the correct anticommutation for fermionic fields, we also require that the bosonic fields $\phi_P(x)$ satisfy the commutation relations

$$[\phi_{P\sigma},\phi_{P\sigma'}]=iP\pi\delta_{\sigma\sigma'}\varepsilon(x-x'), \tag{24}$$

$$[\phi_{R\sigma},\phi_{L\sigma'}]=i\pi\delta_{\sigma\sigma'}, \tag{25}$$

where $\varepsilon(x)$ is the Heaviside step function.

Using these bosonization rules, we obtain the bosonic description for hopping term (21) as

$$H_h=\frac{v_F}{4\pi}\sum_{\sigma}\int dx[(\partial_x\phi_{R\sigma})^2+(\partial_x\phi_{L\sigma})^2]. \tag{26}$$

For later convenience, let us introduce a pair of conjugate non-chiral bosonic fields for each species

$$\phi_{\sigma}\equiv\phi_{R\sigma}+\phi_{L\sigma}, \tag{27}$$

$$\theta_{\sigma}\equiv\phi_{R\sigma}-\phi_{L\sigma}, \tag{28}$$

which satisfy

$$[\phi_{\sigma}(x),\theta_{\sigma'}(x')]=-i4\pi\delta_{\sigma\sigma'}\varepsilon(x-x'). \tag{29}$$

To organize the spin and charge modes more elegantly, we introduce pairs of dual fields as follows:

$$\phi_c(x)=\frac{1}{\sqrt{2}}(\phi_{\uparrow}+\phi_{\downarrow}),\quad \phi_s(x)=\frac{1}{\sqrt{2}}(\phi_{\uparrow}-\phi_{\downarrow}), \tag{30}$$

$$\theta_c(x)=\frac{1}{\sqrt{2}}(\theta_{\uparrow}+\theta_{\downarrow}),\quad \theta_s(x)=\frac{1}{\sqrt{2}}(\theta_{\uparrow}-\theta_{\downarrow}). \tag{31}$$

Here the introduction of numerical factor $1/\sqrt{2}$ is to maintain the commutation relation in Eq. (29). The subscript s means the spin mode and c the charge mode. Using these spin-charge separated modes, bosonic Hamiltonian (26) can be cast into the following form:

$$H_h=\frac{v_F}{8\pi}\int dx[(\partial_x\phi_c)^2+(\partial_x\phi_s)^2+(\partial_x\theta_c)^2+(\partial_x\theta_s)^2]. \tag{32}$$

To bosonize the induced interaction terms $H_{1,2}$, we note that, roughly speaking, both the H_1 and H_2 terms are of the type of Hubbard-like on-site interactions in the continuum limit and, therefore, provide interactions to renormalize the charge/spin velocity and the controlling parameters (i.e., $K_{c/s}$; see below) and the cosine term in the spin sector. By taking microscopic details of these two terms into account, we obtain an extra numerical factor $\cos(k_Fa)=\cos(\pi\nu/2)$. Both terms have the same bosonized form. That is, the bosonized form of H_1+H_2 is

$$\begin{aligned}H_1+H_2=2H_1&=\frac{t\Delta a\cos(k_Fa)}{\pi^2}\int dx[(\partial_x\theta_c)^2-(\partial_x\theta_s)^2]\\&\quad+\frac{4t\Delta\mathcal{P}\cos(k_Fa)}{\pi^2a}\int dx\cos(\sqrt{2}\theta_s)\\&=\frac{v_F\Delta\cot(k_Fa)}{2\pi^2}\int dx[(\partial_x\theta_c)^2-(\partial_x\theta_s)^2]\\&\quad+\frac{2v_F\Delta\mathcal{P}\cot(k_Fa)}{\pi^2a^2}\int dx\cos(\sqrt{2}\theta_s),\end{aligned} \tag{33}$$

where $\mathcal{P}=\eta_{R\uparrow}\eta_{L\uparrow}\eta_{R\downarrow}\eta_{L\downarrow}$, since $\mathcal{P}^2=1$, we obtain $\mathcal{P}=\pm1$. In the following, we will take $\mathcal{P}=+1$.

Now we come to discuss the six-fermion term H_3 in the continuum limit; after a straightforward but tedious calculation we obtain

$$H_3=-\frac{\sqrt{2}t\Delta^2a^2\cos(k_Fa)}{4\pi^3}\int dx\partial_x\theta_c(x+a)[(\partial_x\theta_c)^2-(\partial_x\theta_s)^2]-\frac{t\Delta^2\cos(k_Fa)}{\sqrt{2}\pi^3}\int dx\partial_x\theta_c(x+a)\cos(\sqrt{2}\theta_s)+\frac{t\Delta^2\sin(k_Fa)}{\sqrt{2}\pi^3}\int dx\partial_x\theta_s(x+a)\sin(\sqrt{2}\theta_s). \tag{34}$$

To obtain sensible continuum limit, we have to take $a\to 0$, but keep ta finite consistently. An elegant way to accomplish this is to use the operator product expansions (OPE's)

$$\partial_{z_1}\theta_c(z_1)\partial_{z_2}\theta_c(z_2)\sim-\frac{1}{(z_1-z_2)^2},$$

$$\partial_z\theta_s(z)\sin[\sqrt{2}\theta_s(0)]\sim-\frac{\sqrt{2}}{z}\cos[\sqrt{2}\theta_s(0)], \tag{35}$$

with all other OPE's being regular. We finally get the continuum limit of the six-fermion term to be

$$H_3=-\frac{v_F\Delta^2}{2\pi^3a^2}\int dx\cos(\sqrt{2}\theta_s). \tag{36}$$

The last thing in bosonizing the t-J_z model is to bosonize the magnetic interaction $U(J_z)$ in Eq. (15). We decompose it into the following combinations:

$$U(J_z)=\frac{J_z}{4}\sum_j(n_{j\uparrow}-n_{j\downarrow})(n_{j+1\uparrow}-n_{j+1\downarrow})=\frac{J_z}{4}\sum_j(n_{j\uparrow}n_{j+1\uparrow}+n_{j\downarrow}n_{j+1\downarrow})-\frac{J_z}{4}\sum_j(n_{j\uparrow}n_{j+1\downarrow}+n_{j\downarrow}n_{j+1\uparrow}). \tag{37}$$

The terms in the first bracket are Coulomb interactions between the electrons on different sites; in the continuum limit we obtain its bosonized form as

$$\frac{J_z}{4}\sum_j(n_{j\uparrow}n_{j+1\uparrow}+n_{j\downarrow}n_{j+1\downarrow})=\frac{J_za}{16\pi^2}\int dx[(\partial_x\theta_c)^2+(\partial_x\theta_s)^2]. \tag{38}$$

The terms in the second bracket of Eq. (37) are the Hubbard-like interactions in the continuum limit; the bosonization procedure gives

$$\frac{J_z}{4}\sum_j(n_{j\uparrow}n_{j+1\downarrow}+n_{j\downarrow}n_{j+1\uparrow})=\frac{J_za}{16\pi^2}\int dx[(\partial_x\theta_c)^2-(\partial_x\theta_s)^2]+\frac{J_z\cos(2k_Fa)}{4\pi^2a}\int dx\cos(\sqrt{2}\theta_s). \tag{39}$$

Combining Eqs. (38) and (39) with Eq. (37), we find that the terms involving the charge variable θ_c exactly cancel and we get the bosonized form of $U(J_z)$ as

$$U(J_z)=\frac{J_za}{8\pi^2}\int dx(\partial_x\theta_s)^2-\frac{J_z\cos(2k_Fa)}{4\pi^2a}\int dx\cos(\sqrt{2}\theta_s). \tag{40}$$

It is worth noting that the absence of charge variables in the $U(J_z)$ term is natural, since we are dealing with pure magnetic interactions.

Therefore, after collecting all the results, the low energy effective Hamiltonian for the bosonized form of the t-J_z model is nicely written as

$$\mathcal{H}_{tJ_z}=\int dx(H_c+H_s), \tag{41}$$

where the Hamiltonian for the spin sector (H_s) and the charge sector (H_c) are given by

$$H_c=\frac{v_c}{2}\left[K_c\Pi_c^2+\frac{1}{K_c}(\partial_x\theta_c)^2\right], \tag{42}$$

$$H_s=\frac{v_s}{2}\left[K_s\Pi_s^2+\frac{1}{K_s}(\partial_x\theta_s)^2\right]+\frac{g_\theta}{8\pi^2a^2}\cos(\sqrt{2}\theta_s), \tag{43}$$

where $\Pi_c=(1/4\pi)\partial_x\phi_c$ and $\Pi_s=-(1/4\pi)\partial_x\phi_s$ are the conjugate momenta for the charge field θ_c and spin field θ_s respectively. The effective coupling constant $g_{\theta s}$ is given by

$$g_\theta=v_F\left[8\Delta\cot\left(\frac{\pi}{2}\nu\right)+\frac{J_za}{v_F}\cos(\pi\nu)-\frac{2\Delta^2}{\pi}\right]. \tag{44}$$

The velocities $v_{c/s}$ are renormalized by magnetic interactions and the interactions induced by the no-double-occupancy conditions:

$$v_c=v_F\sqrt{1+\frac{4\Delta}{\pi}\cot\left(\frac{\pi}{2}\nu\right)}, \tag{45}$$

$$v_s=v_F\sqrt{1+\frac{J_za}{\pi v_F}-\frac{4\Delta}{\pi}\cot\left(\frac{\pi}{2}\nu\right)}. \tag{46}$$

The controlling parameters $K_{c/s}$ are given by

$$K_c=\frac{4\pi}{\sqrt{1+\frac{4\Delta}{\pi}\cot\left(\frac{\pi}{2}\nu\right)}}, \tag{47}$$

$$K_s=\frac{4\pi}{\sqrt{1+\frac{J_z a}{\pi v_F}-\frac{4\Delta}{\pi}\cot\left(\frac{\pi}{2}\nu\right)}}. \tag{48}$$

In passing, we would like to stress that the above results are derived for small Δ and $J_z a$. However, the general result of a renomalization of $v_{c/s}$ and $K_{c/s}$, but with no other changes, is expected to be valid more generally.[27,28] In other words, the functional forms of the low energy effective Hamiltonians $H_{c/s}$, being basically dictated by the symmetry requirements, survive even if the interactions are strong, while the above values of $v_{c/s}$ and $K_{c/s}$ are not universal. Therefore, we conclude that if we adiabatically continue the value of Δ to unity, the low energy effective Hamiltonian of the t-J_z model should be of the same form as the above charge and spin Hamiltonians $H_{c/s}$, with the renormalized values of $v_{c/s}$ and $K_{c/s}$ not restricted to those given by Eqs. (45) and (47).

C. Phase diagram

Now we are in the position to discuss the possible phase diagram for the t-J_z model. For convenience, we only discuss the antiferromagnetic case, namely, we assume $J_z>0$.

At first, we notice that the spin and charge degrees of freedom are well separated just like what happened in other 1D interacting models. However, from the expression for the controlling parameters $K_{c/s}$, we have already seen the interesting interplay between hopping and magnetic interactions. The phase diagram is determined by the competition of above two energy scales (t and J_z). This is quite different from the case of the Hubbard model or the XXZ model where the controlling parameter is only determined by the interaction strength. However, the charge sector is massless, described by a quadratic Hamiltonian with no mass term. This means that charge excitations are gapless and the charged sector of the system is metallic. In contrast, the knowledge on the fate of the spin sector needs more work. The situation is similar to that of the Hubbard model.

The fate of the spin sector is determined by the well-studied sine-Gordon Hamiltonian. In the spin sector we have the renormalization group equations (RGE's) (Refs. 28 and 27)

$$\frac{dg_\theta}{dl}=\left(2-\frac{K_s}{2\pi}\right)g_\theta, \tag{49}$$

$$\frac{dK_s}{dl}=-\delta g_\theta^2, \tag{50}$$

where δ is a positive, regularization dependent parameter. With these two RG equations in hand, we can readily analyze the phase diagram for the spin sector.

(i) When $K_s>4\pi$ and $|g_\theta|\leq[(K_s/4\pi)-1]/\sqrt{\delta}$, the spin sector flows to the fixed point line:

$$g_\theta^*=0, \tag{51}$$

$$K_s>4\pi. \tag{52}$$

Thus we obtain the Luttinger liquid behavior for the spin sector. Following the Balents-Fisher's notation,[5] we say that the system is in the $C1S1$ phase; here more generally a $CmSn$ phase means a phase with m massless charge modes and n massless spin modes respectively.

(ii) When parameters K_s and g_θ satisfy one of the conditions

$$K_s\leq 4\pi,\quad g_\theta>0 \tag{53}$$

or

$$K_s>4\pi,\quad g_\theta<\left(\frac{K_s}{4\pi}-1\right)/\sqrt{\delta}, \tag{54}$$

then the RGE flows toward $g_\theta=+\infty$. In this case, the behavior of the system is overwhelmingly determined by the minima of the cosine term. For $g_\theta>0$, these minima are given by

$$\theta_s=\sqrt{2}\left(n+\frac{1}{2}\right)\pi, \tag{55}$$

but due to the angular nature of the variable θ_s, we can have only two distinct ground states, distinguished by the even and odd values of n. This state is identified to be Peierls ordering of spin degrees of freedom. Due to quantum tunneling, the degeneracy of the ground state is removed. Consequently, the excitations above either ground state are gapful. The dominant contributions to the mass gap come from the topological soliton excitations in the dilute gas approximation of solitons and antisolitons. Therefore, in this phase, the spin sector is gapful, and we classify the phase of the system as a $C1S0$ phase.

(iii) In contrast to case (ii), if the parameter K_s and g_θ satisfy one of the following two conditions

$$K_s\leq 4\pi,\quad g_\theta<0 \tag{56}$$

or

$$K_s>4\pi,\quad g_\theta<-\left(\frac{K_s}{4\pi}-1\right)/\sqrt{\delta}, \tag{57}$$

then the RGE flows toward $g_\theta=-\infty$. An argument similar to that in case (ii) gives the ground states determined by

$$\theta_s=\sqrt{2}n\pi. \tag{58}$$

In this state, we have a staggered expectation value for the z component of the spin. Therefore, the spin ordering is Neel-like. In summary, we construct the phase diagram for the t-J_z model in Fig. 1.

III. t-$J_\perp$-J_Z MODEL

In this section, we will discuss the modified version [Eq. (7)] of the t-J model. Again the change to make is in the magnetic interactions. In addition to the $U(J_z)$ term discussed in Sec. II, we now add the XY part, $U(J_\perp)$, of the antiferromagenetic interactions:

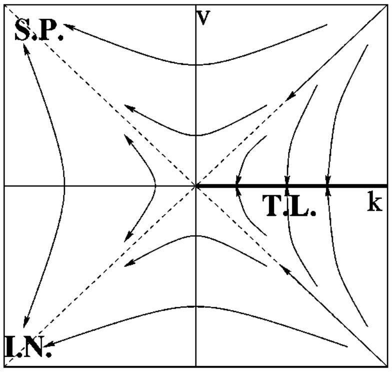

FIG. 1. The schematic phase diagram. The RGE flow gives the possible fate of the t-J_z model as the spin-Peiers phase (S.P.), Ising-Neel phase (I.N.), and the Tomonoga-Luttinger phase (T.L.).

$$U(J_\perp)=J_\perp\sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y)$$
$$=\frac{J_\perp}{2}\sum_j (c_{j\uparrow}^\dagger c_{j\downarrow} c_{j+1,\uparrow}^\dagger c_{j+1,\downarrow} + c_{j\downarrow}^\dagger c_{j\uparrow} c_{j+1,\downarrow}^\dagger c_{j+1,\uparrow}). \tag{59}$$

Following the bosonization procedure presented in Sec. II, we obtain the bosonized form for $U(J_\perp)$ as

$$U(J_\perp)=\frac{J_\perp}{2\pi^2 a}\int dx\cos(\sqrt{2}\phi_s)\cos(\sqrt{2}\theta_s)$$
$$-\frac{J_\perp}{2\pi^2 a}[1-\cos(2k_F a)]\int dx\cos(\sqrt{2}\phi_s). \tag{60}$$

Therefore, for the modified t-J model [Eq. (7)], we have the bosonized low energy effective Hamiltonian

$$\mathcal{H}=\int dx(H_c+\tilde{H}_s), \tag{61}$$

where the Hamiltonian of the charge sector, H_c, is still given by Eq. (42), since the XY part of magnetic interactions only changes spin dynamics. In contrast, due to the extra $U(J_\perp)$, the Hamiltonian H_s of the spin sector has been drastically modified to

$$\tilde{H}_s=\frac{v_s}{2}\left[K_s\Pi_s^2+\frac{1}{K_s}(\partial_x\theta_s)^2\right]$$
$$+\frac{g_\theta}{8\pi^2a^2}\int dx\cos(\sqrt{2}\theta_s)-\frac{g_\phi}{8\pi^2a^2}\int dx\cos(\sqrt{2}\phi_s)$$
$$+\frac{g_{\theta\phi}}{8\pi^2a^2}\int dx\cos(\sqrt{2}\phi_s)\cos(\sqrt{2}\theta_s). \tag{62}$$

Compared to the t-J_z model, the Hamiltonian H_s now has two extra terms with the coupling constants g_ϕ and $g_{\theta\phi}$, respectively. The spin velocity (v_s) and controlling parameter (K_s) are still the same as those in Eqs. (45)–(47). Using Eq. (60), the coupling constants g_ϕ and $g_{\theta\phi}$ are determined to be

$$g_\phi=8J_\perp a\sin^2\left(\frac{\pi}{2}\nu\right), \tag{63}$$

$$g_{\phi\theta}=4J_\perp a. \tag{64}$$

Due to the appearance of the interaction term $g_{\theta\phi}$, which has a nonzero conformal spin, the dynamics for the spin sector becomes much more involved. When we use the scaling arguments to discuss the relevance of the interaction terms, we need to be more careful. We had better use the RG flow for Hamiltonian (62) to discuss the details of the spin dynamics. Fortunately, up to one loop level, the RGE for a Hamiltonian like Eq. (62) have been studied thoroughly, though in a quite different context.[29,30,27] The resulting RGE for the double cosine term $g_{\theta\phi}$ is

$$\frac{dg_{\theta\phi}}{dl}=2\left[1-\left(\frac{K_s}{4\pi}+\frac{4\pi}{K_s}\right)\right]g_{\theta\phi}. \tag{65}$$

Since we know

$$\frac{K_s}{4\pi}+\frac{4\pi}{K_s}\geqslant 2, \tag{66}$$

the double cosine term in Hamiltonian (62) is always irrelevant. Of course, the action of the RG will generate more terms, such as single cosine terms. However, the arguments of these single cosine terms are twice as big and these terms are more irrelevant than the existing terms. Thus we can neglect them. This situation is quite different from that of the two coupled Luttinger liquid case.[29,30,27] Therefore, we only need to focus on the effective Hamiltonian

$$\tilde{H}_s=\frac{v_s}{2}\left[K_s\Pi_s^2+\frac{1}{K_s}(\partial_x\theta_s)^2\right]+\frac{g_\theta}{8\pi^2a^2}\int dx\cos(\sqrt{2}\theta_s)$$
$$-\frac{g_\phi}{8\pi^2a^2}\int dx\cos(\sqrt{2}\phi_s)$$
$$=\frac{v_s}{2}[\tilde{\Pi}_s^2+(\partial_x\tilde{\theta}_s)^2]+\frac{g_\theta}{8\pi^2a^2}\int dx\cos(\beta_s\tilde{\theta}_s)$$
$$-\frac{g_\phi}{8\pi^2a^2}\int dx\cos(\tilde{\beta}_s\tilde{\phi}_s), \tag{67}$$

where we have introduced $\tilde{\Pi}_s=\sqrt{K_s}\Pi_s$ and $\tilde{\theta}_s=\theta_s/\sqrt{K_s}$. The definitions of β_s and $\tilde{\beta}_s$ are

$$\beta_s=\sqrt{2K_s},\quad \tilde{\beta}_s=\frac{16\pi}{\sqrt{2K_s}}. \tag{68}$$

It is now easy to observe that the low energy effective Hamiltonian possesses the following duality property: That is, the Hamiltonian (67) is invariant under the following transformation:

$$\beta_s \leftrightarrow \tilde{\beta}_s, \quad g_\theta \leftrightarrow -g_\phi. \tag{69}$$

Note that such a duality does not appear in the t-J_z model or in the Hubbard model. But it is also interesting to note that it appeared in the 1D XYZ Thirring model[27] and in the case of two coupled Luttinger liquids.[29,30,27]

Compared with the sine-Gordon system, the symmetry of Eq. (67) is discrete, while there is a hidden $U(1)$ symmetry in the sine-Gordon system, which reflects the $U(1)$ invariance of its dual fermionic model (the massive Thirring model).

It is also easy to get the scaling dimension for the cosine terms of the field θ_s and its conjugate ϕ_s as

$$\Delta_\theta = \frac{K_s}{2\pi}, \quad \Delta_\phi = \frac{8\pi}{K_s}. \tag{70}$$

Therefore, one of the two cosine terms is always relevant, which is associated with the ordering of the θ or ϕ field. Let us discuss the following two different cases separately.

(i) When the scaling dimension $\Delta_\theta < 2$, the $\cos(\sqrt{2}\theta_s)$ term is relevant. This case is similar to the t-J_z case, and the system eventually flows to the spin-Peierls phase for $g_\theta > 0$ or the Ising-Neel order for $g_\theta < 0$ respectively.

(ii) When the scaling dimension $\Delta_\phi < 2$, the $\cos(\sqrt{2}\phi_s)$ term is relevant. In this case, since g_ϕ is always negative, therefore the system flows toward the Ising-Neel phase only.

In summary, we see that the phase diagram in Fig. 1 can only be partially accessed in the t-J model. The difference in the two cases reflects the fact that the duality transformation (69) can only be realized in part of the parameter space, since the coupling constant g_ϕ is definitely negative, while the coupling constant g_θ can be either negative or positive, depending on the interplay between t and J_z.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, the phase diagram of the most general 1D t-$J_\perp$-J_z model is discussed based on bosonization and the RGE. To make sense of the bosonization procedure for the interactions induced by no double occupancy constraints, we have introduced a deformation of Hubbard operators [Eq. (6)], which contain a deformation parameter Δ. While at $\Delta = 1$ the no double occupancy constraints at each site are recovered, the case with a small positive Δ is accessible to perturbative RG analysis. Since the basic structure of the bosonized low energy effective Hamiltonian is argued to be determined only by the symmetry requirements, the bosonized form of the low energy effective Hamiltonian with a small deformation parameter Δ is expected to survive the limit $\Delta \rightarrow 1$. However, we cannot simply use the values of $v_{c/s}$ and $K_{c/s}$ to make precise predictions on the phase diagram, since these values are not reliable at $\Delta = 1$. We should take the strategy in which both $v_{c/s}$ and $K_{c/s}$ are considered as phenomenological parameters.

For the case with $J_z \gg J_\perp$, the model is reduced to the so-called t-J_z model. In this case, the spin sector can flow to three distinct phases: the gapless phase, the spin-Peierls phase, and the Ising-Neel phase, depending on the range of the parameters, meanwhile the charge dynamics remains always gapless. In the case with $J_z > J_{zc}$, where J_{zc} represents the value to make $g_\theta = 0$ and $K_s < 4\pi$, the system flows to the Ising-Neel ordering in spin dynamics. We identify this phase as the so-called phase separation phase. For the case with $J_z < J_{zc}$ and $K_s < 4\pi$, the spin sector eventually flows to the spin-Peierls phase which is gapful. We can identify this phase as a superconducting phase. Finally, for $K_s > 4\pi$, the spin sector flows toward a gapless phase and thus the system flows toward the Tomonoga-Luttinger liquid phase. Such a phase is consistent with the phase diagram constructed by the Los Alamos group in Ref. 19, where the authors mapped the t-J_z model into the 1D XXZ model and constructed the phase diagram from the knowledge of exact solutions for the 1d XXZ model. This consistency also helps us to justify our proposal to use the deformed Hubbard operators and the continuation from the case of $\Delta \ll 1$ to the desired case $\Delta = 1$.

In the opposite limit, namely, $J_\perp \gg J_z$, the modified t-J model can be reduced the the t-$J_\perp$ model. In this case, we still have g_θ generally nonzero due to the no-double-occupancy induced interactions. Therefore, the phase diagram of the t-$J_\perp$ model is expected to be similar to the case of the most general t-$J_\perp$-J_z model. That is, the system is generically gapful in the spin sector and thus cannot flow toward the Tomonoga-Luttinger phase. This result is a little bit different from the naive speculation that the t-J_z model should be basically similar to the t-J model. From our study, we conclude that there are some delicate differences between the two cases, since the XY part and the Ising part of the magnetic interactions play different roles in the spin ordering.

In summary, the key physical idea of our analytical study of the phase diagram in the one dimensional t-J model is the introduction of a deformed Hubbard operator [Eq. (6)], with the deformation parameter Δ treated as a small parameter. Therefore, the four- and six-fermion interactions induced by the no double occupancy constraints can be treated as perturbations. This allows us to do two things: (1) a bosonization of the deformed model, and (2) a perturbative RG analysis. Finally we continue Δ to its physical value $\Delta = 1$. The consistency has been verified *a posteriori*, and the results justify the approach. Why is the continuation from $\Delta \ll 1$ to $\Delta = 1$ so good? We think the powerfulness of this approach lies in the fact that it combines together three powerful techniques, each having a wide range of validity: The first one is the idea of deformation, underlying which is the principle of *adiabatic continuity*. The second is bosonization, by which the form of the low energy effective Hamiltonian is determined *only by symmetry requirements*. Finally, the technique of perturbative RG analysis is known to be powerful in providing a *classification of universality classes* for quantum phases and quantum phase transitions. Any of the above three techniques does not depend on the precise value of Δ, so neither will a combination of them, as one might expect *a priori*. Of course, whether the continuation indeed works all way up to $\Delta = 1$ can only be tested *a posteriori*, by comparison either with other reliable methods or, eventually, with experimental results if available.

Finally we would like to mention that the technique of introducing deformed Hubbard operators may work in other one-dimensional models in which the no double occupancy constraints play an important role. There may be a chance for this deformation technique to work as well in *higher* dimensions, when combined with other techniques.

ACKNOWLEDGMENTS

This research was supported in part by the U.S. National Science Foundation under Grant No. PHY-9970701.

[1] P. W. Anderson, Science **235**, 1196 (1987).
[2] F. C. Zhang and T. M. Rice, Phys. Rev. B **37**, 3759 (1988); **41**, 7243 (1990).
[3] For a review, see E. Dagotto and T. M. Rice, Science **271**, 618 (1996); E. Dagotto, *ibid.* **293**, 2430 (2001).
[4] H. Mayaffre *et al.*, Science **279**, 345 (1998); J. H. Schön *et al.*, *ibid.* **293**, 2430 (2001).
[5] L. Balents and M. P. A. Fisher, Phys. Rev. B **53**, 12 133 (1996); H. H. Lin, L. Balents, and M. P. A. Fisher, *ibid.* **56**, 6569 (1998); **58**, 1794 (1998).
[6] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
[7] E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968).
[8] H. A. Bethe, Z. Phys. **71**, 205 (1931).
[9] C. N. Yang and C. P. Yang, Phys. Rev. **150**, 321 (1966).
[10] C. K. Lai, J. Math. Phys. **15**, 1675 (1974); B. Sutherland, Phys. Rev. B **12**, 3795 (1975); P. Schlottmann, Phys. Rev. B **36**, 5177 (1987); P. A. Bares, G. Blatter, and M. Ogatta, *ibid.* **44**, 130 (1991).
[11] P. B. Wiegmann, Phys. Rev. Lett. **60**, 821 (1988).
[12] P. Coleman, Phys. Rev. B **29**, 3035 (1984).
[13] N. Read, D. M. Newns, and S. Doniach, Phys. Rev. B **30**, 3841 (1984).
[14] P. Coleman, C. Pepin, and A. M. Tsvelik, Phys. Rev. B **62**, 3852 (2000).
[15] P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928).
[16] E. H. Lieb, T. D. Schultz, and D. C. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).
[17] J. Hubbard, Proc. R. Soc., Dondon, Ser. A **277**, 237 (1964).
[18] M. Ogata *et al.*, Phys. Rev. Lett. **66**, 2388 (1991).
[19] D. Batista and G. Ortiz, Phys. Rev. Lett. **85**, 4755 (2000).
[20] E. Dagotta, Rev. Mod. Phys. **66**, 763 (1994).
[21] N. E. Bonsteel *et al.*, Phys. Rev. Lett. **68**, 2684 (1992).
[22] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982).
[23] Y.-S. Wu, Phys. Rev. Lett. **52**, 2103 (1984); **53**, 111 (1984).
[24] D. M. Haldane, J. Phys. C **14**, 2585 (1981).
[25] A. Luther and V. J. Emery, Phys. Rev. Lett. **33**, 589 (1974).
[26] For a review, see, H. J. Shulz, G. Cuniberti, and P. Pieri, cond-mat/9807366 (unpublished); J. von Delft and H. Schoeller, cond-mat/9805275 (unpublished).
[27] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, New York, 1998).
[28] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, New York, 1999).
[29] F. V. Kusmartsev, A. Luther, and A. A. Nersesyan JETP Lett. **55**, 724 (1992).
[30] V. M. Yakovenko, JETP Lett. **56**, 510 (1992).